16.32 Project: Continuous Low-Thrust Asteroid Mission

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May 14, 2018

1 Problem

The goal of this project was to optimise a continuous low-thrust trajectory from geostationary orbit to landing on an asteroid. Low-thrust propulsion is appealing for small satellite missions as their increased efficiency, versus chemical propulsion alternatives, allows for reasonable fuel mass fractions and therefore the ability to carry scientific payload. However, the increased efficiency comes at a cost in terms of the mission time. Therefore, this project looked at the minimum time problem for a low-thrust asteroid mission to see just how quickly missions could be performed and at what cost in terms of payload mass. Since the problem constrained the control to continuous thrust, the minimum time optimisation is equivalent to the minimum fuel optimisation.

A 3U CubeSat (4kg) spacecraft was used as a representative mass and the ion Electrospray Propulsion System (iEPS) under development in the SPL was used as a representative low-thrust propulsion system. In its current state, the iEPS performance characteristics in a 3U configuration (0.5mN thrust, 1000s I_{sp}) give GPOPS-II problems in terms of converging to a feasible solution during the escape from Earth. Therefore, near-term developments for the iEPS were considered that will increase the thrust to 10mN and the I_{sp} to 2500s.

The target asteroid was characterised by three parameters: the semi-major axis of its orbit, the eccentricity of its orbit, and its radius. A density of 3.2g/cm^3 was assumed based on typical asteroid densities assumed in the literature which allows for calculation of the asteroid's gravitational parameter and the approximate radius of its sphere of influence through

$$r_{\mathrm{SOI}} \approx a \left(\frac{\mu_a}{\mu_s}\right)^{2/5} \tag{1}$$

where a is the asteroid's semi-major axis in the heliocentric frame, μ_a is the asteroid's gravitational parameter, and μ_s is the Sun's gravitational parameter. For a 10m radius asteroid, the sphere of influence has a radius of approximately 106m.

2 Method

This problem was solved as a three phase problem in GPOPS-II. Each phase corresponds to a different central body for the spacecraft's orbit. Initially, the central body is the Earth. Then the spacecraft moves into a heliocentric orbit before ending in the frame of the asteroid. This allows the problem to be solved as three separate two-body problems which greatly simplifies the dynamics. Between each phase the heliocentric positions of the spacecraft are matched to ensure continuity of the trajectory.

2.1 Dynamics

In each phase there are seven state variables:

$$\begin{aligned}
r &: \text{ radial position} \\
\theta &: \text{ angular position} \\
v_r &: \text{ velocity in the radial direction} \\
v_\theta &: \text{ velocity in the angular direction} \\
m &: \text{ spacecraft mass} \\
\theta_a &: \text{ heliocentric angular position of the asteroid} \\
\theta_e &: \text{ heliocentric angular position of Earth}
\end{aligned}$$

The propulsion system is represented by two control inputs:

$$\begin{aligned}
F_r &: \text{ thrust in the radial direction} \\
F_\theta &: \text{ thrust in the angular direction}
\end{aligned}$$

In the MATLAB code, the variable θ is represented by the variable h ($\theta = h$, $v_\theta = v_h$, etc.) to simplify the variable names.

The first five state variables correspond to the spacecraft and have corresponding dynamics given by

$$\dot{r} = v_r \tag{2}$$

$$\dot{\theta} = v_\theta / r \tag{3}$$

$$\dot{v}_r = v_\theta^2 / r - \mu / r^2 + F_r / m \tag{4}$$

$$\dot{v}_\theta = -v_r v_\theta / r + F_\theta / m \tag{5}$$

$$\dot{m} = -F / I_{sp} / g \tag{6}$$

where F is the total thrust of the propulsion system calculated as

$$F = \sqrt{F_r^2 + F_\theta^2} \tag{7}$$

and g is the gravitational acceleration at the surface of the Earth (9.81m/s^2). The gravitational parameter, μ, is dependent on the central body of the orbit and therefore the phase. However, since μ is the only variable that changes between each phase this allows for the same dynamics equations to be used for all three phases.

