Characterization and Computation of Correlated Equilibria in Polynomial Games

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Joint work with Profs. Pablo Parrilo and Asuman Ozdaglar
Outline

- Introduction
- Characterizing correlated equilibria in finite games
- Characterizing correlated equilibria in continuous games [new]
- An overview of SDP / SOS methods
- Computing correlated equilibria in polynomial games with SDP relaxations [new]
Game theoretic setting

- Standard strategic (normal) form game
- Players (rational agents) numbered \( i = 1, \ldots, n \)
- Each has a set \( C_i \) of strategies \( s_i \)
- Players choose their strategies simultaneously
- Rationality: Each player seeks to maximize his own utility function \( u_i : C \rightarrow \mathbb{R} \), which represents all his preferences over outcomes
Goals

- Much is known theoretically about correlated equilibria: existence (under some topological assumptions), relation to Nash equilibria, etc.
- Efficient algorithms are known for computing these when the strategy sets are finite
- Much less on games with infinite strategy sets
- Our goal is to find classes of infinite games for which correlated equilibria can be computed, as well as associated algorithms
Previous work

- Definition of correlated equilibrium [Aumann 1974]
- Rationality argument for playing correlated equilibria [Aumann 1987]
- Elementary existence proof [Hart & Schmeidler 1989]
- Efficient algorithms for computing correlated equilibria of finite games [Papadimitriou 2005]
- Sum of squares techniques [Parrilo 2000, ...]
- Algorithm for computing minimax strategies of polynomial games [Parrilo 2006]
Chicken and correlated equilibria

<table>
<thead>
<tr>
<th>$(u_1, u_2)$</th>
<th>Wimpy</th>
<th>Macho</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wimpy</td>
<td>(4, 4)</td>
<td>(1, 5)</td>
</tr>
<tr>
<td>Macho</td>
<td>(5, 1)</td>
<td>(0, 0)</td>
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- Nash equilibria (self-enforcing independent distrib.)
  - $(M, W)$ yields utilities $(5, 1)$; $(W, M)$ yields $(1, 5)$
  - $(\frac{1}{2}W + \frac{1}{2}M, \frac{1}{2}W + \frac{1}{2}M)$ yields expected utility $(2\frac{1}{2}, 2\frac{1}{2})$

- Correlated equilibria (self-enforcing joint distrib.)
  - e.g. $\frac{1}{2}(W, M) + \frac{1}{2}(M, W)$ yields $(3\frac{1}{2}, 3\frac{1}{2})$
  - $\frac{1}{3}(W, W) + \frac{1}{3}(W, M) + \frac{1}{3}(M, W)$ yields $(3\frac{2}{3}, 3\frac{2}{3})$
Correlated equilibria in games with finite strategy sets

- $u_i(t_i, s_{-i}) - u_i(s)$ is change in player $i$’s utility when strategy $t_i$ replaces $s_i$ in $s = (s_1, \ldots, s_n)$

- A probability distribution $\pi$ is a correlated equilibrium if

$$\sum_{s \in \{r_i\} \times C_{-i}} [u_i(t_i, s_{-i}) - u_i(s)]\pi(s | s_i = r_i) \leq 0$$

for all players $i$ and all strategies $r_i, t_i \in C_i$

- No player has an incentive to deviate from his recommended strategy $r_i$
LP characterization

- A probability distribution $\pi$ is a correlated equilibrium if and only if

$$\sum_{\{s \in C_i\}} [u_i(t_i, s_{-i}) - u_i(s)]\pi(s) \leq 0$$

for all players $i$ and all strategies $s_i, t_i \in C_i$

- Proof: Use definition of conditional probability, pull denominator out of sum, and cancel it (reversible).

