## 1.1 *Stress, strain, and displacement → wave equation*

From the relationship between stress, strain, and displacement, we can derive a 3D elastic wave equation. Figure 1.1 shows relationships between each pair of parameters. In this section, I will show each term in Figure 1.1.

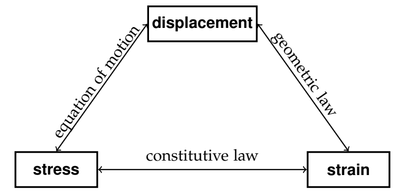


Figure 1.1: Relationship of each parameter.

### 1.1.1 *Displacement*

Displacement, characterizes vibrations, is distance of a particle from its position of equilibrium:

$$\mathbf{u}(\mathbf{x},t) = \begin{pmatrix} u_1(\mathbf{x},t) \\ u_2(\mathbf{x},t) \\ u_3(\mathbf{x},t) \end{pmatrix}. \tag{1.1}$$

### 1.1.2 *Stress*

Stress characterizes forces applied to a material:

$$\sigma_{ij} = \underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \tag{1.2}$$

which is a tensor, and the first subscript indicates the surface applied and the second the direction (Figure 1.2).

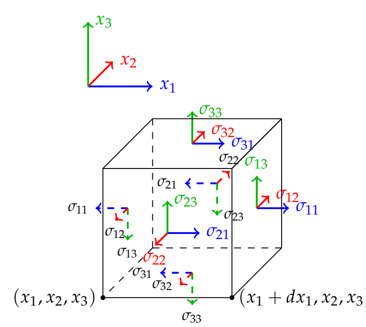


Figure 1.2: Stresses.

### 1.1.3 *Strain*

Strain characterizes deformations under stress. If stresses are applied to a material that is not perfectly rigid, points within it move with respect to each other, and deformation results.

Let us consider an elastic material which moves $\mathbf{u}(\mathbf{x})$ (Figure 1.3). When the original location of the material is $\mathbf{x}$, the displacement of a nearby point originally at $\mathbf{x} + \delta\mathbf{x}$ can be written as

$$u_i(\mathbf{x}+\delta\mathbf{x}) \approx u_i(\mathbf{x}) + \frac{\partial u_i(\mathbf{x})}{\partial x_j}\delta x_j = \underbrace{u_i(\mathbf{x})}_{parallel\ translation} + \underbrace{\delta u_i}_{rotation+deformation}, \tag{1.3}$$

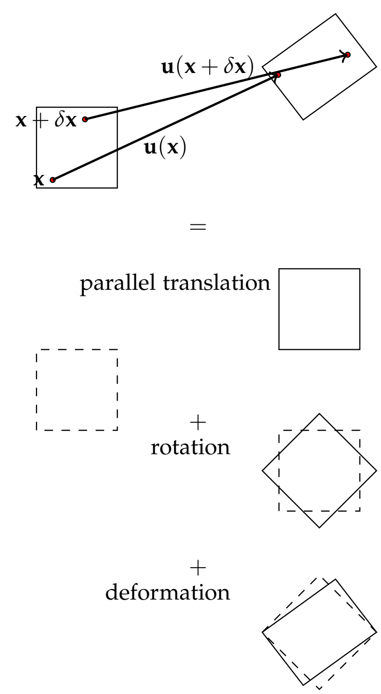


Figure 1.3: Displacement includes parallel translation, rotation, and deformation (strain).

Therefore, in the first-order assumption,

$$
\begin{aligned}
\delta u_i &= \frac{\partial u_i(\mathbf{x})}{\partial x_j}\delta x_j \\
&= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\delta x_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\delta x_j \\
&= \frac{1}{2}(u_{i,j} + u_{j,i})\delta x_j + \frac{1}{2}(\nabla \times \mathbf{u} \times \delta \mathbf{x})_i \\
&= (e_{ij} + \omega_{ij})\delta x_j,
\end{aligned} \tag{1.4}
$$

where $\omega_{ij}$ is a rotational translation term (diagonal term is zero, $\omega_{ij} = -\omega_{ji}$). Then $e_{ij} = \underline{\mathbf{e}}$ is the strain tensor, which contains the spatial derivatives of the displacement field. With the definition of $e_{ij}$, the tensor is symmetric and has 6 independent components.

$$
e_{ij} = \begin{pmatrix} u_{1,1} & 1/2(u_{1,2} + u_{2,1}) & 1/2(u_{1,3} + u_{3,1}) \\ 1/2(u_{2,1} + u_{1,2}) & u_{2,2} & 1/2(u_{2,3} + u_{3,2}) \\ 1/2(u_{3,1} + u_{1,3}) & 1/2(u_{3,2} + u_{2,3}) & u_{3,3} \end{pmatrix} \tag{1.5}
$$

If the diagonal terms of $e_{ij}$ are zero, we do not have volume changes. The volume increase, *dilatation*, is given by the sum of the extensions in the $x_i$ directions:

$$
e_{ii} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = tr(\mathbf{e}) = \nabla \cdot \mathbf{u} = \theta \tag{1.6}
$$

This dilatation gives the change in volume per unit volume associated with the deformation. $\partial u_i / \partial x_i$ mentions displacement of the $x_i$ direction changes along the direction of $x_i$.

$$
\left(1 + \frac{\partial u_1}{\partial x_1}\right) dx_1 \left(1 + \frac{\partial u_2}{\partial x_2}\right) dx_2 \left(1 + \frac{\partial u_3}{\partial x_3}\right) dx_3 \approx \left(1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_1 dx_2 dx_3 = (1+\theta)V = V + \Delta V, \tag{1.7}
$$

where $\theta = \Delta V / V$.