The dynamics of the asteroid are

$$\dot{\theta}_a = v_{\theta,a} / r_a \tag{8}$$

where the radial position of the asteroid can be calculated as

$$r_a = \frac{a(1 - e^2)}{1 + e\cos(\theta_a)} \tag{9}$$

where a is the asteroid's orbital semi-major axis and e is the asteroid's orbital eccentricity. Therefore, the radial position is completely defined by the asteroid's angular position and does not need to be tracked as a separate state variable. The velocity of the asteroid in the angular direction can be calculated as

$$v_{\theta,a} = \frac{\sqrt{a(1 - e^2)\mu_s}}{r_a} \tag{10}$$

which is simply the angular momentum of the asteroid's orbit divided by its radial position. Like the radial position, the velocity in the angular direction is completely defined by the asteroid's angular position and does not need to be tracked as a separate state variable.

The dynamics of the Earth are simply

$$\dot{\theta}_e = \sqrt{\mu_s / r_e^3} \tag{11}$$

where the orbital radius of the Earth, r_e, is assumed constant at 1au.

2.2 Constraints

Throughout all three phases, a path constraint on the total thrust is given by

$$\sqrt{F_r^2(t) + F_\theta^2(t)} = F_{\text{iEPS}} \tag{12}$$

which constrains the total thrust at all times to be equal to the thrust of the iEPS (10mN).

Between each phase event groups are used to ensure continuity of the trajectory. A total of eight constraints are applied between each phase:

$$\begin{aligned} t_{f,i} &= t_{0,i+1} \\ x_{f,i} &= x_{0,i+1} \\ y_{f,i} &= y_{0,i+1} \\ v_{x,f,i} &= v_{x,0,i+1} \\ v_{y,f,i} &= v_{y,0,i+1} \\ m_{f,i} &= m_{0,i+1} \\ \theta_{a,f,i} &= \theta_{a,0,i+1} \\ \theta_{e,f,i} &= \theta_{e,0,i+1} \end{aligned}$$

where the subscript f, i represents the final state of phase i and subscript $0, i+1$ represents the initial state of phase $i+1$. t represents the time which must be continuous between phases with appropriate unit conversions. x and y represent the heliocentric position of the spacecraft. In phase one this is calculated as

$$x_{f,1} = r_{f,1} \cos\theta_{f,1} + r_e \cos\theta_{e,f,1} \tag{13}$$

$$y_{f,1} = r_{f,1} \sin\theta_{f,1} + r_e \sin\theta_{e,f,1} \tag{14}$$

In phase two this is calculated as

$$x_{0,2} = r_{0,2} \cos\theta_{0,2} \tag{15}$$

$$y_{0,2} = r_{0,2} \sin\theta_{0,2} \tag{16}$$

$$x_{f,2} = r_{f,2} \cos\theta_{f,2} \tag{17}$$

$$y_{f,2} = r_{f,2} \sin\theta_{f,2} \tag{18}$$

while in phase three this is calculated as

$$x_{0,3} = r_{0,3} \cos\theta_{0,3} + r_e \cos\theta_{a,0,3} \tag{19}$$

$$y_{0,3} = r_{0,3} \sin\theta_{0,3} + r_e \sin\theta_{a,0,3} \tag{20}$$

v_x and v_y represent the heliocentric velocity of the spacecraft. In phase one this is calculated as

$$v_{x,f,1} = v_{r,f,1} \cos\theta_{f,1} - v_{\theta,f,1} \sin\theta_{f,1} - v_e \sin\theta_{e,f,1} \tag{21}$$

$$v_{y,f,1} = v_{r,f,1} \sin\theta_{f,1} + v_{\theta,f,1} \cos\theta_{f,1} + v_e \cos\theta_{e,f,1} \tag{22}$$

where v_e, the velocity of the Earth, is calculated as

$$v_e = \sqrt{\mu_s / r_e} \tag{23}$$

In phase two the heliocentric velocities are calculate as

$$v_{x,0,2} = v_{r,0,2} \cos\theta_{0,2} - v_{\theta,0,2} \sin\theta_{0,2} \tag{24}$$

$$v_{y,0,2} = v_{r,0,2} \sin\theta_{0,2} + v_{\theta,0,2} \cos\theta_{0,2} \tag{25}$$

$$v_{x,f,2} = v_{r,f,2} \cos\theta_{f,2} - v_{\theta,f,2} \sin\theta_{f,2} \tag{26}$$

$$v_{y,f,2} = v_{r,f,2} \sin\theta_{f,2} + v_{\theta,f,2} \cos\theta_{f,2} \tag{27}$$

