MATH348: Advanced Engineering Mathematics

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Sep. 7, 2012

1 Fourier Series (sec: 11.1)

1.1 General concept of Fourier Series (10 mins)

Show some figures by using a projector. Fourier analysis is a method to decompose a function into sine and cosine functions. Explain a little bit about Gibbs phenomenon.

1.2 Who cares?

- frequency domain (spectral analysis, noise separation, convolution)
- ODE and PDE

1.3 Periodic function

When the period is p,

$$f(x+p) = f(x). \tag{1}$$

The period of trigonometric series 2π.

1.4 Fourier Series (10 mins)

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{2}$$

where Fourier coefficients a_0, a_n, and b_n are

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

The meaning of Fourier coefficients:

a_0 average of the function in the interval
a_n amplitude for each cosine wave (frequency)
b_n amplitude for each sine wave (frequency)

I will do the derivation of these coefficients on Sep. 17.

1.5 Examples (25 mins)

In all examples, $-\pi < x < \pi$.

Even function

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi/2 < x < \pi/2 \\ 0 & \text{if} \quad x < -\pi/2, \pi/2 < x \end{cases} \tag{3}$$

ANSWER

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \frac{1}{2} \\
a_n &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nx dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{n\pi}\left\{\sin n\left(\frac{\pi}{2}\right) - \sin n\left(-\frac{\pi}{2}\right)\right\} = \frac{2}{n\pi}\sin\frac{n\pi}{2} \\
b_n &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nx dx = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{-1}{n\pi}\left\{\cos n\left(\frac{\pi}{2}\right) - \cos n\left(-\frac{\pi}{2}\right)\right\} = 0
\end{aligned}$$

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos nx\right) \\
&= \frac{1}{2} + \sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{2}{(2n-1)\pi}\cos(2n-1)x\right)
\end{aligned}$$

Odd function

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2 \\ -1 & \text{if} \quad -\pi/2 < x < 0 \\ 0 & \text{if} \quad x < -\pi/2, \pi/2 < x \end{cases} \tag{4}$$

ANSWER

$$a_0 = \frac{1}{2\pi}\left\{-\int_{-\frac{\pi}{2}}^{0} dx + \int_{0}^{\frac{\pi}{2}} dx\right\} = 0$$

$$a_n = \frac{1}{\pi}\left\{-\int_{-\frac{\pi}{2}}^{0} \cos nx dx + \int_{0}^{\frac{\pi}{2}} \cos nx dx\right\} = \frac{1}{\pi}\left\{-\left[\frac{\sin nx}{n}\right]_{-\frac{\pi}{2}}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\frac{\pi}{2}}\right\}$$

$$= \frac{1}{n\pi}\left\{-\left(-\sin(-n\frac{\pi}{2})\right) + \sin(n\frac{\pi}{2})\right\} = 0$$

$$b_n = \frac{1}{\pi}\left\{-\int_{-\frac{\pi}{2}}^{0} \sin nx dx + \int_{0}^{\frac{\pi}{2}} \sin nx dx\right\} = \frac{1}{\pi}\left\{-\left[\frac{-\cos nx}{n}\right]_{-\frac{\pi}{2}}^{0} + \left[\frac{-\cos nx}{n}\right]_{0}^{\frac{\pi}{2}}\right\}$$

$$= \frac{1}{n\pi}\left\{\left(\cos 0 - \cos(-\frac{n\pi}{2})\right) - \left(\cos(\frac{n\pi}{2}) - \cos 0\right)\right\} = 0$$

$$= \frac{1}{n\pi}\left\{\left(1 - \cos(\frac{n\pi}{2})\right) - \left(\cos(\frac{n\pi}{2}) - 1\right)\right\} = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin nx\right)$$

Compare these two examples and mention odd and even functions.

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Worksheet: Fourier Series

Sep. 7, 2012

Find the Fourier series of the function $f(x)$ of period $p = 2\pi$.

1)

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < \pi/2 \\ 0 & \text{if} \quad x < 0, \pi/2 < x \end{cases}$$

2)

$$f(x) = \begin{cases} 1 & \text{if} \quad -\pi < x < -\pi/2 \\ -1 & \text{if} \quad -\pi/2 < x < 0 \\ 1 & \text{if} \quad 0 < x < \pi/2 \\ -1 & \text{if} \quad \pi/2 < x < \pi \end{cases}$$

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Worksheet: Fourier Series (Answer)

Sep. 7, 2012

1)

$$a_0 = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} dx = \frac{1}{4}$$
$$a_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos nx dx = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$
$$b_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin nx dx = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$
$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\left\{\sin\frac{n\pi}{2}\cos nx + \left(1-\cos\frac{n\pi}{2}\right)\sin nx\right\}\right]$$

2)

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} \sin nx dx - \int_{-\frac{\pi}{2}}^{0} \sin nx dx + \int_0^{\frac{\pi}{2}} \sin nx dx - \int_{\frac{\pi}{2}}^{\pi} \sin nx dx\right] = \frac{2}{n}\left[1+\cos n\pi - 2\cos\frac{n\pi}{2}\right]$$
$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left[1+\cos n\pi - 2\cos\frac{n\pi}{2}\right]\sin nx\right)$$

Sep. 10, 2012

2 Functions of any period $p = 2L$ (sec: 11.2)

2.1 Worksheet on Sep. 7 (15 mins)

Maybe ask students to solve.

2.2 Even & Odd Functions (10 mins)

Even function ($f(-x) = f(x)$):

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x \right) \tag{5}$$

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x dx$$

e.x., $\cos x$, x^2, $2^x + 2^{-x}$, $\cdots$

Odd function ($f(-x) = -f(x)$):

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi}{L} x \right) \tag{6}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

e.x., $\sin x$, x, $2^x - 2^{-x}$, $\cdots$

$$(Even) \cdot (Even) = (Even)$$
$$(Even) \cdot (Odd) = (Odd)$$
$$(Odd) \cdot (Odd) = (Even)$$

Therefore, we need to compute only the half range.

2.3 $p = 2L$ (15 mins)

Exercise (2) can solve $p = \pi$ rather than $p = 2\pi$. Introduce $x \to \frac{\pi}{L}x$ of equation 2:

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \qquad (7) \\
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx
\end{aligned}
$$

Explain why we want to solve of period $p = 2L$.

2.4 Who cares?

The period of most of periodic functions is not 2π. If $L \to \infty$ ...

2.5 Exercises (20 mins)

Solve Exercise (2) with $p = \pi$ and get the same answer. Solve Exercise (1) with $-5 < x < 5$ (take 10 mins for students to solve it) and get the different answer.

ANSWER

$$
\begin{aligned}
a_0 &= 0, \quad a_n = 0 \\
b_n &= \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin 2nx dx = \frac{2}{n\pi} (1 - \cos n\pi) \\
f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - \cos n\pi) \sin 2nx
\end{aligned}
$$

Calculations with $p = 2\pi$ and $p = \pi$ are the same.

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \left[1 + \cos n\pi - 2 \cos \frac{n\pi}{2} \right] \sin nx \right) \\
&= 0 + \frac{4}{\pi} \sin 2x + 0 + 0 + 0 + \frac{4}{3\pi} \sin 6x + \cdots \\
f(x) &= \sum_{m=1}^{\infty} \frac{2}{m\pi} (1 - \cos m\pi) \sin 2mx \\
&= \frac{4}{\pi} \sin 2x + 0 + \frac{4}{3\pi} \sin 6x + \cdots
\end{aligned}
$$

Exercise (1) with $-5 < x < 5$ ($p = 2L = 10$):

$$\begin{aligned}
a_0 &= \frac{1}{10}\int_0^1 dx = \frac{1}{10} \\
a_n &= \frac{1}{5}\int_0^1 \cos\frac{n\pi}{5}x dx = \frac{1}{n\pi}\sin\frac{n\pi}{5} \\
b_n &= \frac{1}{5}\int_0^1 \sin\frac{n\pi}{5}x dx = \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{5}\right) \\
f(x) &= \frac{1}{10} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\left\{\sin\frac{n\pi}{5}\cos\frac{n\pi}{5}x + \left(1-\cos\frac{n\pi}{5}\right)\sin\frac{n\pi}{5}x\right\}\right]
\end{aligned}$$

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Worksheet: Fourier series of period $p = 2L$

Sep. 10, 2012

Find the Fourier series of function $f(x)$ of period $p = 2L$.