### 1.1.4 Geometric law

Relationship between displacement and strain, which represents geometric properties (deformation).

As we have already found in equation 1.4,

$$
\underline{\mathbf{e}} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \tag{1.8}
$$

### 1.1.5 Equation of motion

Relationship between displacement and stress, which represents dynamic properties (motion).

We write Newton's second law in terms of body forces and stresses. When I consider the stresses in the $x_2$ direction (the red arrows in Figure 1.2),

$$\begin{aligned} &\{\sigma_{12}(\mathbf{x}+dx_1\hat{\mathbf{n}}_1)-\sigma_{12}(\mathbf{x})\}\,dx_2dx_3 \\ &+\{\sigma_{22}(\mathbf{x}+dx_2\hat{\mathbf{n}}_2)-\sigma_{22}(\mathbf{x})\}\,dx_1dx_3 \\ &+\{\sigma_{32}(\mathbf{x}+dx_3\hat{\mathbf{n}}_3)-\sigma_{32}(\mathbf{x})\}\,dx_1dx_2 \\ &\qquad\qquad +f_2dV=\rho\frac{\partial^2 u_2}{\partial t^2}dV \end{aligned} \tag{1.9}$$

where $dV = dx_1dx_2dx_3$. With a Taylor expansion,

$$\left(\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{32}}{\partial x_3}\right)dV+f_2dV=\rho\frac{\partial^2 u_2}{\partial t^2}dV \tag{1.10}$$

We also have similar equations for $x_1$ and $x_2$ directions, and by using the summation convention,

$$\begin{aligned} \underbrace{\sigma_{ij,j}(\mathbf{x},t)}_{surface\ forces}+\underbrace{f_i(\mathbf{x},t)}_{body\ forces}&=\rho\frac{\partial^2 u_i(\mathbf{x},t)}{\partial t^2} \\ \nabla\cdot\underline{\sigma}+\mathbf{f}&=\rho\ddot{\mathbf{u}}. \end{aligned} \tag{1.11}$$

This is the equation of motion, which is satisfied everywhere in a continuous medium. When the right-hand side in equation 1.11 is zero, we have the equation of equilibrium,

$$\sigma_{ij,j}(\mathbf{x},t)=-f_i(\mathbf{x},t), \tag{1.12}$$

and if no body forces are applied, we have the homogeneous equation of motion

$$\sigma_{ij,j}(\mathbf{x},t)=\rho\frac{\partial^2 u_i(\mathbf{x},t)}{\partial t^2}. \tag{1.13}$$

### *1.1.6 Constitutive equations*

Relationship between stress and strain, which represents material properties (strength, stiffness). Here, we consider the material has a linear relationship between stress and strain (linear elastic). Linear elasticity is valid for the short time scale involved in the propagation of seismic waves.

Based on Hooke's law, the relationship between stress and strain is

$$\begin{aligned} \sigma_{ij}&=c_{ijkl}e_{kl} \\ \underline{\sigma}&=\underline{\underline{\mathbf{c}}}\,\underline{\mathbf{e}}, \end{aligned} \tag{1.14}$$

where constant $c_{ijkl}$ is the elastic moduli, which describes the properties of the material.

Not all components of $c_{ijkl}$ are independent. Because stress and strain tensors are symmetric and thermodynamic consideration;

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{1.15}$$

Therefore, we have 21 independent components in $c_{ijkl}$. With Voigt recipe, we change the subscripts with

$$11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5,\ 12 \to 6,$$

and we can write the elastic moduli as $c_{ij}$ $(i, j = 1, 2, \cdots, 6)$. With these 21 components, we can describe general anisotropic media.

Strain energy is defined by

$$W = \frac{1}{2}\int \sigma_{ij} e_{ij} dV$$
$$= \frac{1}{2}\int c_{ijkl} e_{ij} e_{kl} dV,$$

Therefore, $c_{ijkl} = c_{klij}$.

### 1.1.7 *Wave equation (general anisotropic media)*

Wave equation describes vibrations ($\mathbf{u}$) at each space ($\mathbf{x}$) and time ($t$) under material properties ($\underline{\underline{\mathbf{c}}}$, $\rho$);

$$f(\mathbf{u}, \mathbf{x}, t, \rho, \underline{\underline{\mathbf{c}}}) = \mathbf{F}. \tag{1.16}$$

In homogeneous case ($\mathbf{F} = 0$),

$$f(\mathbf{u}, \mathbf{x}, t, \rho, \underline{\underline{\mathbf{c}}}) = 0. \tag{1.17}$$

We eliminate $\underline{\boldsymbol{\sigma}}$ and $\underline{\mathbf{e}}$ by plugging in equations 1.8, 1.11, and 1.14.

$$\nabla \cdot \left\{ \underline{\underline{\mathbf{c}}} \left( \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \right) \right\} = \rho \ddot{\mathbf{u}} \tag{1.18}$$

This is a general wave equation for anisotropic elastic media.

