Adaptive Discretization Methods for Computing Correlated Equilibria of Polynomial Games

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Joint work with Profs. Pablo Parrilo and Asuman Ozdaglar
Game theoretic setting

- Standard strategic (normal) form game
- Players (rational agents) numbered $i = 1, \ldots, n$
- Each has a set $C_i$ of strategies $s_i$
- Players choose their strategies simultaneously
- Rationality: Each player seeks to maximize his own utility function $u_i : C \rightarrow \mathbb{R}$, which represents all his preferences over outcomes
Chicken and correlated equilibria

<table>
<thead>
<tr>
<th>((u_1, u_2))</th>
<th>Wimpy</th>
<th>Macho</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wimpy</td>
<td>(4, 4)</td>
<td>(1, 5)</td>
</tr>
<tr>
<td>Macho</td>
<td>(5, 1)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

- Nash equilibria (self-enforcing independent distrib.)
  - \((M, W)\)
  - \((W, M)\)

- Correlated equilibria (self-enforcing joint distrib.)
  - \(\frac{1}{2}(W, M) + \frac{1}{2}(M, W)\)
  - \(\frac{1}{3}(W, W) + \frac{1}{3}(W, M) + \frac{1}{3}(M, W)\)
Correlated equilibria in games with finite strategy sets

• $u_i(t_i, s_{-i}) - u_i(s)$ is change in player $i$’s utility when strategy $t_i$ replaces $s_i$ in $s = (s_1, \ldots, s_n)$

• A probability distribution $\pi$ is a correlated equilibrium if

$$\sum_{s \in \{r_i\} \times C_{-i}} [u_i(t_i, s_{-i}) - u_i(s)] \pi(s|s_i = r_i) \leq 0$$

for all players $i$ and all strategies $r_i, t_i \in C_i$

• No player has an incentive to deviate from his recommended strategy $r_i$
LP characterization

• A probability distribution $\pi$ is a correlated equilibrium if and only if

$$\sum_{s_{-i} \in C_{-i}} \left[ u_i(t_i, s_{-i}) - u_i(s) \right] \pi(s) \leq 0$$

for all players $i$ and all strategies $s_i, t_i \in C_i$

• Set of correlated equilibria of a finite game is a polytope
Polynomial games

- Strategy space is $C_i = [-1, 1]$ for all players $i$
- Utilities $u_i$ are multivariate polynomials
- Finitely supported equilibria always exist
Finitely supported $\epsilon$-correlated equilibria

- A probability measure $\pi$ with finite support contained in $\tilde{\mathcal{C}} = \prod \tilde{C}_i$ is an $\epsilon$-correlated equilibrium if

$$\sum_{s_{-i} \in \tilde{\mathcal{C}}_{-i}} [u_i(t_i, s_{-i}) - u_i(s)] \pi(s) \leq \epsilon_{i,s_i}$$

for all $i$, $s_i \in \tilde{C}_i$, and $t_i \in \mathcal{C}_i$ and

$$\sum_{s_i \in \tilde{C}_i} \epsilon_{i,s_i} \leq \epsilon$$

for all $i$. 
Adaptive discretization

- Given $\tilde{C}_i^k$, optimize the following (as an SDP)
  
  $$\min \epsilon$$
  
  s.t. $\pi$ is an $\epsilon$-correlated equilibrium
  
  which is a correlated equilibrium
  
  when deviations are restricted to $\tilde{C}_i^k$

- Let $\epsilon^k$ and $\pi^k$ be an optimal solution
- If $\epsilon^k = 0$ then halt
- Otherwise, compute $\tilde{C}_i^{k+1}$ (next slide) and repeat
- Convergence theorem: $\epsilon^k \rightarrow 0$
Adaptive discretization (II)

- Steps to compute $\tilde{C}^{k+1}$
  - For some player $i$, the $\epsilon$-correlated equilibrium constraints are tight
  - Find values of $t_i$ making these tight (free with SDP duality), add these into $\tilde{C}_i^k$ to get $\tilde{C}_i^{k+1}$
  - For $j \neq i$, let $\tilde{C}_j^{k+1} = \tilde{C}_j^k$
Applying SOS / SDP

- Given a polynomial game and a finite support set \( \tilde{C}_i \subset [-1, 1] \) for each player, the condition that \( \pi \) be a probability measure on \( \tilde{C} \) and an \( \epsilon \)-correlated equilibrium can be written in an SDP.

- First constraint says a univariate polynomial in \( t_i \) with coefficients linear in the \( \pi(s) \) and \( \epsilon_{i,s_i} \) is \( \geq 0 \) on \( [-1, 1] \), hence is expressible exactly in an SDP.

- Remaining constraints are linear, so usable in SDP.

- Always feasible since \( \epsilon \) can vary and finite games have correlated equilibria.
Random example

- Three players, random polynomial utilities (deg. 4)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\epsilon^k$</th>
<th>$\tilde{C}_x^k \setminus \tilde{C}_x^{k-1}$</th>
<th>$\tilde{C}_y^k \setminus \tilde{C}_y^{k-1}$</th>
<th>$\tilde{C}_z^k \setminus \tilde{C}_z^{k-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.99</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>4.16</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5.76</td>
<td>{−1}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.57</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.28</td>
<td>{0.53}</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.16</td>
<td></td>
<td>{0.49, 0.70}</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$10^{-7}$</td>
<td></td>
<td>{−1, 0.60}</td>
<td>{−0.60, 0.47}</td>
</tr>
</tbody>
</table>
Closing remarks

• Can you remove the condition that $\pi$ is an exact correlated equilibrium when deviations are restricted to $\tilde{C}_i^k$ from the optimization problem?
  – This seems to work well in practice
  – We have explicit counterexamples showing it doesn’t work in general

• For a completely different approach to computing correlated equilibria of polynomial games that does not use discretization, see [SPO 2007]