1)

$$f(x) = x, p = 6$$

2)

$$f(x) = x^2, p = 6$$

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Worksheet: Fourier series of period $p = 2L$ (ANSWER)

Sep. 10, 2012

1)

$$
\begin{aligned}
a_0 &= 0,\ a_n = 0 \\
b_n &= \frac{2}{3}\int_0^3 x \sin\frac{n\pi}{3}x dx \\
&= \frac{2}{3}\left(\left[\frac{3x}{n\pi}\cos\frac{n\pi}{3}x\right]_0^3 + \int_0^3 \frac{3}{n\pi}\cos\frac{n\pi}{3}x dx\right) \\
&= \frac{2}{3}\left(-\frac{9}{n\pi}\cos n\pi + \left(\frac{3}{n\pi}\right)^2\left[\sin\frac{n\pi}{3}x\right]_0^3\right) \\
&= -\frac{6}{n\pi}\cos n\pi \\
f(x) &= \sum_{n=1}^{\infty} -\frac{6}{n\pi}\cos n\pi \sin\frac{n\pi}{3}x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{6}{n\pi}\sin\frac{n\pi}{3}x
\end{aligned}
$$

2)

$$
\begin{aligned}
b_n &= 0,\ a_0 = \frac{1}{3}\int_0^3 x^2 dx = 3 \\
a_n &= \frac{2}{3}\int_0^3 x^2 \cos\frac{n\pi}{3} x dx \\
&= \frac{2}{3}\left(\left[\frac{3}{n\pi}x^2 \sin\frac{n\pi}{3}x\right]_0^3 - int_0^3 \frac{6}{n\pi} x \sin\frac{n\pi}{3} x dx\right) \\
&= -\frac{4}{n\pi}\left(\left[-\frac{3x}{n\pi}\cos\frac{n\pi}{3}x\right]_0^3 + \int_0^3 \frac{3}{n\pi}\cos\frac{n\pi}{3}x dx\right) \\
&= -\frac{4}{n\pi}\left(-\frac{9}{n\pi}\cos n\pi + \frac{3}{n\pi}\left[\frac{3}{n\pi}\sin\frac{n\pi}{3}x\right]_0^3\right) \\
&= \frac{36}{(n\pi)^2}(-1)^n \\
f(x) &= 3 + \sum_{n=1}^{\infty}\frac{36}{(n\pi)^2}(-1)^n \cos\frac{n\pi}{3}x = 3 - \frac{36}{\pi^2}\cos\frac{\pi}{3}x + \frac{9}{\pi^2}\cos\frac{2\pi}{3}x
\end{aligned}
$$

Sep. 12, 2012

3 Even & Odd Functions. Half-range Expansions

3.1 In-class exercise (20 mins)

3.2 Review Even & Odd Functions (10 mins)

Even function ($f(-x) = f(x)$) (Fourier cosine series):

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x \right) \tag{8}$$
$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$
$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x dx$$

e.x., $\cos x$, x^2, $2^x + 2^{-x}$, $\cdots$

Odd function ($f(-x) = -f(x)$) (Fourier sine series):

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi}{L} x \right) \tag{9}$$
$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

e.x., $\sin x$, x, $2^x - 2^{-x}$, $\cdots$

Think a little bit about

$$f(x) = 2x + 3x^2 \tag{10}$$

3.3 Half-range Expansions (10 mins)

We can pretend a function ($0 < x < L$) is odd or even functions and find Fourier series. For example,

$$f(x) = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi}{L} x. \tag{11}$$

If we assume $f(x)$ is a periodic function ($-L < x < L$: we extend a string to $-L < x < 0$ area), we know

$$K_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx. \tag{12}$$

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In-class Exercise: Even & Odd Functions.

Sep. 12, 2012

Find the Fourier series of periodic function $f(x)$.

1)

The period of $f(x)$ is $p = 2\pi$.

$$f(x) = \begin{cases} 3 & \text{if} \quad -\pi < x < 0 \\ -3 & \text{if} \quad 0 < x < \pi \end{cases}$$

2)

The period of $f(x)$ is $p = 20$.

$$f(x) = \begin{cases} 3 & \text{if} \quad -5 < x < 5 \\ 0 & \text{if} \quad -10 < x < -5,\ 5 < x < 10 \end{cases}$$

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In-class Exercise: Even & Odd Functions. (ANSWER)

Sep. 12, 2012

1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} -3\sin nx dx = \frac{6}{n\pi}(\cos n\pi - 1)$$

$$f(x) = \sum_{n=1}^{\infty}\frac{6}{n\pi}(\cos n\pi - 1)\sin nx$$

2)

$$a_0 = \frac{1}{10}\int_0^5 3dx = \frac{3}{2}$$

$$a_n = \frac{1}{5}\int_0^5 3\cos\frac{n\pi}{10}dx = \frac{6}{n}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{6}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{10}$$

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Worksheet: Even & Odd Functions. Half-range Expansions

Sep. 12, 2012

1)

Find the Fourier sine and cosine series of function $f(x)$.
1.1

$$f(x) = 1 \quad (0 < x < 2)$$

1.2

$$f(x) = 2 - x \quad (0 < x < 2)$$

2)

Find the Fourier series of fuction $f(x)$ of period $p = 2\pi$.

$$f(x) = \begin{cases} x + \frac{\pi}{2} & \text{if } -\pi < x < 0 \\ -x + \frac{\pi}{2} & \text{if } 0 < x < \pi \end{cases}$$

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Worksheet: Even & Odd Functions. Half-range Expansions (ANSWER)

Sep. 12, 2012

1.1

sine series

$$b_n = \int_0^2 \sin \frac{n\pi}{2} x dx = \frac{2}{n\pi}\left(1 - \cos n\pi\right)$$
$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - \cos n\pi\right) \sin \frac{n\pi}{2} x$$

cosine series

$$a_0 = \frac{1}{2}\int_0^2 dx = 1$$
$$a_n = \int_0^2 \cos \frac{n\pi}{2} x dx = 0$$
$$f(x) = 1$$

1.2

sine series

$$\begin{aligned} b_n &= \int_0^2 (2-x) \sin \frac{n\pi}{2} x dx \\ &= \left[\frac{-2}{n\pi}(2-x)\cos \frac{n\pi}{2} x\right]_0^2 + \int_0^2 \frac{-2}{n\pi} \cos \frac{n\pi}{2} x dx \\ &= \frac{4}{n\pi} \end{aligned}$$
$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{2} x$$

cosine series

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_0^2 (2-x)dx = 1 \\
a_n &= \int_0^2 (2-x)\cos\frac{n\pi}{2}xdx \\
&= \left(\frac{2}{n\pi}\right)^2 (1-\cos n\pi) \\
f(x) &= 1 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)^2 (1-\cos n\pi)\cos\frac{n\pi}{2}x
\end{aligned}$$

2)

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)dx = 0 \\
a_n &= \frac{2}{\pi}\int_0^{\pi}\left(-x+\frac{\pi}{2}\right)\cos nxdx \\
&= \frac{2}{\pi}\left(\left[\frac{1}{n}\left(-x+\frac{\pi}{2}\right)\sin nx\right]_0^{\pi} + \int_0^{\pi}\frac{1}{n}\sin nxdx\right) \\
&= \frac{2}{n^2\pi}(1-\cos n\pi) \\
f(x) &= \sum_{n=1}^{\infty}\frac{2}{n^2\pi}(1-\cos n\pi)\cos nx \\
&= \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)
\end{aligned}$$

Sep. 14, 2012

4 Fourier-series exercise

4.1 Review half-range expansion (10 mins)

Explain Fourier sine and cosine functions. Connect to physics (string with length L).

4.2 Review in-class exercise (10 mins)

- Students forget formula of Fourier series when they stop using.
- For exam or quiz, they need to remember correctly, but otherwise, they can check a book or Internet, or ask someone.
- They need to remember when we can do by using Fourier analysis.
- Solving PDE is for next month. Spectral analysis.

Good

- Students have some sense for Fourier series.

Need to improve

- Practice for integration
- "Find the Fourier series" means "$f(x) = \cdots$", not a_0, a_n, and b_n.
- If $p = 2L$, don't forget $\cos(n\pi x)/L$ (not $\cos nx$).

Emphasize what one does by finding Fourier series (draw figures for $n = 1, 2, 3, \cdots$).

4.3 Fourier-series exercise using a handout (25 mins)

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Worksheet: Fourier Series

Sep. 14, 2012

1) Find the Fourier series. $p = 6$.

$$f(x) = |x| - 1, \ \ (-3 < x < 3)$$

2) Find the Fourier series. $p = 2\pi$

$$f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad -\pi < x < 0 \end{cases}$$

3) Find the Fourier series. $p = 6$

$$f(x) = 1 - x^2, \ \ (-3 < x < 3)$$

4) Find the Fourier series. $p = 4$

$$f(x) = e^x, \ \ (-2 < x < 2)$$

5) $-\pi < x < \pi$. Show

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} (\pi - x)/2 & \text{if} \quad 0 < x < \pi \\ -(\pi + x)/2 & \text{if} \quad -\pi < x < 0 \end{cases}$$

6) Show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Hint: Find the Fourier series of $f(x) = 1$ in $-\pi/2 < x < \pi/2$ with $p = 2\pi$.