In phase three the heliocentric velocities are calculated as

$$v_{x,0,3} = v_{r,0,3}\cos\theta_{0,3} - v_{\theta,0,3}\sin\theta_{0,3} + v_{a,r,0,3}\cos\theta_{a,0,3} - v_{\theta,a,0,3}\sin\theta_{a,0,3} \tag{28}$$

$$v_{y,0,3} = v_{r,0,3}\sin\theta_{0,3} + v_{\theta,0,3}\cos\theta_{0,3} + v_{a,r,0,3}\sin\theta_{a,0,3} + v_{\theta,a,0,3}\cos\theta_{a,0,3} \tag{29}$$

where the velocity of the asteroid in the angular direction is calculated as before in the dynamics as

$$v_{\theta,a,0,3} = \frac{\sqrt{a(1-e^2)\mu_s}}{r_{a,0,3}} \tag{30}$$

where the radial position is also calculated as before in the dynamics as

$$r_{a,0,3} = \frac{a(1-e^2)}{1+e\cos(\theta_{a,0,3})} \tag{31}$$

The radial velocity of the asteroid can then be calculated based on the specific energy of the asteroid's orbit

$$\frac{1}{2}\left(v_{r,a,0,3}^2 + v_{\theta,a,0,3}^2\right) - \frac{\mu_s}{r_{a,0,3}} = -\frac{\mu_s}{2a} \tag{32}$$

which gives the magnitude of the radial velocity as

$$v_{r,a,0,3} = \sqrt{\mu_s\left(\frac{2}{r_{a,0,3}} - \frac{1}{a}\right) - v_{\theta,a,0,3}^2} \tag{33}$$

The direction is given by the true anomaly of the asteroid (equivalent to the angular position in this problem). If the true anomaly is in the range $[0, \pi]$ then the radial velocity is positive and if the true anomaly is in the range $[\pi, 2\pi]$ then the radial velocity is negative.

The initial state of the spacecraft is constrained to be in geostationary orbit while the final state of the spacecraft (in phase 3) is constrained to be landed on the asteroid with zero velocity.

2.3 Initial Guess

One of the difficulties with solving this problem is providing a good initial guess as without a good initial guess, GPOPS-II struggles to converge to a feasible solution. An initial guess can be provided by an analytical estimate of the spacecraft state. To develop the analytical estimate start with the specific energy of the spacecraft

$$\epsilon = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{34}$$

Under the assumption that the orbit is approximately circular then the specific energy can be expressed as

$$\epsilon = \frac{1}{2}\frac{\mu}{r} - \frac{\mu}{r} \tag{35}$$

$$= -\frac{1}{2}\frac{\mu}{r} \tag{36}$$

The time derivative is therefore

$$\frac{d\epsilon}{dt} = \frac{1}{2}\frac{\mu}{r^2}\frac{dr}{dt} \tag{37}$$

which is related to the power input of the propulsion system by

$$\frac{d\epsilon}{dt} = \frac{1}{m}\vec{F}\cdot\vec{v} \tag{38}$$

$$= \frac{1}{m}\sqrt{\frac{\mu}{r}}F \tag{39}$$

if we assume that the thrust is applied in the direction of velocity. Combining the two equations for the time rate of change of the specific energy we get

$$\frac{1}{2}\frac{\mu}{r^2}\frac{dr}{dt} = \frac{1}{m}\sqrt{\frac{\mu}{r}}F \tag{40}$$