*geometric law* $\underline{\mathbf{e}} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ (eq 1.8)

- small perturbation

*equation of motion* $\nabla \cdot \underline{\boldsymbol{\sigma}} = \rho \ddot{\mathbf{u}}$ (eq 1.11)

- small perturbation
- continuous material

*constitutive law* $\underline{\boldsymbol{\sigma}} = \underline{\underline{\mathbf{c}}}\ \underline{\mathbf{e}}$ (eq 1.14)

- small perturbation
- continuous material
- elastic material

### 1.1.8 *Elastic moduli in isotropic media*

On a large scale (compared with wave length), the earth has approximately the same physical properties regardless of orientation, which is called *isotropic*. In the isotropic case, $c_{ijkl}$ has only two independent components. One pair of the components are called the Lamé constants $\lambda$ and $\mu$, which are defined as

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.19}$$

$\mu$ is called the shear modulus, but $\lambda$ does not have clear physical explanation. By using the Voigt recipe, equation 1.18 can be written with a matrix form;

$$c_{ij} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \tag{1.20}$$

In the isotropic media, equation 1.14 becomes

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$
$$\underline{\sigma} = \lambda tr(\underline{\mathbf{e}})\mathbf{I} + 2\mu\underline{\mathbf{e}} \qquad (1.21)$$

where $\theta$ is the dilatation.

There are other elastic moduli, which are related to the Lamé constants, such as bulk modulus ($K$), Poisson's ratio ($\nu$), and Young's modulus ($E$) (Table 1.1.8).

Table 1.1: Elastic moduli

| | $(\lambda,\mu)$ | $(\lambda,\nu)$ | $(K,\lambda)$ | $(E,\mu)$ | $(K,\mu)$ | $(E,\nu)$ | $(\mu,\nu)$ | $(K,\nu)$ | $(K,E)$ |
|---|---|---|---|---|---|---|---|---|---|
| $K$ | $\lambda+\frac{2}{3}\mu$ | $\frac{\lambda(1+\nu)}{3\nu}$ | | $\frac{E\mu}{3(3\mu-E)}$ | | $\frac{E}{3(1-2\nu)}$ | $\frac{2\mu(1+\nu)}{3(1-2\nu)}$ | | |
| $\nu$ | $\frac{\lambda}{2(\lambda+\mu)}$ | | $\frac{\lambda}{3K-\lambda}$ | $\frac{E}{2\mu}$ | $\frac{3K-2\mu}{2(3K+\mu)}$ | | | | $\frac{3K-E}{6K}$ |
| $E$ | $\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ | $\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$ | $\frac{9K(K-\lambda)}{3K-\lambda}$ | | $\frac{9K\mu}{3K+2\mu}$ | | $2\mu(1+\nu)$ | $3K(1-2\nu)$ | |
| $\lambda$ | | | | | | | | | |
| $\mu$ | | | | | | | | | |

### 1.1.9 *Wave equation in isotropic media*

Using equation 1.21 instead of equation 1.14, we can derive the wave equation in an isotropic medium.

From equations 1.8, 1.11, and 1.21, the isotropic wave equation is

$$\rho\ddot{\mathbf{u}} = (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times\nabla\times\mathbf{u}, \qquad (1.22)$$

with an assumption of slowly-varying material ($\nabla\lambda \approx 0$ and $\nabla\mu \approx 0$).

$\nabla\cdot\mathbf{u}$ volumetric deformation

$\nabla\times\mathbf{u}$ shearing deformation

*geometric law* $\underline{\mathbf{e}} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$ (eq 1.8)

*equation of motion* $\nabla\cdot\underline{\sigma} = \rho\ddot{\mathbf{u}}$ (eq 1.11)

*constitutive law* $\underline{\sigma} = \lambda tr(\underline{\mathbf{e}})\mathbf{I} + 2\mu\underline{\mathbf{e}}$ (eq 1.21)

### 2.3.11 *Principal stresses*

For any stress tensor, we can always find a direction of $\hat{\mathbf{n}}$ that defines the plane of no shear stresses. This is important for earthquake source mechanisms.

To find the direction $\hat{\mathbf{n}}$ is an eigenvalue problem:

$$\begin{aligned} \underline{\sigma}\hat{\mathbf{n}} &= \lambda\hat{\mathbf{n}} \\ (\underline{\sigma} - \lambda\underline{\mathbf{I}})\hat{\mathbf{n}} &= 0, \end{aligned} \quad (2.57)$$

where $\lambda$ is eigenvalues, not a Lamé constant. To find $\lambda$, we need to solve

$$\det[\underline{\sigma} - \lambda\underline{\mathbf{I}}] = 0, \quad (2.58)$$

and obtain three eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$ ($|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$), which are the *principal stresses* ($\sigma_1$, $\sigma_2$, and $\sigma_3$, respectively). Corresponding eigenvectors for each eigenvalue define the *principal stress axes* ($\hat{\mathbf{n}}^{(1)}$, $\hat{\mathbf{n}}^{(2)}$, and $\hat{\mathbf{n}}^{(3)}$).