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Worksheet: Fourier Series (ANSWER)

Sep. 14, 2012

1)

$$f(x) = \begin{cases} x-1 & \text{if} \quad 0 < x < 3 \\ -x-1 & \text{if} \quad -3 < x < 0 \end{cases}$$

$f(x)$ is the even function.

$$\begin{aligned}
a_0 &= \frac{1}{3}\int_0^3 (x-1)dx = \frac{1}{3}\left[\frac{x^2}{2} - x\right]_0^3 = \frac{1}{2} \\
a_n &= \frac{2}{3}\int_0^3 (x-1)\cos\frac{n\pi x}{3}dx \\
&= \frac{2}{3}\left(\left[\frac{3}{n\pi}(x-1)\sin\frac{n\pi x}{3}\right]_0^3 - \int_0^3 \frac{3}{n\pi}\sin\frac{n\pi x}{3}dx\right) \\
&= \frac{2}{3}\left(\frac{6}{n\pi}\sin n\pi + \left(\frac{3}{n\pi}\right)^2\left[\cos\frac{n\pi x}{3}\right]_0^3\right) \\
&= \frac{2}{3}\left(\frac{3}{n\pi}\right)^2(\cos n\pi - 1) \\
&= \frac{6}{(n\pi)^2}(\cos n\pi - 1) \\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{6}{(n\pi)^2}(\cos n\pi - 1)\cos\frac{n\pi}{3}x
\end{aligned}$$

2)

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^\pi \sin x dx = \frac{1}{2\pi}[-\cos x]_0^\pi = \frac{1-\cos\pi}{2\pi} = \frac{1}{\pi} \\
a_1 &= \frac{1}{\pi}\int_0^\pi \sin x \cos x dx = \frac{1}{2\pi}\int_0^\pi \sin 2x dx = \frac{1}{2\pi}\left[-\frac{1}{2}\cos 2x\right]_0^\pi = 0
\end{aligned}
$$

$n \neq 1$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^\pi \sin\cos nx dx = \frac{1}{\pi}\int_0^\pi \frac{\sin(n+1)x - \sin(n-1)x}{2} dx \\
&= \frac{1}{2\pi}\left[\frac{-1}{n+1}\cos(n+1)x - \frac{-1}{n-1}\cos(n-1)x\right]_0^\pi \\
&= \frac{-1}{2\pi}\left(\frac{\cos(n+1)\pi - 1}{n+1} - \frac{\cos(n-1)\pi - 1}{n-1}\right) \\
&= \frac{-1}{2\pi}\left(\frac{-\cos n\pi - 1}{n+1} - \frac{-\cos n\pi - 1}{n-1}\right) \\
&= \frac{1+\cos n\pi}{\pi(1-n^2)} \\
b_1 &= \frac{1}{\pi}\int_0^\pi \sin^2 x dx = \frac{1}{\pi}\int_0^\pi \frac{1-\cos 2x}{2} dx = \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{1}{2}
\end{aligned}
$$

$n \neq 1$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^\pi \sin x \sin nx dx \\
&= \frac{1}{2\pi}\int_0^\pi (\cos(n-1)x - \cos(n+1)x)\, dx \\
&= \frac{1}{2\pi}\left[\frac{1}{n-1}\sin(n-1)x - \frac{1}{n+1}\sin(n+1)x\right]_0^\pi = 0 \\
f(x) &= \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{1+\cos n\pi}{\pi(1-n^2)} \cos nx
\end{aligned}
$$

3)

$f(x)$ is the even function.

$$
\begin{aligned}
b_n &= 0 \\
a_0 &= \frac{1}{3}\int_0^3 (1-x^2)dx = -2 \\
a_n &= \frac{2}{3}\int_0^3 (1-x^2)\cos\frac{n\pi}{3}xdx \\
&= \frac{2}{3}\left[\left[\frac{3}{n\pi}(1-x^2)\sin\frac{n\pi}{3}x\right]_0^3 - \int_0^3 \frac{3}{n\pi}(-2x)\sin\frac{n\pi x}{3}dx\right] \\
&= \frac{2}{3}\left[\frac{6}{n\pi}\int_0^3 x\sin\frac{n\pi x}{3}dx\right] \\
&= \frac{4}{n\pi}\left[\left[\frac{-3}{n\pi}x\cos\frac{n\pi x}{3}\right]_0^3 + \int_0^3 \frac{3}{n\pi}\cos\frac{n\pi x}{3}dx\right] \\
&= \frac{4}{n\pi}\left[\left(\frac{-9}{n\pi}\cos n\pi\right) + \frac{9}{(n\pi)^2}\left[\sin\frac{n\pi x}{3}\right]_0^3\right] \\
&= \frac{-36}{(n\pi)^2}\cos n\pi \\
f(x) &= -2 - \sum_{n=1}^{\infty}\frac{36}{(n\pi)^2}\cos n\pi\cos\frac{n\pi}{3}x
\end{aligned}
$$

4)

$$
\begin{aligned}
a_0 &= \frac{1}{4}\int_{-2}^{2} e^x dx = \frac{1}{4}\left(e^2 - e^{-2}\right) \\
a_n &= \frac{1}{2}\int_{-2}^{2} e^x \cos\frac{n\pi x}{2} dx = \frac{1}{2}\left[\frac{2}{n\pi}\left[e^x \sin\frac{n\pi x}{2}\right]_{-2}^{2} - \frac{2}{n\pi}\int_{-2}^{2} e^x \sin\frac{n\pi x}{2} dx\right] \\
&= -\frac{1}{n\pi}\int_{-2}^{2} e^x \sin\frac{n\pi x}{2} dx \\
&= -\frac{1}{n\pi}\left[\left[\frac{-2}{n\pi} e^x \cos\frac{n\pi x}{2}\right]_{-2}^{2} - \int_{-2}^{2} \frac{-2}{n\pi} e^x \cos\frac{n\pi x}{2} dx\right] \\
&= -\frac{1}{n\pi}\left[(-e^2 + e^{-2})\frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi}\int_{-2}^{2} e^x \cos\frac{n\pi x}{2} dx\right] \\
&= -\frac{1}{n\pi}\left[(-e^2 + e^{-2})\frac{2\cos n\pi}{n\pi} + \frac{2}{n\pi}\int_{-2}^{2} e^x \cos\frac{n\pi x}{2} dx\right] \\
&= \frac{2(e^2 - e^{-2})\cos n\pi}{(n\pi)^2 + 4} \\
b_n &= \frac{1}{2}\int_{-2}^{2} e^x \sin\frac{n\pi x}{2} dx \\
&= \frac{-(e^2 - e^{-2})n\pi\cos n\pi}{(n\pi)^2 + 4} \\
f(x) &= \frac{e^2 - e^{-2}}{4} + \sum_{n=1}^{\infty}\left[\frac{2(e^2 - e^{-2})\cos n\pi}{(n\pi)^2 + 4}\cos\frac{n\pi x}{2} - \frac{(e^2 - e^{-2})n\pi\cos n\pi}{(n\pi)^2 + 4}\sin\frac{n\pi x}{2}\right] \\
&= (e^2 - e^{-2})\left[\frac{1}{4} + \sum_{n=1}^{\infty}\frac{\cos n\pi}{(n\pi)^2 + 4}\left(2\cos\frac{n\pi x}{2} - n\pi\sin\frac{n\pi x}{2}\right)\right]
\end{aligned}
$$

5)

$$
f(x) = \begin{cases} (\pi - x)/2 & \text{if} \quad 0 < x < \pi \\ -(\pi + x)/2 & \text{if} \quad -\pi < x < 0 \end{cases}
$$

$f(x)$ is the odd function.

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\pi - x)\sin nx dx = \frac{1}{n} \\
f(x) &= \sum_{n=1}^{\infty}\frac{\sin nx}{n}
\end{aligned}
$$

6)

$$f(x) = 1 \ \ (-\pi/2 < x\pi/2, p = 2\pi)$$

$f(x)$ is the even function.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi/2} dx = \frac{1}{2} \\
a_n &= \frac{2}{\pi}\int_0^{\pi/2} \cos nx dx = \frac{2}{n\pi}\sin\frac{n\pi}{2} \\
f(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi}{2}\cos n\pi \\
&= \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right)
\end{aligned}$$

If $x = 0$, $f(x) = 1$ and

$$\begin{aligned}
1 &= \frac{1}{2} + \frac{2}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) \\
\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots
\end{aligned}$$

You can use different functions for deriving this expression.

Name:

MATH348A AEM

Homework: Fourier Series

Due date: Sep. 17, 2012

1)

Find the Fourier series. $p = 6$

$$f(x) = x^3, \ \ (-3 < x < 3)$$

Draw the first four Fourier series ($n = 1 \sim 4$).