The solution to this differential equation is

$$r(t) = \frac{r_0}{\left(1 \pm \frac{a_p}{v_0}t\right)^2} \tag{41}$$

where r_0 is the initial orbital radius, v_0 is the initial orbital velocity, and a_p is the acceleration provided by the propulsion system. The sign of the $\pm$ depends on if the trajectory is outbound (-) or inbound (+). The other state variables can then be calculated as (only the outbound forms of the states are given)

$$v_r = \frac{dr}{dt} = \frac{2r_0 a_p/v_0}{\left(1 - \frac{a_p}{v_0}t\right)^3} \tag{42}$$

$$v_\theta = \sqrt{\frac{\mu}{r}} = v_0\left(1 - \frac{a_p}{v_0}t\right) \tag{43}$$

$$\theta = \theta_0 + \int_0^t \frac{v_\theta}{r}dt = \theta_0 + \frac{1}{4}\frac{v_0^2}{r_0 a_p}\left[1 - \left(1 - \frac{a_p}{v_0}t\right)^4\right] \tag{44}$$

The time for phase one can then be calculated by solving

$$r(t_{f,1}) = r_{\mathrm{SOI},e} \tag{45}$$

which gives the time at which the spacecraft leaves the sphere of influence of the Earth. We can then calculate the time for phase two by solving

$$r(t_{f,2}) = a \tag{46}$$

which gives the time at which the spacecraft intersects the semi-major axis of the asteroid. With the time in hand we can generate analytical initial guesses for the solution along phases one and two. For phase three we assume that the spacecraft is in orbit around the asteroid at the radius of the asteroid and calculate the outbound trajectory for the spacecraft to escape from the asteroid. To finalise the initial guess we reverse the state variables in time.

The analytical estimates don't provide a perfect initial guess but it provides a good enough initial guess that GPOPS-II is able to converge to a solution.

3 Results

Given the GPOPS-II problem framework developed above we can solve for time/fuel optimal trajectories with either the angular positions of the asteroid and Earth free or fixed. A typical heliocentric trajectory for an asteroid with a semi-major axis of 1.1au and eccentricity of 0 is shown in Figure 1. Figure 2 shows the trajectories from escape from Earth and landing at the asteroid. The line perpendicular to the trajectory indicates the transition between phases. We can see that the position of the spacecraft as well as its velocity (at least in direction) are continuous between phases. Figure 3 shows the control for all three phases.

3.1 Asteroid 2001 AV43

One use of the GPOPS-II problem framework is to analyse the optimal mission start time for an asteroid mission. For this analysis we will consider the asteroid 2001 AV43 which has an inclination similar to that

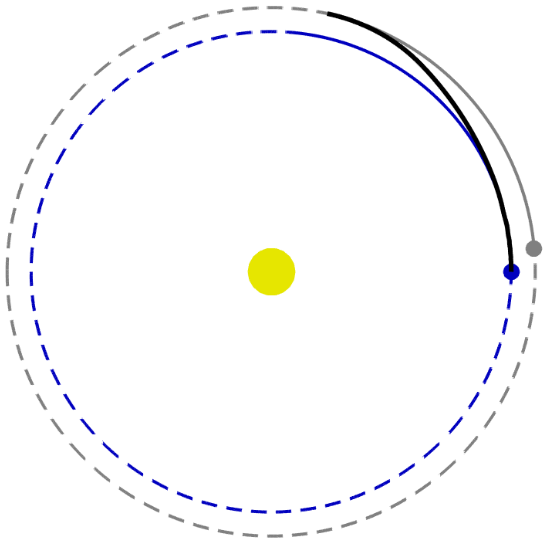

Figure 1: Trajectory to an asteroid in a circular orbit with radius 1.1au.

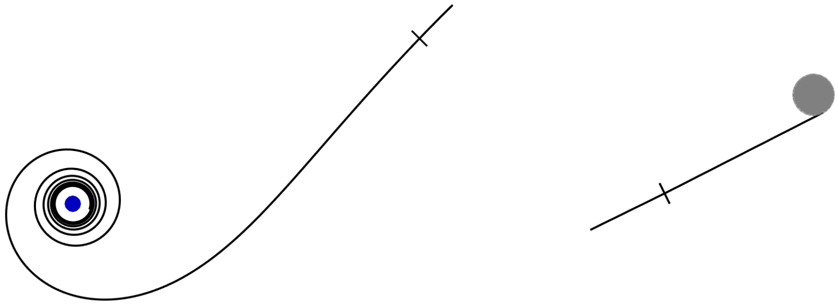

Figure 2: Trajectories for escape from Earth and landing at asteroid with transition between phases marked with the perpendicular line.