- Set of correlated equilibria of a finite game is a polytope
Departure function characterization

- A probability distribution $\pi$ is a correlated equilibrium if and only if
  \[ \sum_{s \in C} [u_i(\zeta_i(s_i), s_{-i}) - u_i(s)] \pi(s) \leq 0 \]
  for all players $i$ and all departure functions $\zeta_i : C_i \rightarrow C_i$

- Proof: Let $t_i = \zeta_i(s_i)$ on previous slide and sum over $s_i \in C_i$. Conversely, define $\zeta_i(r_i) = r_i$ for all $r_i \neq s_i$ and $\zeta_i(s_i) = t_i$, then cancel terms.

- Interpretation as Nash equilibria of extended game
Continuous games

- Finitely many players (still)
- Strategy spaces $C_i$ are compact metric spaces
- Utility functions $u_i : C \rightarrow \mathbb{R}$ are continuous
- Examples:
  - Finite games: $|C_i| < \infty$, $u_i$ arbitrary
  - Polynomial games: $C_i = [-1, 1]$, $u_i$ polynomial
  - …
- Main property of continuous games: Correlated (and Nash) equilibria always exist
Defining correlated equilibria in continuous games

- Definition in literature: The probability measure $\pi$ is a correlated equilibrium if

$$\int [u_i(\zeta_i(s_i), s_{-i}) - u_i(s)] d\pi \leq 0$$

for all $i$ and all (measurable) departure functions $\zeta_i$

- Equivalent to above def. if strategy sets are finite
- Quantifier ranging over large set of functions
- Unknown function $\zeta_i$ inside $u_i$
- Is there a characterization without these problems?
An instructive failed attempt

- The following “characterization” fails:

\[
\int_{s_i} \left[ u_i(t_i, s_{-i}) - u_i(s) \right] \pi(s) \leq 0
\]

for all players \( i \) and all strategies \( s_i, t_i \in S_i \)

- Holds for any continuous probability distribution \( \pi \)

- This condition is much weaker than correlated equilibrium
Simple departure functions

- A probability measure $\pi$ is a correlated equilibrium if and only if

$$\int [u_i(\zeta_i(s_i), s_{-i}) - u_i(s)]d\pi \leq 0$$

for all $i$ and all simple (having finite range) measurable departure functions $\zeta_i$

- Proof: Approximate any $\zeta_i$ as a pointwise limit of simple functions $\xi_{i}^{k}$ (possible because $C_i$ is compact metric). Then $u_i(\xi_{i}^{k}(s_i), s_{-i}) \to u_i(\zeta_i(s_i), s_{-i})$ pointwise by continuity of $u_i$ and the result follows by Lebesgue’s dominated convergence theorem.
No departure functions

- A probability measure $\pi$ is a correlated equilibrium if and only if

$$\mu_{i,t_i}(B_i) := \int_{B_i \times C_{-i}} [u_i(t_i, s_{-i}) - u_i(s)] d\pi \leq 0$$

for all $i, t_i \in C_i$, and measurable subsets $B_i \subseteq C_i$

- Equivalently, $-\mu_{i,t_i}$ is a positive measure for all $i, t_i$

- Proof: The above integral is the corresponds to the departure function given by $\zeta_i(r_i) = r_i$ for $r_i \not\in B_i$ and $\zeta_i(r_i) = t_i$ for $r_i \in B_i$. Conversely, the integral for any simple departure function is a finite sum of terms of this type for some values of $B_i$ and $t_i$. 
Integration against test functions

- A probability measure $\pi$ is a correlated equilibrium if and only if

$$\int f_i(s_i) d\mu_{i,t_i} = \int f_i(s_i) [u_i(t_i, s_{-i}) - u_i(s)] d\pi \leq 0$$

for all $i, t_i \in C_i$, and $f_i : C_i \to [0, \infty)$ in some sufficiently rich class of test functions, e.g.