2)

Show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

using the Fourier series of the function

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad -\pi < x < 0 \end{cases},$$

where $p = 2\pi$.

MATH348A AEM

Homework: Fourier Series (ANSWER)

Due date: Sep. 17, 2012

1)

$$
\begin{aligned}
a_0 &= 0, \quad a_n = 0 \\
b_n &= \frac{2}{3}\int_0^3 x^3 \sin\frac{n\pi x}{3}dx \\
&= \frac{2}{3}\left[\left[\frac{-3}{n\pi}x^3\cos\frac{n\pi x}{3}\right]_0^3 + \frac{3}{n\pi}\int_0^3 3x^2\cos\frac{n\pi x}{3}dx\right] \\
&= \frac{2}{3}\left[\frac{-81}{n\pi}\cos n\pi + \frac{9}{n\pi}\left[\left[\frac{3x^2}{n\pi}\sin\frac{n\pi x}{3}\right]_0^3 - \frac{3}{n\pi}\int_0^3 2x\sin\frac{n\pi x}{3}dx\right]\right] \\
&= \frac{-54}{n\pi}\cos n\pi + \frac{6}{n\pi}\left[\frac{-6}{n\pi}\int_0^3 x\sin\frac{n\pi x}{3}dx\right] \\
&= \frac{-54}{n\pi}\cos n\pi - \frac{36}{(n\pi)^2}\left\{\left[\frac{-3x}{n\pi}\cos\frac{n\pi x}{3}\right]_0^3 + \int_0^3 \frac{3}{n\pi}\cos\frac{n\pi x}{3}dx\right\} \\
&= \frac{-54}{n\pi}\cos n\pi + \frac{36}{(n\pi)^2}\frac{9}{n\pi}\cos n\pi - \frac{108}{(n\pi)^3}\int_0^3 \cos\frac{n\pi x}{3}dx \\
&= \frac{-54}{n\pi}\cos n\pi + \frac{324}{(n\pi)^3}\cos n\pi - \frac{108}{(n\pi)^3}\left[\frac{3}{n\pi}\sin\frac{n\pi x}{3}\right]_0^3 \\
&= \frac{324 - 54(n\pi)^2}{(n\pi)^3}\cos n\pi \\
f(x) &= \sum_{n=1}^{\infty}\frac{324 - 54(n\pi)^2}{(n\pi)^3}\cos n\pi \sin\frac{n\pi x}{3}
\end{aligned}
$$

2)

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{\pi} x dx = \frac{\pi}{4} \\
a_n &= \frac{1}{\pi}\int_0^{\pi} x\cos nx dx = \frac{1}{\pi}\left[\left[\frac{1}{n}x\sin nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx dx\right] \\
&= \frac{1}{n^2\pi}(\cos n\pi - 1) \\
b_n &= \frac{1}{\pi}\int_0^{\pi} x\sin nx dx = \frac{1}{\pi}\left[\left[\frac{-1}{n}\cos n\pi\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx dx\right] \\
&= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos n\pi + \frac{1}{n^2}[\sin nx]_0^{\pi}\right] \\
&- \frac{1}{n}\cos n\pi \\
f(x) &= \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(\cos n\pi - 1)\cos nx - \frac{1}{n}\cos n\pi\sin nx\right] \\
f(x) &= \frac{\pi}{4} + \frac{1}{\pi}(-2)\cos x - (-1)\sin x + \frac{1}{4\pi}(0)\cos 2x - \frac{1}{2}\sin 2x + \frac{1}{9\pi}(-2)\cos 3x - \frac{1}{3}(-1)\sin 3x \\
&+ \frac{1}{16\pi}(0)\cos 4x - \frac{1}{4}\sin 4x + \frac{1}{25\pi}(-2)\cos 5x - \frac{1}{5}(-1)\sin 5x \\
x &= \frac{\pi}{2} \\
\frac{\pi}{2} &= \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \cdots \\
\frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \cdots
\end{aligned}
$$

MATH348A AEM

Feedback to Nori Nakata

Your comments are so important for me (Nori). Thank you for your participation.

What are good points of my teaching? As a teacher, what are my strengths? What have I done as a teacher that has helped you learn in this class?

Do you have any suggestion for me to improve my teaching skills? What could I do in this class to better help you understand the material or improve your skills?

Sep. 17, 2012

5 Derivation of Euler Formulas. Orthogonal Functions

5.1 Review homework and worksheet (10 mins)

5.2 Derivation of Fourier series

I gave students the concept of Fourier series. Here, I teach how to derive the formula.

5.3 Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \quad (13)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx$$

We need to show a_0, a_n, and b_n can be shown by these expressions.

5.4 For a_0

We integrate from $-L$ to L on both sides of expression 14,

$$\int_{-L}^{L} f(x) dx = \int_{-L}^{L} a_0 dx + \int_{-L}^{L} \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) dx$$

$$\int_{-L}^{L} f(x) dx = a_0 \int_{-L}^{L} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-L}^{L} \cos \frac{n\pi}{L} x dx + b_n \int_{-L}^{L} \sin \frac{n\pi}{L} x dx \right)$$

$$\int_{-\pi}^{\pi} \cos nx dx = 0, \quad \int_{-\pi}^{\pi} \sin nx dx = 0$$

Therefore,

$$\int_{-L}^{L} f(x) dx = 2La_0$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

5.5 For a_n and b_n

We first need to show orthogonal functions.

5.6 Orthogonal functions

Show

$$\int_{-L}^{L} \cos\frac{n\pi}{L}x \sin\frac{m\pi}{L}x dx = 0, \tag{14}$$

$$\int_{-L}^{L} \cos\frac{n\pi}{L}x \cos\frac{m\pi}{L}x dx = 0, \quad (n \neq m) \tag{15}$$

$$\int_{-L}^{L} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x dx = 0, \quad (n \neq m) \tag{16}$$

$$\int_{-L}^{L} \cos^2\frac{n\pi}{L}x dx = \int_{-L}^{L} \sin^2\frac{n\pi}{L}x dx = L \tag{17}$$

where $n, m = 1, 2, 3 \cdots$.

Kronecker delta

$$\int_{-L}^{L} \cos\frac{n\pi}{L}x \cos\frac{m\pi}{L}x dx = \int_{-L}^{L} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x dx = L\delta_{mn}$$

ANSWER

Equation 14 Solution 1

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \frac{1}{2}\int_{-L}^{L} \left[\sin\frac{(m+n)\pi x}{L} - \sin\frac{(m-n)\pi x}{L}\right] dx = 0$$

Solution 2

$$\begin{aligned}
I &= \int_{-L}^{L} \cos\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx \\
&= \left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right]_{-L}^{L} + \int_{-L}^{L} \frac{L}{n\pi}\frac{L}{m\pi}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} dx \\
&= 0 + \frac{L^2}{mn\pi^2}\left\{\left[\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{x}\right]_{-L}^{L} + \int_{-L}^{L} \frac{L}{n\pi}\frac{L}{m\pi}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L} dx\right\} \\
&= \left(\frac{L^2}{mn\pi^2}\right)^2 I \\
I &= 0
\end{aligned}$$

Solution 3

$$
\begin{aligned}
e^{ix} = \cos x + i\sin x,\quad e^{-ix} = \cos x - i\sin x \\
\sin x = \frac{e^{ix}-e^{-ix}}{2},\quad \cos x = \frac{e^{ix}+e^{-ix}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx &= \frac{1}{4}\int_{-L}^{L}\left(e^{i\frac{n\pi x}{L}}+e^{-i\frac{n\pi x}{L}}\right)\left(e^{i\frac{m\pi x}{L}}-e^{-i\frac{m\pi x}{L}}\right)dx \\
&= \frac{1}{4}\int_{-L}^{L}\left(e^{i\frac{(n+m)\pi x}{L}}-e^{i\frac{(n-m)\pi x}{L}}+e^{i\frac{(-n+m)\pi x}{L}}-e^{i\frac{-(n+m)\pi x}{L}}\right)dx \\
\int_{-L}^{L}e^{i\frac{(n+m)\pi x}{L}}dx &= \left[\frac{L}{i(n+m)\pi}e^{i\frac{(n+m)\pi x}{L}}\right]_{-L}^{L} \\
&= e^{i(n+m)\pi}-e^{-(n+m)\pi} = 2\sin(n+m)\pi = 0
\end{aligned}
$$

therefore,

$$
\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = 0
$$

Equation 15

$$
\begin{aligned}
\cos\alpha\cos\beta &= \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \\
\int_{-L}^{L}\cos\frac{n\pi}{L}x\cos\frac{m\pi}{L}xdx &= \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(n+m)\pi x}{L}+\cos\frac{(n-m)\pi x}{L}\right)dx = 0
\end{aligned}
$$