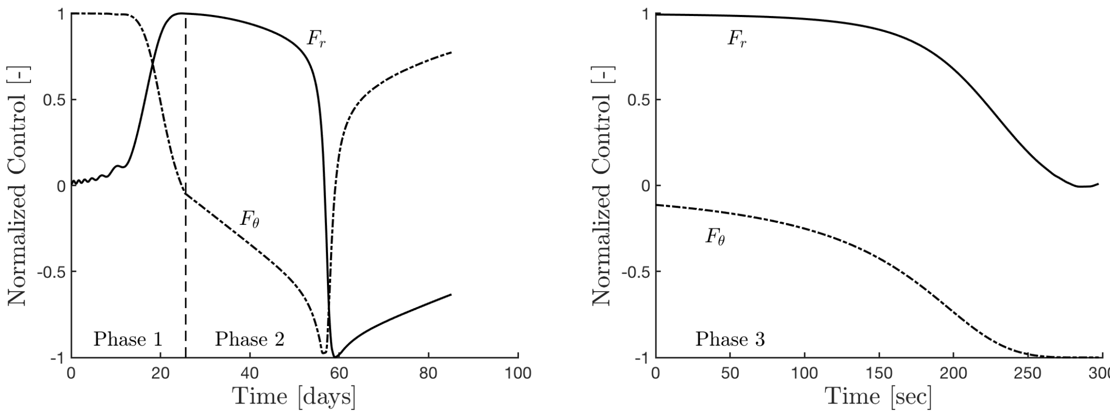

Figure 3: Control for phases 1 and 2 (left) and phase 3 (right).

of Earth and therefore allows us to use the 2D dynamics. 2001 AV43 has a semi-major axis of 1.28au and an eccentricity of 0.241. It had a closest approach to Earth in mid November of 2013. Figure 4 shows the orbit of 2001 AV43 as well as positions of the Earth and 2001 AV43 on August 1, 2013 and February 1, 2014.

To find the optimal mission start date, a solution was found on October 1, 2013 using the analytical estimate for the initial guess. The solution on October 1 was then used as the initial guess to find the solution on October 2, and so on. In this manner, we can (relatively) quickly determine the payload mass fraction for each mission start date by propagating solutions for initial guesses.

The payload mass fraction landed on the asteroid for each mission start day is shown in Figure 5. Starting from October 1, 2013 solutions were found backwards in time to August 1, 2013. Solutions were then found forwards in time until early December. GPOPS-II was unable to find solutions in early December (reason unknown) so a new solution was generated at February 1, 2014 and solutions found backwards in time.

We can see that the payload mass fraction increases linearly from August 1, 2013 to November 15, 2013 where it reaches a maximum (approximately 0.65). Beyond November 15, 2013 the payload mass fraction drops quickly before levelling out at around 0.3 in December.

3.2 Optimal Eccentricity

Another use of the GPOPS-II problem framework is to analyse the optimal eccentricity for a given semi-major axis. If we leave the initial Earth and asteroid angular positions free and vary the eccentricity we notice that an optimal eccentricity exists. Figure 6 shows the payload mass fraction versus eccentricity for a semi-major axis of 1.21au. We can see that the payload mass fraction is optimised around an eccentricity of 0.17. The existence of an optimal eccentricity makes intuitive sense. As the eccentricity is increased from zero, the periapsis of the asteroid orbit is lowered which decreases the time required for the spacecraft to travel from Earth's orbit to that of the asteroid. However, if the eccentricity is increased too much then the velocity at periapsis of the asteroid becomes too fast and the spacecraft is unable to catch the asteroid to land on it.

To solve for the optimal eccentricity, a steepest descent (actually ascent in this case) algorithm was implemented with a constrained bisection line search over the GPOPS-II problem framework. The optimal trajectory was solved for an eccentricity of zero using the analytical estimate for the initial guess. For subsequent solutions, the previous solution was used as the initial guess to improve computation time. At each iteration, the line search distance was constrained in order to allow the previous solution to act as a good initial guess for the next eccentricity. Convergence was determined based on the change in eccentricity for the next guess.