- Measurable characteristic functions
- Measurable functions
- Continuous functions
- Squares of polynomials (if $C_i \subset \mathbb{R}^k$)
Semidefinite programming

- A semidefinite program (SDP) is an optimization problem of the form

\[
\min \quad L(S) \quad \leftarrow L \text{ is a given linear functional}
\]

s.t. \quad T(S) = v \quad \leftarrow T \text{ is a given linear transformation,}

\quad v \text{ is a given vector}

\quad S \succeq 0 \quad \leftarrow S \text{ is a symmetric matrix of decision variables}

- SDPs generalize linear programs and can be solved efficiently using interior point methods
Sums of squares + SDP

- A polynomial $p(x)$ is $\geq 0$ for all $x \in \mathbb{R}$ iff it is a sum of squares of polynomials $q_k$ (SOS)

$$p(x) = \sum q_k^2(x) \text{ for all } x \in \mathbb{R}$$

- A polynomial $p(x)$ is $\geq 0$ for all $x \in [-1, 1]$ iff there are SOS polynomials $s(x), t(x)$ such that

$$p(x) = s(x) + (1 - x^2)t(x)$$

- Coefficients of SOS polynomials can be described in an SDP
Multivariate SOS + SDP

- A polynomial $p(x_1, \ldots, x_k)$ is $\geq 0$ for all $x \in \mathbb{R}^k$ iff there is an integer $r(p) \geq 0$ such that
  $$(x_1^2 + \ldots + x_k^2)^r p(x_1, \ldots, x_k)$$
is a sum of squares of polynomials $q_l(x_1, \ldots, x_k)$

- Sequence of sufficient SDP conditions characterizing multivariate nonnegative polynomials exactly “in the limit”

- Similar conditions for nonnegativity on $[-1, 1]^k, \ldots$

- Such conditions are generally not exact for a fixed $r$ independent of $p$, with some important exceptions
Moments of measures + SDP

• If $\tau$ is a measure on $[-1, 1]$, then for a polynomial $p$:
  \[ \int p^2(x)d\tau(x) \geq 0 \quad \text{and} \quad \int (1 - x^2)p^2(x)d\tau(x) \geq 0 \]

• $\tau_0, \ldots, \tau_{2m}$ are the moments of a measure $\tau$ on $[-1, 1]$ (i.e. $\tau_k = \int x^k d\tau(x)$) iff
  \[
  \begin{bmatrix}
  \tau_0 & \tau_1 & \tau_2 \\
  \tau_1 & \tau_2 & \tau_3 \\
  \tau_2 & \tau_3 & \tau_4
  \end{bmatrix} \preceq 0, \quad \begin{bmatrix}
  \tau_0 - \tau_2 & \tau_1 - \tau_3 \\
  \tau_1 - \tau_3 & \tau_2 - \tau_4
  \end{bmatrix} \succeq 0 \quad (m = 2 \text{ case})
  \]

• Moments of measures on $[-1, 1]$ can be described in an SDP
Moments of multivariate measures

• If $\tau$ is a measure on $[-1,1]^k$, then for a polynomial $p(x_1,\ldots,x_k)$:

\[
\int p^2(x)d\tau(x) \geq 0 \quad \text{and} \quad \int (1-x_i^2)p^2(x)d\tau(x) \geq 0
\]

• Requiring these conditions for all $p$ up to a fixed degree gives a necessary semidefinite condition the joint moments of a measure must satisfy.

• Exact “in the limit”
Polynomial games

- Strategy space is $C_i = [-1, 1]$ for all players $i$
- Utilities $u_i$ are multivariate polynomials
- Finitely supported equilibria always exist, with explicit bounds on support size [1950s; SOP 2006]
- Minimax strategies and values can be computed by semidefinite programming [Parrilo 2006]
Computing corr. equil. in poly. games: Naive attempt (LP)

- Intended as a benchmark to judge other techniques
- Ignore polynomial structure
- Restrict strategy choices (and deviations) to fixed finite sets $\tilde{C}_i \subset C_i$
- Compute exact correlated equilibria of approximate game
- This is a sequence of LPs which converges (slowly!) to the set of correlated equilibria as the discretization gets finer.
Computing corr. equil. in poly. games by SDP relaxation

- Describe moments of measures $\pi$ on $[-1, 1]^n$ with a sequence of necessary SDP conditions which become sufficient in the limit.