Equation 16

$$
\begin{aligned}
\sin\alpha\sin\beta &= \frac{-\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2} \\
\int_{-L}^{L}\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}xdx &= \frac{1}{2}\int_{-L}^{L}\left(-\cos\frac{(n+m)\pi x}{L}+\cos\frac{(n-m)\pi x}{L}\right)dx = 0
\end{aligned}
$$

Equation 17

$$
\begin{aligned}
\cos^2\alpha = \frac{1+\cos 2\alpha}{2},\quad \sin^2\alpha = \frac{1-\cos 2\alpha}{2} \\
\int_{-L}^{L}\cos^2\frac{n\pi}{L}xdx = \int_{-L}^{L}\frac{1+\cos\frac{2n\pi x}{L}}{2}dx = L \\
\int_{-L}^{L}\sin^2\frac{n\pi}{L}xdx = \int_{-L}^{L}\frac{1-\cos\frac{2n\pi x}{L}}{2}dx = L
\end{aligned}
$$

$$
f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad -\pi < x < 0 \end{cases} \tag{18}
$$

5.7 For a_n and b_n

We multiply $\cos(m\pi x)(L)$ and integrate from $-L$ to L on both sides of expression 14,

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \int_{-L}^{L}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]\cos\frac{m\pi x}{L}dx$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = a_0\int_{-L}^{L}\cos\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx + b_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx\right)$$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx\right)$$

$$n = m$$

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}dx$$

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = La_n$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

We multiply $\sin(m\pi x)(L)$ and integrate from $-L$ to L on both sides of expression 14,

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \int_{-L}^{L}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]\sin\frac{m\pi x}{L}dx$$

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = a_0\int_{-L}^{L}\sin\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx + b_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right)$$

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty}\left(b_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right)$$

$$n = m$$

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = b_n\int_{-L}^{L}\sin^2\frac{n\pi x}{L}dx$$

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = Lb_n$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Name:

MATH348A AEM

Homework: Fourier series

Due date: Sep. 19, 2012

Find the Fourier series of function $f(x)$. $p = 2\pi$ and $-\pi < x < \pi$.

$$f(x) = \sin|x|$$

Hint: $\int_0^\pi \sin x \cos nx dx \neq 0$.

MATH348A AEM

Homework: Fourier series (ANSWER)

Due date: Sep. 19, 2012

$f(x)$ is an even function.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^\pi \sin x dx \\
a_1 &= \frac{2}{\pi}\int_0^\pi \sin x \cos x dx = \frac{1}{\pi}\left[\frac{-1}{2}\cos 2x\right]_0^\pi = 0 \\
n &\neq 1 \\
a_n &= \frac{2}{\pi}\int_0^\pi \sin x \cos nx dx \\
&= \frac{1}{\pi}\left[\int_0^\pi \{\sin(n+1)x - \sin(n-1)x\}\, dx\right] \\
&= \frac{-2(\cos n\pi + 1)}{\pi(n^2-1)} \\
f(x) &= \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{-2(\cos n\pi + 1)}{\pi(n^2-1)} \cos nx
\end{aligned}
$$

Sep. 19, 2012

6 Fourier integral transform

$$p = 2\pi(Real) \to p = 2L(Real) \to p = 2L(complex) \to p = \infty(complex)$$

6.1 Complex Fourier series

Real Fourier series

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \\
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx
\end{aligned}$$

Euler formula

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i \sin\theta \\
\cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{aligned}$$

$$\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2} + b_n \frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} e^{\frac{in\pi x}{L}} + \frac{a_n + ib_n}{2} e^{-\frac{in\pi x}{L}} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} e^{\frac{in\pi x}{L}} + \frac{a_{|-n|} + ib_{|-n|}}{2} e^{\frac{i(-n)\pi x}{L}} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n - ib_n}{2} e^{\frac{in\pi x}{L}} \right) + \sum_{n=-\infty}^{-1} \left(\frac{a_{|n|} + ib_{|n|}}{2} e^{\frac{in\pi x}{L}} \right)
\end{aligned}$$

We define c_n as

$$
\begin{aligned}
c_0 &\equiv a_0 \\
c_n &\equiv \frac{a_n - ib_n}{2}, \quad n > 0 \\
c_n &\equiv \frac{a_{|n|} + ib_{|n|}}{2}, \quad n < 0
\end{aligned}
$$

Then $f(x)$ is

$$
\begin{aligned}
f(x) &= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{L}} + \sum_{n=-\infty}^{-1} c_n e^{\frac{in\pi x}{L}} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}
\end{aligned}
$$

What is c_n?

$$
\begin{aligned}
c_0 = a_n &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
n &> 0 \\
c_n = \frac{a_n - ib_n}{2} &= \frac{1}{2L}\left(\int_{-L}^{L} f(x)\frac{e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}}{2}dx - i\int_{-L}^{L} f(x)\frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2i}dx\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx \\
n &< 0 \\
c_n = \frac{a_{|n|} + ib_{|n|}}{2} &= \frac{1}{2L}\left(\int_{-L}^{L} f(x)\frac{e^{\frac{i|n|\pi x}{L}} + e^{-\frac{i|n|\pi x}{L}}}{2}dx + i\int_{-L}^{L} f(x)\frac{e^{\frac{i|n|\pi x}{L}} - e^{-\frac{i|n|\pi x}{L}}}{2i}dx\right) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{i|n|\pi x}{L}}dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx
\end{aligned}
$$

Therefore, complex Fourier series is

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \\
c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}dx
\end{aligned}
$$

6.2 Fourier transform

$$k \equiv \frac{n\pi}{L}$$
$$\Delta k = \frac{\Delta n\pi}{L} = \frac{\pi}{L}$$

We multiply this to the complex Fourier series, then

$$f(x) = \frac{L}{\pi}\sum_{-\infty}^{\infty} c_n e^{ikx}\Delta k$$

$L \to \infty$ (partly: just for summation to integral)

$$f(x) = \frac{L}{\pi}\int_{-\infty}^{\infty} c_n e^{ikx} dk$$
$$c_n = \frac{1}{2L}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

We define $F(k)$ as

$$F(k) \equiv \sqrt{\frac{2}{\pi}}Lc_n$$
$$c_n = \frac{1}{L}\sqrt{\frac{\pi}{2}}F(k)$$

$$f(x) = \frac{L}{\pi}\frac{1}{L}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$
$$F(k) = \sqrt{\frac{2}{\pi}}L\frac{1}{2L}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

Inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}dk$$

If x is ”time”, k is ”frequency”.

6.3 time (s)$\longleftrightarrow$frequency (1/s), space (m)$\longleftrightarrow$wavenumber (1/m)

$$k = \frac{n\pi}{L}$$

Therefore, the unit of k is the unit of $1/x$. Fourier analysis is important because we can understand which frequency waves are including. Then we can analyze predominant frequency of a structure.

6.4 Several interesting functions

- Ricker wavelet

Sep. 21, 2012

7 Fourier Integral: spectrum

$$F(\omega) = ReF(\omega) + iImF(\omega)$$

7.1 Amplitude spectrum

$$|F(\omega)| = \sqrt{F(\omega)F*(\omega)}$$

7.2 Power spectrum

$$|F(\omega)|^2 = F(\omega)F*(\omega)$$

Total energy

$$\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

7.3 Phase spectrum

Find the similarity. Sometimes amplitude is not reliable.

$$argF(\omega) = arctan\frac{ReF(\omega)}{ImF(\omega)}$$

Name:

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In-class Exercise: Fourier series

Sep. 21, 2012

Find the Fourier series of function $f(x)$ in Figure 1. The period of function $f(x)$ is $-9 < x < 9$ (Hint: Can you find a shorter period?).

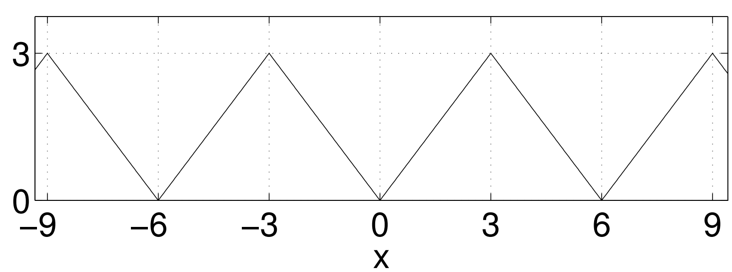

Figure 1. Function $f(x)$.

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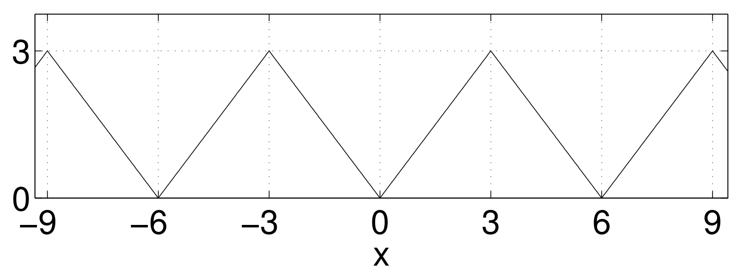
In-class Exercise: Fourier series (ANSWER)

Sep. 21, 2012

Find the Fourier series of function $f(x)$ in Figure 1. The period of function $f(x)$ is $-9 < x < 9$ (Hint: Can you find a shorter period?).



