Exchangeable Equilibria in Symmetric Bimatrix Games

N. D. Stein  P. A. Parrilo  A. Ozdaglar

Laboratory for Information and Decision Systems
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

Brazilian Workshop on Game Theory, 2010
Thought experiment

Setup

- Pick two random Bayesian rational agents off the street
- Put them in separate rooms
- Give them each the table for a symmetric bimatrix game:
  \[
  \begin{pmatrix}
  (0, 0) & (1, 1) \\
  (1, 1) & (0, 0)
  \end{pmatrix}
  \]
- Tell them this is what you have done
- Ask each what strategy he would play

Main question

- What should we expect to happen?
Main idea

More formal setup
- Population of interchangeable players
- Two play a game with symmetric payoffs
- We are outside observers predicting play
- Environment gives no way to break symmetry

Immediate implications
- Bayesian rationality $\Rightarrow$ play is a correlated equilibrium $W$
- Interchangeability $\Rightarrow$ $W = W^T$

Our claim
- Not all symmetric correlated equilibria are reasonable
- Some are “more symmetric” than others

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Thought experiment, continued

**Sneaky trick**
- Suppose we pick three people
- Again put each in a room
- Give all the same bimatrix game
- Ask what they would do
- Call their responses $X_1$, $X_2$, and $X_3$

**Implications**
- Ignoring $X_3$, $X_1$ and $X_2$ should be a correlated equilibrium
- Joint distribution of the $X_i$ invariant under relabeling
Non-example

Game: \[
\begin{pmatrix}
(0, 0) & (1, 1) \\
(1, 1) & (0, 0)
\end{pmatrix}
\]
Correlated equilibrium: \[
\begin{pmatrix}
0 & 0.5 \\
0.5 & 0
\end{pmatrix}
\]

Is this a reasonable joint distribution for \(X_1\) and \(X_2\)?

Claim

No symmetric distribution of \(X_1, X_2, X_3\) has marginal \[
\begin{pmatrix}
0 & 0.5 \\
0.5 & 0
\end{pmatrix}
\].

Proof.

- With probability one \(X_i \neq X_j\) for all \(i \neq j\).
- By the pigeonhole principle, \(X_i = X_j\) for some \(i \neq j\).
Exchangeability

**Definition**
A sequence of random variables $X_1, X_2, \ldots$ is **exchangeable** if permuting finitely many of the $X_k$ doesn’t affect its distribution.

**Properties**
- i.i.d. $\implies$ exchangeable
- $X_j, X_k$ marginal is symmetric, fixed for any $j \neq k$
- identically distributed

**Exchangeable but not independent examples**
- Distribution of $X_1$ arbitrary, all $X_k = X_1$ almost surely
- Repeated flips of a coin with a random bias
Exchangeable equilibria

Definition
An exchangeable equilibrium is a correlated equilibrium which is extendable to an exchangeable distribution.

Remarks
- Natural limit of thought experiment
- Correlated equilibrium \(\Leftrightarrow\) Bayesian rationality
- Exchangeable distribution \(\Leftrightarrow\) Bayesian model for interchangeable members of population
- Symmetric Nash equilibria are i.i.d. distributions
- \(\text{NE}_{\text{Sym}} \subset \text{XE}_{\text{Sym}} \subset \text{CE}_{\text{Sym}}\)
Theorem (de Finetti)

A sequence $X_1, X_2, \ldots$ is exchangeable if and only if it is i.i.d. conditioned on some random parameter $\Lambda$.

Interpretation

- In exchangeable equilibria players react symmetrically to noisy measurement of environment.
- If parameter $\Lambda$ were common knowledge play would be a (random, symmetric) Nash equilibrium.
- This corresponds to perfect measurements, but in general exchangeable equilibria measurements may be noisy.
- E.g.: Sunspots may or may not occur; if they do players may or may not notice.
- Standard game theoretic insight: Players may be better off with less info, i.e., noisier measurements.
### Definition

Let \( Z = \{zz^T \mid z \in \mathbb{R}^{m \times 1} \geq 0\} \) be the set of symmetric, rank 1, nonnegative matrices. The set of **completely positive (CP)** matrices is \( \text{conv}(Z) \).

### Observation

The probability matrices in \( Z \) (those whose entries sum to 1) are the joint distributions of i.i.d. random variables.

### Corollary (of de Finetti’s theorem)

*The joint distribution of random variables \( X_1, X_2 \) is completely positive if and only if it can be extended to an exchangeable sequence \( X_1, X_2, \ldots \).*
Corollary

The exchangeable equilibria are the correlated equilibria which are completely positive as matrices.