- For fixed $d$, we want to use SDP to express

$$\int p^2(s_i)[u_i(t_i, s_{-i}) - u_i(s)]d\pi \leq 0$$

for all $i$, $t_i \in [-1, 1]$, and polys. $p$ of degree $< d$.

- Shown above: for each $d$ this is a necessary condition for $\pi$ to be a correlated equilibrium which becomes sufficient as $d \to \infty$. 
Computing corr. equil. in poly. games by SDP relaxation (II)

- Define \( S_d^i \in \mathbb{R}^{d \times d} [s_i] \) by \( \left[ S_d^i \right]_{jk} = s_i^{j+k}, \ 0 \leq j, k < d \)

- A polynomial is a square of a polynomial if and only if it can be written as \( c^\top S_d^i c \) for some \( c \in \mathbb{R}^d \)

- Let \( M_d^i (t_i) = \int S_d^i [u_i(t_i, s_{-i}) - u_i(s)] d\pi \in \mathbb{R}^{d \times d} [t_i] \)

- Then we wish to constrain \( \pi \) to satisfy \( c^\top M_d^i (t_i)c \leq 0 \) for all \( c \in \mathbb{R}^d \) and \( t_i \in [-1, 1] \), i.e. \( M_d^i (t_i) \preceq 0 \) for \( t_i \in [-1, 1] \)
Computing corr. equil. in poly. games by SDP relaxation (III)

- $M^d_i(t_i)$ is a matrix whose entries are univariate polynomials in $t_i$ with coefficients which are affine in the decision variables of the problem (the joint moments of $\pi$)

- This can be expressed exactly as an SDP constraint for any $d$ (this is one of those special cases in which the condition is exact, even though there are multiple variables, $t_i$ and $c$)

- Putting it all together we get a nested sequence of SDPs converging to the set of correlated equilibria
Comparison of Static Discretization, Adaptive Discretization, and SDP Relaxation

\[ u_x(x,y) = 0.554 + 1.36x - 1.2y + 0.596x^2 + 2.072xy - 0.394y^2 \]

\[ u_y(x,y) = -1.886 - 1.232x + 0.842y - 0.108x^2 + 1.918xy - 1.044y^2 \]
Conclusions

• New characterizations of correlated equilibria in infinite games
• First algorithms for computing correlated equilibria in any class of infinite games

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Time for a better discretization algorithm?
Approximate correlated equilibria

- The probability measure $\pi$ is an $\epsilon$-correlated equilibrium if
  \[ \int [u_i(\zeta_i(s_i), s_{-i}) - u_i(s)]d\pi \leq \epsilon \]
  for all $i$ and all (measurable) departure functions $\zeta_i$
- Same as definition of correlated equilibrium if $\epsilon = 0$
- We will be interested in the case when the support of $\pi$ is a finite set
Characterizing finitely supported $\epsilon$-correlated equilibria

- A probability measure $\pi$ with finite support contained in $\tilde{C} = \prod \tilde{C}_i$ is an $\epsilon$-correlated equilibrium if and only if

$$\sum_{\{s_{-i} \in C_{-i}\}} \sum_{\{s_i \in \tilde{C}_i\}} \left[ u_i(t_i, s_{-i}) - u_i(s_i) \right] \pi(s) \leq \epsilon_{i,s_i}$$

for all $i$, $s_i \in \tilde{C}_i$, and $t_i \in C_i$ and

$$\sum_{s_i \in \tilde{C}_i} \epsilon_{i,s_i} \leq \epsilon$$

for all $i$. 
Applying SOS / SDP

- Given a polynomial game and a finite support set \( \tilde{C}_i \subset [-1, 1] \) for each player, the condition that \( \pi \) be a probability measure on \( \tilde{C} \) and an \( \epsilon \)-correlated equilibrium can be written in an SDP.