ANSWER

Period is 6 ($p = 6$) and $f(x)$ is an even function.

$$
\begin{aligned}
a_0 &= \frac{1}{3}\int_0^3 x\,dx = \frac{3}{2} \\
a_n &= \frac{2}{3}\int_0^3 x\cos\frac{n\pi}{3}x\,dx = \frac{6}{(n\pi)^2}(\cos n\pi - 1) \\
f(x) &= \frac{3}{2} + \sum_{n=1}^{\infty}\left\{\frac{6}{(n\pi)^2}(\cos n\pi - 1)\cos\frac{n\pi}{3}x\right\}
\end{aligned}
$$

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Homework: Fourier transform

Due date: Sep. 24, 2012

1)

Find the Fourier seires of function $f(x) = e^x$ if $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

2)

Find the Fourier transform ($F(k)$) of function $f(x)$.

1

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

2

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x < 0, 1 < x \end{cases}$$

MATH348A AEM

Worksheet: Fourier transform (ANSWER)

Due date: Sep. 24, 2012

1)

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx} dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e(1-in)xdx \\
&= \frac{1}{2\pi(1-in)}\left(e^{(1-in)\pi} - e^{(1-in)(-\pi)}\right) \\
&= \frac{1+in}{2\pi(1+n^2)}\left(e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right) \\
&= \frac{1+in}{2\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right)\cos n\pi \\
f(x) &= \sum_{-\infty}^{\infty}\frac{1+in}{2\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right)\cos n\pi e^{inx} \\
&= \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=1}^{\infty}\frac{1+in}{1+n^2}(-1)^n e^{inx} + \sum_{n=1}^{\infty}\frac{1-in}{1+n^2}(-1)^{-n}e^{-inx}\right) \\
&= \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left\{(1+in)e^{inx} + (1-in)e^{-inx}\right\}\right) \\
&= \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left\{e^{inx} + e^{-inx} + ine^{inx} - ine^{-inx}\right\}\right) \\
&= \frac{e^{\pi}-e^{-\pi}}{2\pi}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\left\{2\cos nx - 2n\sin nx\right\}\right)
\end{aligned}
$$

2)

1

$$
F(k) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-ikx}}{-ik}\right]_{-1}^{1} = \frac{e^{-ik}-e^{ik}}{-ik\sqrt{2\pi}} = \frac{-2i\sin k}{-ik\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\frac{\sin k}{k}
$$

2

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ikx} dx = \frac{1 - e^{-ik}}{ik\sqrt{2\pi}}$$

Sep. 24, 2012

8 Exercise

Delta function

Gauss function

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Worksheet: Fourier transform

Sep. 24, 2012

$\mathfrak{F}$: Fourier transform, $\mathfrak{F}^{-1}$: Inverse Fourier transform.

1)

Find the Fourier transform ($F(k)$) of function $f(x)$.

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ -1 & \text{if} \quad -1 < x < 0 \\ 0 & \text{if} \quad x < -1, 1 < x \end{cases}$$

2)

When $\mathfrak{F}(f(t)) = F(\omega)$ and $f(t)$ is a real function, show $\mathfrak{F}(f(-t)) = F^*(\omega)$ where $*$ is the complex conjugate.

3)

When $\mathfrak{F}(f(t)) = F(\omega)$, show $\mathfrak{F}(f(t-a)) = e^{-i\omega a}\mathfrak{F}(f(t))$.

4)

Find the Fourier transform ($F(\omega)$) of the function

$$f(t) = \begin{cases} \cos(5t) & \text{if} \quad -10 < t < 10 \\ 0 & \text{if} \quad t < -10, 10 < t \end{cases}$$

5)

Find the Fourier transform ($F(\omega)$) of the function

$$f(t) = \begin{cases} \sin(5t) & \text{if} \quad -10 < t < 10 \\ 0 & \text{if} \quad t < -10, 10 < t \end{cases}$$

6)

Find the Fourier transform ($F(k)$) of the function $f(x) = e^{-3x^2}$ (Gauss function). Also, apply the inverse transform to $F(k)$ and obtain $f(x)$.

Hint: $\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$. $F(k)$ is also a Gauss function.

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Worksheet: Fourier transform: ANSWER

Sep. 24, 2012

1)

$$F(k) = \frac{1}{\sqrt{2\pi}}\left(\int_0^1 e^{-ikx}dx + \int_{-1}^0 (-1)e^{-ikx}dx\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-ik}(e^{-ik}-1) - \frac{1}{-ik}(1-e^{ik})\right) = \sqrt{\frac{2}{\pi}}\frac{i(\cos k - 1)}{k}$$

2)

This shows what the time-reversed Fourier transform is.

$$\mathfrak{F}(f(-t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(-t)e^{-i\omega t}dt$$

$\tau = -t$

$$= \frac{1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(\tau)e^{i\omega\tau}(-d\tau)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}(d\tau)$$

Complex conjugate twice ($f(\tau)$ is a real function)

$$= \left[\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{i\omega\tau}(d\tau)\right]^*\right]^*$$
$$= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}(d\tau)\right]^* = F^*(\omega)$$

Therefore,

$$\mathfrak{F}(f(t)) = F(\omega)$$
$$\mathfrak{F}(f(-t)) = F^*(\omega)$$

3)

This shows what the time-shifted Fourier transform is.

$$\begin{aligned}\mathfrak{F}(f(t-a)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-a)e^{-i\omega t}dx \\ T &= t-a \\ \mathfrak{F}(f(t-a)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(T)e^{-i\omega(T+a)}dT \\ &= e^{-i\omega a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(T)e^{-i\omega T}dT \\ &= e^{-i\omega a}\mathfrak{F}(f(t))\end{aligned}$$

4)

$$\begin{aligned}F(\omega) &= \sqrt{\frac{1}{2\pi}}\int_{-10}^{10}\cos(5t)e^{-i\omega t}dx \\ &= \sqrt{\frac{1}{2\pi}}\int_{-10}^{10}\frac{e^{i5t}+e^{-i5t}}{2}e^{-i\omega t}dt \\ &= \frac{1}{2\sqrt{2\pi}}\int_{-10}^{10}\left(e^{i(5-\omega)t}+e^{-i(5+\omega)t}\right)dt \\ &= \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{i(5-\omega)}\left[e^{i(5-\omega)t}\right]_{-10}^{10}+\frac{1}{-i(5+\omega)}\left[e^{-i(5+\omega)t}\right]_{-10}^{10}\right) \\ &= \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{i(5-\omega)}\left[e^{i(5-\omega)t}\right]_{-10}^{10}+\frac{1}{-i(5+\omega)}\left[e^{-i(5+\omega)t}\right]_{-10}^{10}\right) \\ &= \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{i(5-\omega)}\left(e^{i(5-\omega)10}-e^{-i(5-\omega)10}\right)+\frac{1}{-i(5+\omega)}\left(e^{-i(5+\omega)10}-e^{i(5+\omega)10}\right)\right) \\ &= \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{i(5-\omega)}(2i\sin((5-\omega)10))+\frac{1}{-i(5+\omega)}(2i\sin(-(5+\omega)10))\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{\sin((5-\omega)10)}{5-\omega}+\frac{\sin((5+\omega)10)}{5+\omega}\right)\end{aligned}$$

5)

$$
\begin{aligned}
F(\omega) &= \sqrt{\frac{1}{2\pi}} \int_{-10}^{10} \sin(5t) e^{-i\omega t} dx \\
&= \sqrt{\frac{1}{2\pi}} \int_{-10}^{10} \frac{e^{i5t} - e^{-i5t}}{2i} e^{-i\omega t} dt \\
&= \frac{1}{2i\sqrt{2\pi}} \int_{-10}^{10} \left(e^{i(5-\omega)t} - e^{-i(5+\omega)t} \right) dt \\
&= \frac{1}{2i\sqrt{2\pi}} \left(\frac{1}{i(5-\omega)} \left[e^{i(5-\omega)t} \right]_{-10}^{10} - \frac{1}{-i(5+\omega)} \left[e^{-i(5+\omega)t} \right]_{-10}^{10} \right) \\
&= \frac{1}{2i\sqrt{2\pi}} \left(\frac{1}{i(5-\omega)} \left[e^{i(5-\omega)t} \right]_{-10}^{10} - \frac{1}{-i(5+\omega)} \left[e^{-i(5+\omega)t} \right]_{-10}^{10} \right) \\
&= \frac{1}{2i\sqrt{2\pi}} \left(\frac{1}{i(5-\omega)} \left(e^{i(5-\omega)10} - e^{-i(5-\omega)10} \right) - \frac{1}{-i(5+\omega)} \left(e^{-i(5+\omega)10} - e^{i(5+\omega)10} \right) \right) \\
&= \frac{1}{2i\sqrt{2\pi}} \left(\frac{1}{i(5-\omega)} (2i\sin((5-\omega)10)) - \frac{1}{-i(5+\omega)} (2i\sin(-(5+\omega)10)) \right) \\
&= \frac{i}{\sqrt{2\pi}} \left(\frac{\sin((5+\omega)10)}{5+\omega} - \frac{\sin((5-\omega)10)}{5-\omega} \right)
\end{aligned}
$$

6)

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-3x^2} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(3x^2+ikx)} dx
\end{aligned}
$$

$$
-3x^2 - ikx = -\left(\sqrt{3}x + \frac{ik}{2\sqrt{3}} \right)^2 - \frac{k^2}{12}
$$

Define $u = \sqrt{3}x + (ik)/(2\sqrt{3})$.