Consequences

- The set of exchangeable equilibria is convex and compact
- \( \text{NE}_{\text{Sym}} \subset \text{conv}(\text{NE}_{\text{Sym}}) \subset \text{XE}_{\text{Sym}} \subset \text{CE}_{\text{Sym}} \)
  - These sets can all be different (example soon)

Sidenote

- Can also use CP matrices to characterize \( \text{conv}(\text{NE}_{\text{Sym}}) \)
- Can then prove that \( \text{conv}(\text{NE}_{\text{Sym}}) = \text{XE}_{\text{Sym}} \) for 2 \( \times \) 2 games
Example game

\[
\begin{pmatrix}
(u_1, u_2) & a & b & c \\
\hline
a & (5, 5) & (5, 4) & (0, 0) \\
b & (4, 5) & (4, 4) & (4, 5) \\
c & (0, 0) & (5, 4) & (5, 5)
\end{pmatrix}
\]

- Symmetric Nash equilibria:
  \[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.6 & 0.2 \end{bmatrix}\]

- Non-exchangeable correlated equilibrium:
  \[W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \text{ (zero diagonal)}\]

- Exchangeable equilibrium not in \(\text{conv}(\text{NE}_{\text{Sym}})\):
  \[W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & 1 & 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\]
Comparison of equilibrium sets for the game on the previous slide

- **Correlated equil.** which is not exchangeable:
  \[ W^1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \]

- **Exchangeable equil.** not in conv(Nash):
  \[ W^2 = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \]

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Exchangeable Equilibria in Symmetric Bimatrix Games
Existence

Theorem (Nash)

A symmetric bimatrix game has a symmetric Nash equilibrium.

Remarks

- In particular this implies exchangeable equilibria exist
- There are several more elementary proofs
- One is an adaptation of Hart and Schmeidler’s proof of existence of correlated equilibria
  - Adding a limiting argument we can prove Nash’s theorem itself in full generality
- We give a different proof based on the statement of Hart and Schmeidler’s result
Definition

The \textbf{n-player extension} of a symmetric bimatrix game $\Gamma$ is the \textit{n-player game} $\Gamma^n$ in which each pair of players plays $\Gamma$ and each player’s utility is the sum of his utilities from these subgames.

Notation

- Set of strategy profiles: $(C_1)^n = C_1 \times \cdots \times C_1$
- Set of correlated strategies symmetric under permuting the players: $\Delta_{\text{Sym}}((C_1)^n)$
- Call the symmetric correlated equilibria $\text{CE}_{\text{Sym}}(\Gamma^n)$
- Marginalization onto first $m$ players: $\mu^n_m : \Delta_{\text{Sym}}((C_1)^n) \to \Delta_{\text{Sym}}((C_1)^m)$
Multiplayer extension lemma

Lemma

Let $\pi \in \Delta_{\text{Sym}}((C_1)^n)$. Then $\pi \in \text{CE}_{\text{Sym}}(\Gamma^n)$ if and only if $\mu_2^n(\pi) \in \text{CE}_{\text{Sym}}(\Gamma)$. In particular $\mu_m^n : \text{CE}_{\text{Sym}}(\Gamma^n) \to \text{CE}_{\text{Sym}}(\Gamma^m)$.

Proof.

\[
\mathbb{E}_\pi u_1^n(f(X_1), X_2, \ldots, X_n) = \mathbb{E}_\pi \sum_{i=2}^{n} u_1(f(X_1), X_i) \\
= \sum_{i=2}^{n} \mathbb{E}_\pi u_1(f(X_1), X_i) \\
= \sum_{i=2}^{n} \mathbb{E}_{\mu_2^n(\pi)} u_1(f(X_1), X_2) \\
= (n - 1) \mathbb{E}_{\mu_2^n(\pi)} u_1(f(X_1), X_2) \qed
\]

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Equivalence with original definition

Original definition
- An XE is a CE which extends to an exchangeable distribution.

Alternative definition
- An XE is a CE which extends to $\Delta_{\text{Sym}}((C_1)^n)$ for all $n$.

Corollary
- An XE is a CE which extends to $\text{CE}_{\text{Sym}}(\Gamma^n)$ for all $n$.

Interpretation
- Exchangeable equilibria are symmetric correlated equilibria of large games with many identical interactions.
Theorem

Any symmetric bimatrix game admits an exchangeable equilibrium.

Proof.

- For all $n$, $\text{CE}(\Gamma^n)$ is compact, convex, nonempty (HS ’89)
- Average over permutations of the players: so is $\text{CE}_{\text{Sym}}(\Gamma^n)$
- For $m < n$:
  - $\mu^n_m : \text{CE}_{\text{Sym}}(\Gamma^n) \rightarrow \text{CE}_{\text{Sym}}(\Gamma^m)$
  - $\mu^n_2(\text{CE}_{\text{Sym}}(\Gamma^n)) = \mu^m_2(\mu^n_m(\text{CE}_{\text{Sym}}(\Gamma^n))) \subseteq \mu^m_2(\text{CE}_{\text{Sym}}(\Gamma^m))$
- $\text{XE}(\Gamma) = \bigcap_{n=2}^{\infty} \mu^n_2(\text{CE}_{\text{Sym}}(\Gamma^n))$
- Nested intersection of convex sets is nonempty
Closing remarks

Interpretations of exchangeable equilibria
- Natural objects between Nash and correlated equilibria
- Right way to maintain symmetry under correlation
- Coordination on noisy measurements of the environment
- Equilibria of games with many simultaneous interactions

Other results
- Extension to multiplayer games / general symmetries
- Can be used to prove Nash’s theorem via the separation techniques of HS ’89 without fixed point theorems