- First constraint says a univariate polynomial in \( t_i \) with coefficients linear in the \( \pi(s) \) and \( \epsilon_{i,s_i} \) is \( \geq 0 \) on \( [-1, 1] \), hence is expressible exactly in an SDP.

- Second constraint is linear, so usable in SDP.

- Conditions to make \( \pi \) a prob. measure also linear.

- Always feasible if \( \epsilon \) can vary.
Adaptive discretization

- Given $\tilde{C}_i^k$, optimize the following (as an SDP)

  $$\min \epsilon$$

  s.t. $\pi$ is an $\epsilon$-correlated equilibrium

  which is a correlated equilibrium

  when deviations are restricted to $\tilde{C}_i^k$

- Let $\epsilon^k$ and $\pi^k$ be an optimal solution
- If $\epsilon^k = 0$ then halt
- Otherwise, compute $\tilde{C}_i^{k+1}$ as described on next slide and repeat
Adaptive discretization (II)

- Steps to compute $\tilde{C}_k^{i+1}$
  - For some player $i$, the $\epsilon$-correlated equilibrium constraints are tight
  - Find values of $t_i$ making these tight (free with SDP duality), add these into $\tilde{C}_i^k$ to get $\tilde{C}_i^{k+1}$
  - For $j \neq i$, let $\tilde{C}_j^{k+1} = \tilde{C}_j^k$

- By construction $\tilde{C}_i^{k+1} \supsetneq \tilde{C}_i^k$ because if not then $\epsilon = 0$, a contradiction
Proof that $\epsilon^k \to 0$

- If not, there is a subseq. along which $\epsilon^k \geq \epsilon > 0$.

- There is some player $i$ for whom $\tilde{C}_i^{k+1} \supsetneq \tilde{C}_i^k$ for infinitely many $l$.

- Assume WLOG that $\tilde{C}_i^{k+1} \supsetneq \tilde{C}_i^k$ for all $l$.

- Cover $C_i$ with finitely many open balls $B_i^j$ such that $|u_i(s_i, s_{-i}) - u_i(t_i, s_{-i})| \leq \frac{\epsilon}{2}$ for all $s_{-i} \in C_{-i}$ whenever $s_i$ and $t_i$ are in the same $B_i^j$.

- For some $l$, all the balls $B_i^j$ which will ever contain a point of some $\tilde{C}_i^k$ already do.
Proof that $\epsilon^k \to 0$ (II)

- Let $t_{i,s_i} \in \tilde{C}_{i}^{k_l+1}$ make the $\epsilon^{k_l}$-correlated equilibrium constraints tight.
- Let $r_{i,s_i} \in \tilde{C}_i^{k_l}$ be in the same $B^j_i$ as $t_{i,s_i}$ for each $s_i$.
- Then we get a contradiction:

$$\epsilon \leq \sum_{s \in \tilde{C}_i^{k_l}} [u_i(t_{i,s_i}, s_{-i}) - u_i(s)] \pi^{k_l}(s)$$

$$- \sum_{s \in \tilde{C}_i^{k_l}} [u_i(r_{i,s_i}, s_{-i}) - u_i(s)] \pi^{k_l}(s)$$

$$= \sum_{s \in \tilde{C}_i^{k_l}} [u_i(t_{i,s_i}, s_{-i}) - u_i(r_{i,s_i}, s_{-i})] \pi^{k_l}(s) \leq \sum_{s \in \tilde{C}_i^{k_l}} \frac{\epsilon}{2} \pi^{k_l}(s) = \frac{\epsilon}{2}$$
Adaptive discretization (III)

- We did not use the polynomial structure of the $u_i$ in the convergence proof, just continuity
- Used polynomiality to convert the optimization problem into an SDP
- Can also do this conversion if the $u_i$ are rational or even piecewise rational (and continuous)
- Solutions of such games are surprisingly complex – the Cantor measure arises as the unique Nash equilibrium of a game with rational $u_i$ [Gross 1952]
- Now we have a way to approximate these!