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\sqrt{3}x+\frac{ik}{2\sqrt{3}}\right)^2 - \frac{k^2}{12}} dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-k^2/12} \int_{-\infty}^{\infty} e^{-\left(\sqrt{3}x+\frac{ik}{2\sqrt{3}}\right)^2} dx \\
&= \frac{1}{\sqrt{6\pi}} e^{-k^2/12} \int_{-\infty}^{\infty} e^{-u^2} du \\
&= \frac{1}{\sqrt{6}} e^{-k^2/12}
\end{aligned}
$$

Inverse Fourier transform

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}\sqrt{6}} \int_{-\infty}^{\infty} e^{-k^2/12} e^{ikx} dx \\
&= \frac{1}{\sqrt{12\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{k}{\sqrt{12}} - \sqrt{3}ix\right)^2 - 3x^2} dk
\end{aligned}
$$

Define $u = \frac{k}{\sqrt{12}} - \sqrt{3}ix$.

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{12\pi}} e^{-3x^2} \int_{-\infty}^{\infty} e^{-\left(\frac{k}{\sqrt{12}} - \sqrt{3}ix\right)^2} dk \\
&= \frac{1}{\sqrt{12\pi}} e^{-3x^2} \sqrt{12} \int_{-\infty}^{\infty} e^{-u^2} du \\
&= e^{-3x^2}
\end{aligned}
$$

Name:

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Homework: Fourier transform

Due date: Sep. 26, 2012

Make an own problem by using Fourier transform and solve it. For example, "show $\cdots$" or "Find the Fourier transform of a function $\cdots$". The quality of the problem should be good for the quiz.

Sep. 26, 2012

9 Fourier integral: Fourier transform of derivative

9.1 Linearity of the Fourier transform

Fourier transform: $\mathfrak{F}$. We want to show

$$\mathfrak{F}(af(x)+bg(x)) = a\mathfrak{F}(f(x)) + b\mathfrak{F}(g(x))$$

$$\begin{aligned}\mathfrak{F}(af(x)+bg(x)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{-ikx}dx\\ &= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-ikx}dx\\ &= a\mathfrak{F}(f(x)) + \mathfrak{F}(g(x))\end{aligned}$$

9.2 Fourier transform of the derivative of $f(x)$

$f(x)$ is continuous and $f(x) \to 0$ as $|x| \to \infty$. We want to show

$$\mathfrak{F}(f'(x)) = ik\mathfrak{F}(f(x))$$

$$\begin{aligned}\mathfrak{F}(f'(x)) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f'(x)e^{-ikx}dx\\ &= \frac{1}{\sqrt{2\pi}}\left\{\Big[f(x)e^{-ikx}\Big]_{-\infty}^{\infty} - (-ik)\int_{-\infty}^{\infty}f(x)e^{-ikx}dx\right\}\\ &= 0 + ik\mathfrak{F}(f(x))\end{aligned}$$

$$\mathfrak{F}(f''(x)) = ik\mathfrak{F}(f'(x)) = -k^2\mathfrak{F}(f(x))$$

9.3 Review all topics

- Meaning of Fourier analysis
- Fourier series
- Fourier transform

Name:

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Quiz II

Sep. 28, 2012

Instructions:

- Put your name on the top of this page.
- For this quiz, you may only use a pencil and eraser.
- This quiz is closed book, closed notes, but you can bring one sheet of paper.
- No calculators, no cellphones, no electronic devices are allowed during the quiz.
- Do your own work - in order to get full scores, **show the details of your work on each problem**.
- At the end of the quiz, sign the honor statement at the bottom of this page, if it pertains to you.

In taking this exam, I have adhered to the instructions listed above. I have not used any unauthorized aids or help. I have neither given help of any kind to, nor taken help of any kind from, anyone else.

SIGNATURE:

1)

Evaluate

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx$$

where $n, m = 1, 2, 3\cdots$. Show details of your evaluation.

2)

Show

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

Hint: Find the Fourier series of $f(x) = x^2$ in $-1 < x < 1$ with period $p = 2$.

3)

Find the Fourier transform $(F(k))$ of the function

$$f(x) = \begin{cases} 1 - |x| & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

Write $F(k)$ in a pure real or imaginary function.

Name:

MATH348A AEM

Quiz II (ANSWER)

Sep. 28, 2012

1)

2)

$$
\begin{aligned}
f(x) &= x^2 \\
b_n &= 0, a_0 = \int_0^1 x^2 dx = \frac{1}{3} \\
a_n &= 2\int_0^1 x^2 \cos n\pi x dx \\
&= 2\left(\left[\frac{1}{n\pi}x^2 \sin n\pi x\right]_0^1 - \int_0^1 \frac{2x}{n\pi}\sin n\pi x dx\right) \\
&= -2\left[\frac{-2x}{(n\pi)^2}\cos n\pi x\right]_0^1 \\
&= \frac{4}{(n\pi)^2}\cos n\pi \\
f(x) &= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}\cos n\pi \cos n\pi x
\end{aligned}
$$

When $x = 0$

$$
\begin{aligned}
0 &= \frac{1}{3} - \frac{4}{\pi^2} + \frac{4}{4\pi^2} - \frac{4}{9\pi^2} + \frac{4}{16\pi^2} - \cdots \\
\frac{\pi^2}{12} &= 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots
\end{aligned}
$$

3)

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{0}(1+x)e^{-ikx}dx + \int_{0}^{1}(1-x)e^{-ikx}dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\left[(1+x)\frac{1}{-ik}e^{-ikx}\right]_{-1}^{0} + \frac{1}{ik}\int_{-1}^{0}e^{-ikx}dx + \left[(1-x)\frac{1}{-ik}e^{-ikx}\right]_{0}^{1} - \frac{1}{ik}\int_{0}^{1}e^{-ikx}dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-ik} + \frac{1}{ik}\left[\frac{1}{-ik}e^{-ikx}\right]_{0}^{1} + \frac{1}{ik} - \frac{1}{ik}\left[\frac{1}{-ik}e^{-ikx}\right]_{0}^{1}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1-e^{ik}}{k^2} + \frac{1-e^{-ik}}{k^2}\right) \\
&= \frac{2-(e^{ik}+e^{-ik})}{\sqrt{2\pi}k^2} \\
&= \sqrt{\frac{2}{\pi}}\frac{1-\cos k}{k^2}
\end{aligned}
$$

Name:

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Quiz II (ALTERNATE)

Oct. 5, 2012

Instructions:

- Put your name on the top of this page.
- For this quiz, you may only use a pencil and eraser.
- This quiz is closed book, closed notes, but you can bring one sheet of paper.
- No calculators, no cellphones, no electronic devices are allowed during the quiz.
- Do your own work - in order to get full scores, **show the details of your work on each problem**.
- At the end of the quiz, sign the honor statement at the bottom of this page, if it pertains to you.

In taking this exam, I have adhered to the instructions listed above. I have not used any unauthorized aids or help. I have neither given help of any kind to, nor taken help of any kind from, anyone else.

SIGNATURE:

1)

Evaluate

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx$$

where $n, m = 1, 2, 3\cdots$. Show details of your evaluation.

2)

Show

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

Use the Fourier series of $f(x) = x^2$ in $-2 < x < 2$ with period $p = 4$.

3)

Find the Fourier transform $(F(k))$ of the function

$$f(x) = \begin{cases} 1 - x^2 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

$F(k)$ should be a pure real or imaginary function.

