Computation of $\varepsilon$-equilibria in Separable Games

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Games

- Set $I$ of interacting agents ($I = \{1, 2\}$ throughout)
- Set $C_i$ of strategies for each player $i \in I$.
- Utility function $u_i : C_1 \times C_2 \to \mathbb{R}$.
  - Each player wants as much utility as possible.
  - Utilities capture all strategic interactions.
Equilibrium

- A Nash Equilibrium is a choice of strategy for each player, so that if only one player deviates, he cannot expect to improve his utility.

- An $\epsilon$-equilibrium is weaker – no player can improve his payoff by more than $\epsilon$. 
Rock, Paper, Scissors

<table>
<thead>
<tr>
<th>$(u_1, u_2)$</th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rock</td>
<td>$(0, 0)$</td>
<td>$(-1, +1)$</td>
<td>$(+1, -1)$</td>
</tr>
<tr>
<td>Paper</td>
<td>$(+1, -1)$</td>
<td>$(0, 0)$</td>
<td>$(-1, +1)$</td>
</tr>
<tr>
<td>Scissors</td>
<td>$(-1, +1)$</td>
<td>$(+1, -1)$</td>
<td>$(0, 0)$</td>
</tr>
</tbody>
</table>

- No equilibrium!
- Enlarge the set of strategies (not with dynamite).
- Allow players to choose a **mixed strategy**, i.e. a probability distribution over $C_i$.
- Define utility on these larger strategy spaces as expected utility.
Zero-sum Finite Games

- Both $C_1$ and $C_2$ are finite.
- Utilities satisfy $u_1 = -u_2$.
- Strictly competitive games (Rock, Paper, Scissors).
- Set of mixed strategies for each player is a simplex.
- Prove existence of a Nash Equilibrium via LP duality, and compute it efficiently with interior point methods (von Neumann 1928).
General Finite Games

- Both $C_1$ and $C_2$ are finite.
- Utilities are arbitrary.
- Allows for both competition and cooperation.
- Prove existence of an equilibrium via a non-constructive fixed-point argument (Nash 1951).
- Simple algorithms exist, e.g. the Lemke-Howson algorithm (simplex method with a different pivoting rule).
- PPAD-completeness proven in a Dec. 4, 2005 paper.
Continuous Games

- Both $C_1$ and $C_2$ are compact metric spaces.
- Utilities are continuous.
- An equilibrium always exists (Glicksberg 1952).
- But the probability measures involved can be arbitrarily complicated!
- No hope of computing equilibria in general.
Zero-sum Polynomial Games

- $C_1 = C_2 = [-1, 1]$.
- Utilities satisfy $u_1 = -u_2 = \text{a polynomial}$.
- Space of mixed strategies is infinite-dimensional, but has a finite-dimensional representation (more on next slide).
- Can be cast as an SDP, and computed efficiently with interior point methods (Parrilo 200x).
Separable games

- A continuous game is **separable** if it has payoffs:

  \[ u_i(s_1, s_2) = \sum_{k=1}^{r} a_i^k f_1^k(s_1) f_2^k(s_2) \]

  where \( a_i^k \in \mathbb{R} \) and \( f_j^k : C_j \to \mathbb{R} \) is continuous
  (superscripts are not exponents).

- The separable structure allows for a finite-dimensional representation of the mixed strategy space.

- Can assume WLOG that each player randomizes among at most \( r + 1 \) strategies.
Computing $\epsilon$-equilibria for two-player separable games

- Assume $C_i = [-1, 1]$ and the utilities are Lipschitz.
- Discretize the game: Choose a set $\tilde{C}_i$ of $m \propto \frac{1}{\epsilon}$ equally spaced pure strategies for each player, and sample the utilities to get $\tilde{u}_i : \tilde{C}_1 \times \tilde{C}_2 \rightarrow \mathbb{R}$.
- Compute an equilibrium of this finite game.
- This yields an $\epsilon$-equilibrium of the separable game.
Will this work?

- In general computing an equilibrium of a finite game is not easy.

- But in this case the finite game has the same separable structure as the original game:

\[ \tilde{u}_i(s_1, s_2) = \sum_{k=1}^{r} a_i^k \tilde{f}_1^k(s_1) \tilde{f}_2^k(s_2) \]

- In particular the finite game has an equilibrium in which each player mixes among at most \( r + 1 \) strategies, independent of the choice of \( m \propto \frac{1}{\epsilon} \).
Computing an equilibrium of the finite game

• Given a guess at the support of each player’s mixed strategy (which pure strategies he plays with positive probability) there is a simple LP to find equilibria with that support.

• 

\[
\# \text{ supports} = \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{r+1} \leq \binom{m+r}{1+r}
\]

polynomial in \(m\)
Complexity of the algorithm

- The number of LPs and the time to solve each are both polynomial in $\frac{1}{\epsilon}$.
- Algorithm is polynomial in $\frac{1}{\epsilon}$ and exponential in $r$.
- Dependence on $r$ is no worse than for finite games.
- A recently published $\epsilon$-equilibrium algorithm for finite games is exponential in $\frac{1}{\epsilon^2}$ (LMM 2003).
Conclusion

• Future directions:
  – Apply SDP-related methods to non-zero-sum polynomial games.
  – Consider separable games with additional structure, e.g. graphical separable games.
  – Algorithms for discontinuous games.

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