Relationship between the original stress tensor $\underline{\sigma}$ and principle stresses.

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \underline{\mathbf{R}}^T\underline{\sigma}\underline{\mathbf{R}}$$

where $\underline{\mathbf{R}}$ is the rotational matrix based on the eigenvectors:

$$\underline{\mathbf{R}} = \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix}$$

### 2.3.12 *Traction on a fault*

The traction at an arbitrary plane of orientation ($\sigma$) is obtained by multiplying the stress tensor by $\sigma$:

$$\mathbf{T}(\hat{\mathbf{n}}) = \underline{\sigma}\hat{\mathbf{n}}. \quad (2.59)$$

Using this relationship, we can compute a traction on a fault.

In the 2D case, the stress tensor is

$$\underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \quad (2.60)$$

When the fault is oriented $\theta$ (clockwise) from the $x_1$ axis, the normal vector is

$$\hat{\mathbf{n}} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \quad (2.61)$$

Therefore, from equation 2.59, the traction on the fault is

$$\mathbf{T}(\hat{\mathbf{n}}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \quad (2.62)$$

which indicates the direction and strength of the traction on the fault. We can decompose the traction into normal ($\mathbf{T}_N$) and shear $\mathbf{T}_S$ tractions on the fault:

$$\hat{\mathbf{f}} = \mathbf{R}\hat{\mathbf{n}}$$

where

$$\mathbf{R} = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{T}_N = \mathbf{T}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$
$$\mathbf{T}_S = \mathbf{T}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{f}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \qquad (2.63)$$

where $\hat{\mathbf{f}}$ is the unit vector parallel to the fault direction.

### *2.3.13 Deviatoric stresses*

Because in the deep Earth, compressive stresses are dominant, only considering the deviatoric stresses is useful for many applications. For example, the deviatoric stresses result from tectonic forces and cause earthquake faulting.

When the mean normal stress is given by $M = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$, the deviatoric stress is

$$\underline{\boldsymbol{\sigma}_D} = \underline{\boldsymbol{\sigma}} - M\underline{\mathbf{I}} \qquad (2.64)$$

## 2.4 Seismic waves

With components, the 3D isotropic wave equation can be written as

$$\rho \begin{pmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \\ \frac{\partial^2 u_3}{\partial t^2} \end{pmatrix} = (\lambda + 2\mu) \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) \\ \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) \\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) \end{pmatrix} - \mu \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \end{pmatrix} \tag{2.67}$$

### 2.4.1 P- and S-wave velocities

We can separate equation 2.65 into solutions for P and S waves by calculating the divergence and curl, respectively.

Equation 2.65:

$$\rho\ddot{\mathbf{u}} = (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times \nabla \times \mathbf{u},$$

When we compute the divergence of equation 2.65, we obtain

$$\begin{aligned} \rho\frac{\partial^2(\nabla \cdot \mathbf{u})}{\partial t^2} &= (\lambda + 2\mu)\nabla^2(\nabla \cdot \mathbf{u}) \\ \nabla^2(\nabla \cdot \mathbf{u}) - \frac{1}{\alpha^2}\frac{\partial^2(\nabla \cdot \mathbf{u})}{\partial t^2} &= 0, \end{aligned} \tag{2.68}$$

where $\alpha$ is the P-wave velocity:

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{2.69}$$

By computing the curl of equation 2.65, we obtain

$$\begin{aligned} \nabla \times (\nabla\phi) &= 0 \\ \nabla \cdot (\nabla \times \gamma) &= 0 \\ \nabla \times \nabla \times \mathbf{u} &= \nabla\nabla \cdot \mathbf{u} - \nabla^2\mathbf{u} \end{aligned}$$

$$\begin{aligned} \rho\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} &= -\mu\nabla \times \nabla \times \nabla \times \mathbf{u} \\ \rho\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} &= \mu\nabla^2(\nabla \times \mathbf{u}) \\ \nabla^2(\nabla \times \mathbf{u}) - \frac{1}{\beta^2}\frac{\partial^2(\nabla \times \mathbf{u})}{\partial t^2} &= 0, \end{aligned} \tag{2.70}$$

where $\beta$ is the S-wave velocity:

$$\beta = \sqrt{\frac{\mu}{\rho}}. \tag{2.71}$$

Using $\alpha$ and $\beta$, we can rewrite equation 2.65 as

$$\ddot{\mathbf{u}} = \underbrace{\alpha^2\nabla(\nabla \cdot \mathbf{u})}_{P\ wave} - \underbrace{\beta^2\nabla \times (\nabla \times \mathbf{u})}_{S\ wave} \tag{2.72}$$

### 2.4.2 Potentials

A vector field can be represented as a sum of curl-free and divergence-free forms [1] (so called *Helmholtz decomposition*),