Name:

MATH348A AEM

Quiz II (ANSWER)

Oct. 5, 2012

1)

2)

$$
\begin{aligned}
f(x) &= x^2 \\
b_n &= 0, a_0 = \int_0^2 x^2 dx = \frac{4}{3} \\
a_n &= 2\int_0^2 x^2 \cos\frac{n\pi x}{2} dx \\
&= 2\left(\left[\frac{2}{n\pi}x^2 \sin\frac{n\pi x}{2}\right]_0^2 - \int_0^2 \frac{4x}{n\pi}\sin\frac{n\pi x}{2}dx\right) \\
&= \frac{16}{(n\pi)^2}\cos n\pi \\
f(x) &= \frac{4}{3} + \sum_{n=1}^{\infty}\frac{16}{(n\pi)^2}\cos n\pi \cos\frac{n\pi x}{2}
\end{aligned}
$$

When $x = 0$

$$
\begin{aligned}
0 &= \frac{4}{3} - \frac{16}{\pi^2} + \frac{16}{4\pi^2} - \frac{16}{9\pi^2} + \frac{16}{16\pi^2} - \cdots \\
\frac{\pi^2}{12} &= 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots
\end{aligned}
$$

3)

$$
\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\left[\frac{1 - x^2}{-ik} e^{-ikx} \right]_{-1}^{1} + \frac{1}{ik} \int_{-1}^{1} -2x e^{-ikx} dx \right) \\
&= \frac{-2}{\sqrt{2\pi} ik} \left(\left[\frac{x}{-ik} e^{-ikx} \right]_{-1}^{1} + \frac{1}{ik} \int_{-1}^{1} e^{-ikx} dx \right) \\
&= \frac{-2}{\sqrt{2\pi} ik} \left(\left\{ \frac{e^{-ik}}{-ik} - \frac{e^{ik}}{ik} \right\} + \frac{1}{k^2} \left(e^{-ik} - e^{ik} \right) \right) \\
&= \frac{-2}{\sqrt{2\pi} ik} \left(\frac{-1}{ik} (e^{ik} + e^{-ik}) + \frac{1}{k^2} (e^{-ik} - e^{ik}) \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{-1}{ik} \left(\frac{-2\cos k}{ik} - \frac{2i \sin k}{k^2} \right) \\
&= \sqrt{\frac{2}{\pi}} \frac{2}{k^2} \left(\frac{\sin k}{k} - \cos k \right)
\end{aligned}
$$

I did not use.

Sep. 14, 2012

10 Forced Oscillations

10.1 Review worksheet (10 mins)

10.2 Forced Oscillations (40 mins)

$$y'' + 0.02y' + 25y = r(t) \tag{19}$$

where $r(t)$ is measured in $gm \cdot cm/sec^2$.

$$r(t) = \begin{cases} t + \frac{\pi}{2} & \text{if} \quad -\pi < t < 0 \\ -t + \frac{\pi}{2} & \text{if} \quad 0 < x < \pi \end{cases}$$
$$r(t + 2\pi) = r(t)$$

We decompose $r(t)$ to the Fourier series:

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2} + \cdots\right)$$

Therefore, equation 19 is

$$y'' + 0.02y' + 25y = \frac{4}{n^2\pi}\cos nt. \tag{20}$$

Steady-state solution $y_n(t)$ of equation 20 is of the form

$$y_n = A_n \cos nt + B_n \sin nt. \tag{21}$$

The first and second time derivations are

$$y_n' = -A_n n \sin nt + B_n n \cos nt$$
$$y_n'' = -A_n n^2 \cos nt - B_n n^2 \sin nt$$

By substituting expression 21 to equation 20:

$$(-A_n n^2 \cos nt - B_n n^2 \sin nt) + \frac{1}{50}(-A_n n \sin nt + B_n n \cos nt) + 25(A_n \cos nt + B_n \sin nt) = \frac{4}{n^2\pi}\cos nt$$

$\rightarrow$

$$\left(-A_n n^2 + \frac{B_n n}{50} + 25A_n\right)\cos nt = \frac{4}{n^2\pi}\cos nt$$
$$\left(-B_n n^2 - \frac{A_n n}{50} + 25B_n\right)\sin nt = 0$$

Then

$$A_n = \frac{4(25 - n^2)}{n^2\pi D_n}, \quad B_n = \frac{0.08}{n\pi D_n} \tag{22}$$

where

$$D_n = (25 - n^2)^2 + (0.02n)^2.$$

Because equation 20 is linear, the steady-state solution is

$$y = y_1 + y_3 + y_5 + \cdots \tag{23}$$

The amplitude of y is

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2 \pi \sqrt{D_n}}$$
$$C_1 = 0.053,\ C_3 = 0.009,\ C_5 = 0.509,\ C_7 = 0.001$$

$n = 5$ is the resonance mode.

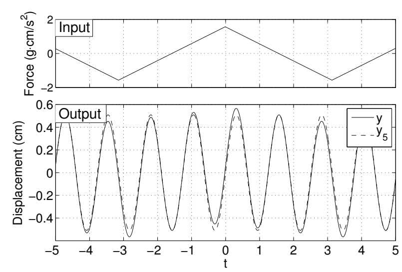

Figure 2. Input ($r(t)$) and Output ($y(t)$).

Name:

MATH348A AEM

In-class Exercise

Sep. 14, 2012

Find the Fourier series of

$$f(x) = 2 - |x|$$

where $-2 < x < 2$ and period $p = 4$.

Name:

MATH348A AEM

In-class Exercise (ANSWER)

Sep. 14, 2012

1)

$$f(x) = \begin{cases} 2-x & \text{if} \quad 0 < x < 2 \\ 2+x & \text{if} \quad -2 < x < 0 \end{cases}$$

The function is an even function.

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_0^2 (2+x)dx = 1 \\
a_n &= \int_0^2 (2-x)\cos\frac{n\pi}{2}xdx \\
&= \left[\frac{2}{n\pi}(2-x)\sin\frac{n\pi}{2}x\right]_0^2 + \int_0^2 \frac{2}{n\pi}\sin\frac{n\pi}{2}xdx \\
&= \left(\frac{2}{n\pi}\right)^2 (1-\cos n\pi) \\
f(x) &= 1 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)^2 (1-\cos n\pi)\cos\frac{n\pi}{2}x
\end{aligned}$$

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Homework: ODE

Due date: Sep. 17, 2012

Find a general solution of ODE $y'' + \omega^2 y = r(t)$ (no damping) with

$$r(t) = \begin{cases} -t - \pi & \text{if} \quad -\pi < t - \pi/2 \\ t & \text{if} \quad -\pi/2 < t < \pi/2 \\ \pi - t & \text{if} \quad \pi/2 < x < \pi \end{cases}$$

$$r(t + 2\pi) = r(t)$$

and $\omega = 2$.

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Homework: ODE (ANSWER)

Due date: Sep. 17, 2012

$r(t)$ is an odd function.

$$b_n = \frac{1}{\pi}\left(\left[-\frac{t}{n}\cos nt\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{1}{n}\cos nt dt + \left[\frac{-\pi+t}{n}\cos nt\right]_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} -\frac{1}{n}\cos nt dt\right)$$
$$= \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$
$$r(t) = \sum_{\infty}^{n=1} \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\sin nt$$
$$= \frac{2}{\pi}\left(\sin t - \frac{1}{3^2}\sin 3t + \frac{1}{5^2}\sin 5t \cdots\right)$$

$$y'' + 4y = r(t)$$
$$-n^2 A_n \cos nt - n^2 B_n \sin nt + 4(A_n\cos nt + B_n \sin nt) = \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\sin nt$$
$$\rightarrow$$
$$-n^2 A_n + 4A_n = 0 \ \rightarrow \ A_n = 0$$
$$-n^2 B_n + 4B_n = \frac{2}{n^2\pi}\sin\frac{n\pi}{2} \ \rightarrow \ B_n = \frac{2}{(4-n^2)n^2\pi}\sin\frac{n\pi}{2}$$
$$y_n = \frac{2}{(4-n^2)n^2\pi}\sin\frac{n\pi}{2}\sin nt$$
$$y(t) = y_1 + y_3 + \cdots$$

References

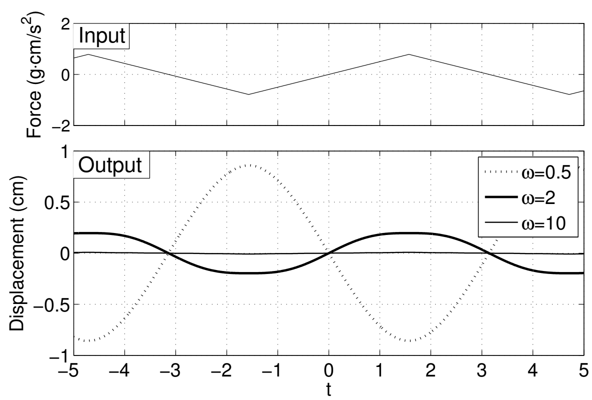

Figure 3. Input ($r(t)$) and Output ($y(t)$).