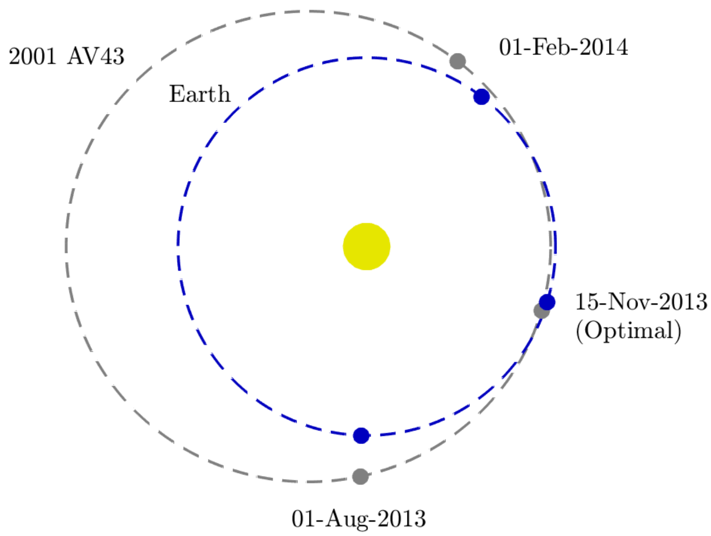

Figure 4: Positions of Earth and asteroid 2001 AV43 near closest approach.

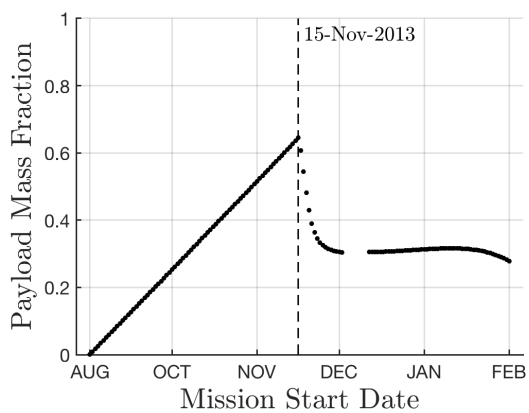

Figure 5: Payload mass fraction versus mission start date.

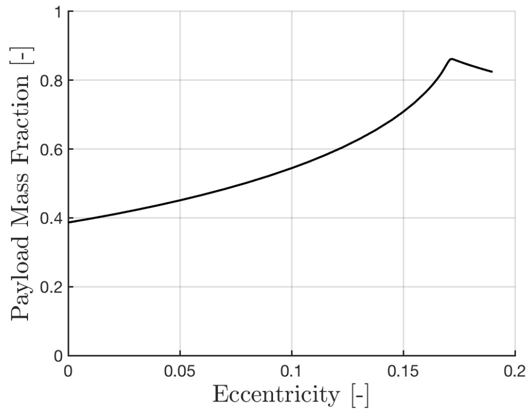

Figure 6: Payload mass fraction versus eccentricity for a semi-major axis of 1.21au.

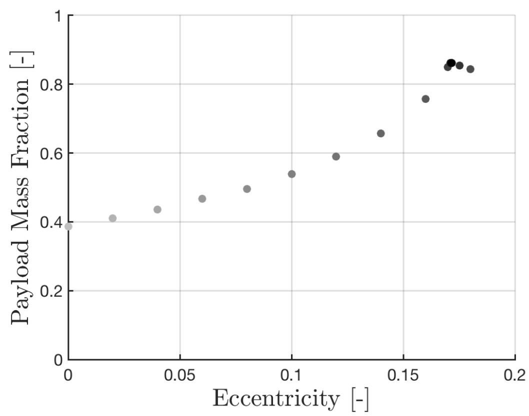

Figure 7: Optimisation of the eccentricity for a semi-major axis of 1.21au.

Figure 7 shows the sequence of iterations for the optimisation run on a semi-major axis of 1.21au with darker dots representing later iterations. We can see that for the earlier iterations the spacing between guesses is even due to the constraint on the change in eccentricity guess. Near the optimal solution the distance between guesses decreases until the convergence tolerance is met. For a semi-major axis of 1.21au the maximum payload mass fraction is 0.863 at an eccentricity of 0.171.