[1] Keiiti Aki and Paul G. Richards. *Quantitative Seismology*. Univ. Science Books, CA, USA, 2 edition, 2002

$$
\begin{aligned}
\mathbf{u} &= \nabla\phi + \nabla \times \mathbf{\Psi} \\
\nabla \cdot \Phi &= 0,
\end{aligned} \tag{2.73}
$$

where $\phi$ is P-wave scalar potential and $\Psi$ is S-wave vector potential. Therefore, we have

$$\nabla \cdot \mathbf{u} = \nabla^2\phi \tag{2.74}$$

$$\nabla \times \mathbf{u} = \nabla \times \nabla \times \mathbf{\Psi} = -\nabla^2\mathbf{\Psi} \tag{2.75}$$

Inserting equations 2.74 and 2.75 into equations 2.68 and 2.70, we obtain two equations for these potentials:

$$\nabla^2\phi - \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.76}$$

$$\nabla^2\mathbf{\Psi} - \frac{1}{\beta^2}\frac{\partial^2\mathbf{\Psi}}{\partial t^2} = 0, \tag{2.77}$$

and P- and S-wave displacements are given by gradient of $\phi$ and curl of $\mathbf{\Psi}$ in equation 2.76.

Equation 2.76 is exactly the same as the 3D scaler wave equation we expected from the 1D one (equation 2.23).

### 2.4.3 *Plane waves*

Because of the shape of wave equations (equations 2.70, 2.76, and 2.77), elastic wave equations also have plane waves as solutions. Plane-wave solution is a solution to the wave equation in which the displacement varies only in the direction of wave propagation and constant in the directions orthogonal to the wave propagation. The solution can be written as

$$
\begin{aligned}
\mathbf{u}(\mathbf{x}, t) &= \mathbf{f}(t - \hat{\mathbf{s}} \cdot \mathbf{x}/c) \\
&= \mathbf{f}(t - \mathbf{s} \cdot \mathbf{x}) \\
&= \mathbf{A}e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}
\end{aligned} \tag{2.78}
$$

where $\mathbf{s}$ is the slowness vector and $c$ is the velocity. The slowness vector shows the direction of the wave propagation. $\mathbf{k} = \omega\mathbf{s}$ is the wavenumber vector.

### 2.4.4 *Spherical waves*

A spherical wave is also a solution for 3D scalar wave equation (equation 2.76). For convenience, we consider the spherical coordinates, and equation 2.76 becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) - \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{2.79}$$

$$\nabla^2\phi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r^2}\right)$$

For $r \neq 0$, a solution of equation 2.79 is

$$\phi(r, t) = \frac{f(t \pm r/\alpha)}{r}, \tag{2.80}$$

which indicates spherical waves.

### 2.4.5 *Polarizations of P and S waves*

Let us consider P plane waves propagating in $x_1$ direction. A plane-wave solution for equation 2.76 is

$$\phi(x_1, t) = Ae^{i(\omega t - kx_1)}, \tag{2.81}$$

and the displacement is

$$\mathbf{u}(x_1, t) = \nabla\phi(x_1, t) = (-ik, 0, 0)Ae^{i(\omega t - kx_1)}. \tag{2.82}$$

Because the compression caused by this displacement is nonzero ($\nabla \cdot \mathbf{u}(x_1, t) \neq 0$), the volume changes. From equation 2.82, the direction of wave propagation and the direction of displacements are the same (*longitudinal wave*).

For S waves, a plane-wave solution for equation 2.77 is a vector:

$$\boldsymbol{\Psi}(\boldsymbol{x_1, t}) = (A_1, A_2, A_3)e^{i(\omega t - kx_1)}, \tag{2.83}$$

and the corresponding displacement is

$$\mathbf{u}(x_1, t) = \nabla \times \boldsymbol{\Psi}(x_1, t) = (0, -ikA_3, ikA_2)e^{i(\omega t - kx_1)}. \tag{2.84}$$

In contrast to P waves, S waves have no volumetric changes ($\nabla \cdot \mathbf{u}(x_1, t) = 0$) and the direction of displacements differ from the direction of wave propagation.

## 1.1 Plane wave reflection and transmission

### 1.1.1 Introduction

When we consider the propagating waves are plane waves, we can find a coordinate system which has $\partial u_i / \partial x_2 = 0$. From equation , if we choose these axes, we obtain

This means that we consider wave propagation on a plane, which is perpendicular to the $x_2$ axis.

$$\rho \begin{pmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \\ \frac{\partial^2 u_3}{\partial t^2} \end{pmatrix} = (\lambda + 2\mu) \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \\ 0 \\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \end{pmatrix} - \mu \begin{pmatrix} -\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \end{pmatrix}$$
$$= (\lambda + 2\mu) \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \\ 0 \\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \end{pmatrix} + \mu \begin{pmatrix} \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_1^2} \\ -\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \end{pmatrix}. \tag{1.1}$$

The displacement on the $x_2$ direction is independent from $x_1$ and $x_3$, and only contain S waves, which are called SH waves. The waves described by $u_1$ and $u_3$ are called P-SV waves.

### 1.1.2 SH wave

From equation 1.1 with replacing $u_2$ to $v$, $\rho/\mu$ as $1/\beta^2$, and $x_1 x_2 x_3$ to $xyz$, we obtain

$$\frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}, \tag{1.2}$$

which is a 2D scaler wave equation. The waves represented by $v$ are called SH wave. We consider a plane-wave solution of equation 1.2 as

$$v = e^{-i\omega(t - px - \eta z)}, \tag{1.3}$$

where $p$ is the ray parameter (and $p$ is the horizontal slowness and $\eta$ the vertical slowness). With $p$ and $\beta$, $\eta$ is

Slownesses and wavenumbers are also related.

$$k_x = p\omega,\ k_z = \eta\omega$$

$$\eta^2 = \frac{1}{\beta^2} - p^2. \tag{1.4}$$

Based on the incident angle of the wave $\phi$ (angle from the $z$ axis), horizontal and vertical slownesses are

$$p = \frac{\sin\phi}{\beta},\ \eta = \frac{\cos\phi}{\beta}. \tag{1.5}$$

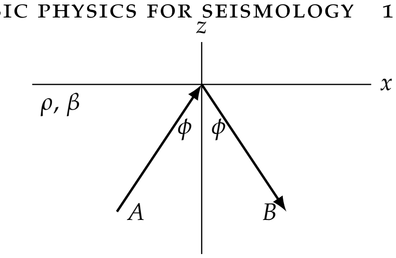


Figure 1.1: Reflection at the free surface.

### 1.1.3 *Reflection and transmission of SH wave*

Let us consider the reflection at the free surface (Figure 1.1). The general solution of SH waves reflected at the free surface is given by

$$v = \underbrace{Ae^{-i\omega(t-px-\eta z)}}_{incoming} + \underbrace{Be^{-i\omega(t-px+\eta z)}}_{reflection}, \quad (1.6)$$

where $A$ and $B$ are constants. As a boundary condition at the free surface, stresses $\sigma_{zx}$, $\sigma_{yz}$, and $\sigma_{zz}$ are zero (because we are considering only the $y$ direction, we use only the condition of $\sigma_{yz}$); therefore at $z = 0$,

$$\sigma_{yz} = \sigma_{zy} = \mu \frac{\partial v}{\partial z}, \quad (1.7)$$

where the first equation naturally satisfies by our coordinate system. From the second equation, we obtain the relationship that

$$\begin{aligned} (A - B)e^{-i\omega(t-px)} &= 0 \\ \frac{B}{A} &= 1, \end{aligned} \quad (1.8)$$

which is the reflection coefficient for SH waves at the free surface. SH waves bounce at the free surface with the same amplitude. From equation 1.8, the displacement at the free surface is $v(z = 0) = 2Aexp(-i\omega(t - px))$, which means twice as large as the incoming wave (and the reflected wave).

Next, we consider the reflections at a boundary (Figure 1.2). This derivation is similar to the string case (1D scaler wave equation). We simply extend it to the 2D case. Now, we set $z = 0$ as a boundary, and medium 1 ($\rho_1$, $\beta_1$) is at $z < 0$ and medium 2 ($\rho_2$, $\beta_2$) $z > 0$. When the incoming wave propagation from medium 1, plane-wave solutions are

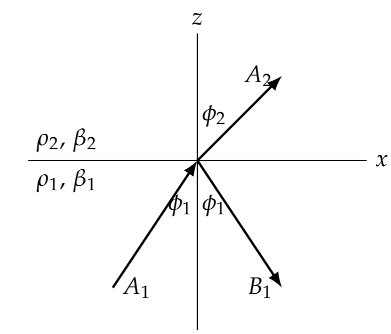


Figure 1.2: Reflection and transmission at a boundary.

why is $p$ in equation 1.9 common for media 1 and 2?

$$\begin{aligned} v_1 &= A_1 e^{-i\omega(t-px-\eta_1 z)} + B_1 e^{-i\omega(t-px+\eta_1 z)}, \ (z < 0) \\ v_2 &= A_2 e^{-i\omega(t-px-\eta_2 z)}, \ (z > 0) \end{aligned} \quad (1.9)$$

where the first term in $v_1$ is the incoming wave, the second term in $v_1$ the reflected wave, and $v_2$ the refracted wave. Define $\phi_1$ and $\phi_2$ are the angle of the incident and refracted waves, respectively, slownesses are

$$p = \frac{\sin\phi_1}{\beta_1} = \frac{\sin\phi_2}{\beta_2}, \ \eta_1 = \frac{\cos\phi_1}{\beta_1}, \ \eta_2 = \frac{\cos\phi_2}{\beta_2} \quad (1.10)$$

At $z = 0$, the displacement satisfies a boundary condition, in which displacements and stresses at the boundary are continuous:

$$v_1 = v_2, \ \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}. \quad (1.11)$$

From these conditions, we obtain

$$A_1 + B_1 = A_2,\ \mu_1\eta_1(A_1 - B_1) = \mu_2\eta_2 A_2 \tag{1.12}$$

and reflection and transmission coefficients are

$\mu/\rho = \beta^2$, $\eta_i = \cos\phi_i/\beta_i$

$$R_{12} = \frac{B_1}{A_1} = \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2} = \frac{\rho_1\beta_1\cos\phi_1 - \rho_2\beta_2\cos\phi_2}{\rho_1\beta_1\cos\phi_1 + \rho_2\beta_2\cos\phi_2}$$
$$T_{12} = \frac{A_2}{A_1} = \frac{2\mu_1\eta_1}{\mu_1\eta_1 + \mu_2\eta_2} = \frac{2\rho_1\beta_1\cos\phi_1}{\rho_1\beta_1\cos\phi_1 + \rho_2\beta_2\cos\phi_2}. \tag{1.13}$$

The impedance for SH waves at media 1 and 2 are $\rho_1\beta_1$ and $\rho_2\beta_2$, respectively.

Now, we show the energy is preserved during these reflection and transmission. The energy at a unit volume (at steady state) can be written by

$$E = \rho\omega^2 X^2, \tag{1.14}$$

where $X$ is the amplitude of waves. When the plane wave propagating with velocity $\beta$, the energy flux at a unit area (perpendicular to the propagation) is

$$F = \beta E = \rho\beta\omega^2 X^2. \tag{1.15}$$

We apply this relationship to the reflection and transmission of SH waves. The energy of the incoming wave at are $S$ is $S\rho_1\beta_1\omega^2\cos\phi_1$ and the sum of the reflection and transmission waves are $S|R_{12}|^2\rho_1\beta_1\omega^2\cos\phi_1 + S|T_{12}|^2\rho_2\beta_2\cos\phi_2$, and these energy should be equal:

$$S\rho_1\beta_1\omega^2\cos\phi_1 = S|R_{12}|^2\rho_1\beta_1\omega^2\cos\phi_1 + S|T_{12}|^2\rho_2\beta_2\cos\phi_2$$
$$1 = |R_{12}|^2 + \frac{\rho_2\beta_2\cos\phi_2}{\rho_1\beta_1\cos\phi_1}|T_{12}|^2, \tag{1.16}$$

where equation 1.13 satisfies equation 1.16.

When medium 2 has a finite thickness ($H$) and the free surface exists on top of it, waves reverberate. The solution in medium 1 is the same as equation equation 1.9. Because we have another reflected waves from the boundary at $z = H$, the solution in medium 2 is

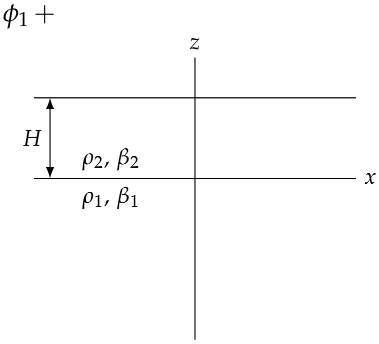


Figure 1.3: Reflection and transmission at a medium which has the free surface and a finite layer.

$$v_2 = A_2 e^{-i\omega(t - px - \eta_2(z-H))} + B_2 e^{-i\omega(t - px + \eta_2(z-H))}. \tag{1.17}$$

Because the stress $\sigma_{yz}$ is 0 at the free surface $z = H$, we obtain $A_2 = B_2$. Therefore, equation 1.17 becomes

$$v_2 = 2A_2 e^{-i\omega(t - px - \eta_2(z-H))}. \tag{1.18}$$

The boundary condition at $z = 0$ is the same as equation 1.11 and we obtain

$$A_1 + B_1 = 2A_2\cos\omega\eta_2 H$$
$$i\mu_1\eta_1(A_1 - B_1) = 2\mu_2\eta_2 A_2\sin\omega\eta_2 H. \tag{1.19}$$

From equation 1.19, we can compute reflection and transmission coefficients:

$$T = \frac{A_2}{A_1} = \frac{\mu_1\eta_1}{\mu_1\eta_1\cos\omega\eta_2 H - i\mu_2\eta_2\sin\omega\eta_2 H},$$
$$R = \frac{B_1}{A_1} = \frac{\mu_1\eta_1\cos\omega\eta_2 H + i\mu_2\eta_2\sin\omega\eta_2 H}{\mu_1\eta_1\cos\omega\eta_2 H - i\mu_2\eta_2\sin\omega\eta_2 H}. \quad (1.20)$$

Different from equations 1.8 or 1.13, equation 1.20 is a function of the frequency. This is because the reflection and transmission depend on the thickness $H$.

Proof $|R| = 1$.

Waves are amplified because of the surface layer. The amplitude ratio between the incident wave and the wave represented by equation 1.17 is

$$\left|\frac{v_2(z=H)}{A_1}\right| = \left|\frac{2A_2}{A_1}\right| = 2|T|. \quad (1.21)$$

Compared with the ratio without the surface layer (2 due to equation 1.8), $|T|$ relates to the amplification of the waves.

If $\eta_i$ is real, the denominator of $T$ is following an ellipse on the real-imaginary domain with principal axes on the real and imaginary axes when $\omega$ changes. Therefore, the maximum and minimum $T$ should be on the real or imaginary axes. On the real axis ($\sin\omega\eta_2 H = 0$ and $\cos\omega\eta_2 H = \pm 1$),

$$|T| = 1, \quad (1.22)$$

and on the imaginary axis ($\sin\omega\eta_2 H = \pm 1$ and $\cos\omega\eta_2 H = 0$),

$$|T| = \frac{\mu_1\eta_1}{\mu_2\eta_2} = \frac{\rho_1\beta_1\cos\phi_1}{\rho_2\beta_2\cos\phi_2}. \quad (1.23)$$

When we consider the vertical incident wave ($\phi_1 = \phi_2 = 0$), the maximum $|T|$ is on the real axis (equation 1.22) when the surface layer is harder than below ($\rho_1\beta_1 < \rho_2\beta_2$). On the other hand, when the surface layer is softer ($\rho_1\beta_1 > \rho_2\beta_2$), the maximum $|T|$ is on the imaginary axis (equation 1.23) and $|T| > 1$, which is the reason of amplification at the soft structure (e.g., figure 1.4). The frequency at the maximum amplification satisfies $\cos\omega\eta_2 H = 0 \rightarrow \omega\eta_2 H = (2n+1)\pi/2$.

The $T$ and $R$ (equation 1.20) include all reverberations (p101-102, Saito).

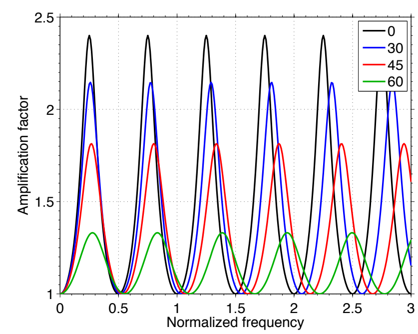


Figure 1.4: Site amplification caused by a soft surface layer for SH waves for different incident angles (line colors). The normalized frequency is $fH/\beta_2$ and the vertical axis $|T|$. In this example, I use $\rho_1/\rho_2 = 1.2$ and $\beta_1/\beta_2 = 2$.

**Postcritical reflection** When $\beta_2 > \beta_1$, $\phi_2$ can be 90° and $\phi_1$ in this condition is called *critical angle*:

$$\phi_c = \sin^{-1}\frac{\beta_1}{\beta_2}. \tag{2.101}$$

When the incident angle is larger than $\phi_c$, we have postcritical reflection, in which waves are perfectly reflected. In this case, $\eta_2 = \sqrt{\beta_2^2 - p^2}$ is imaginary. To avoid divergence of refracted waves of $v_2$ (equation 2.93) at $z \to +\infty$, the sign of $\eta_2$ should be

$$\eta_2 = i\hat{\eta}_2 = i\sqrt{p^2 - \beta_2^{-2}}\ (\omega > 0) \tag{2.102}$$

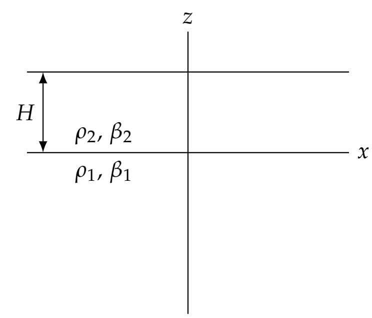


Figure 2.9: Reflection and transmission at a medium which has the free surface and a finite layer.

When medium 2 has a finite thickness ($H$) and the free surface exists on top of it, waves reverberate. The solution in medium 1 is the same as equation equation 2.93. Because we have another reflected waves from the boundary at $z = H$, the solution in medium 2 is

$$v_2 = A_2 e^{-i\omega(t-px-\eta_2(z-H))} + B_2 e^{-i\omega(t-px+\eta_2(z-H))}. \tag{2.103}$$

Because the stress $\sigma_{yz}$ is 0 at the free surface $z = H$, we obtain $A_2 = B_2$. Therefore, equation 2.103 becomes

$$v_2 = 2A_2 \cos\omega\eta_2(z-H) e^{-i\omega(t-px)}. \tag{2.104}$$

The boundary condition at $z = 0$ is the same as equation 2.95 and we obtain

$$\begin{aligned} A_1 + B_1 &= 2A_2\cos\omega\eta_2 H \\ i\mu_1\eta_1(A_1 - B_1) &= 2\mu_2\eta_2 A_2 \sin\omega\eta_2 H. \end{aligned} \tag{2.105}$$

From equation 2.105, we can compute reflection and transmission coefficients:

$$\begin{aligned} T = \frac{A_2}{A_1} &= \frac{\mu_1\eta_1}{\mu_1\eta_1\cos\omega\eta_2 H - i\mu_2\eta_2\sin\omega\eta_2 H}, \\ R = \frac{B_1}{A_1} &= \frac{\mu_1\eta_1\cos\omega\eta_2 H + i\mu_2\eta_2\sin\omega\eta_2 H}{\mu_1\eta_1\cos\omega\eta_2 H - i\mu_2\eta_2\sin\omega\eta_2 H}. \end{aligned} \tag{2.106}$$

Different from equations 2.92 or 2.97, equation 2.106 is a function of the frequency. This is because the reflection and transmission depend on the thickness $H$.
Proof $|R| = 1$.

Waves are amplified because of the surface layer. The amplitude ratio between the incident wave and the wave represented by equation 2.103 is

$$\left|\frac{v_2(z=H)}{A_1}\right| = \left|\frac{2A_2}{A_1}\right| = 2|T|. \tag{2.107}$$

Compared with the ratio without the surface layer (2 due to equation 2.92), $|T|$ relates to the amplification of the waves.

If $\eta_i$ is real, the denominator of $T$ is following an ellipse on the real-imaginary domain with principal axes on the real and imaginary axes when $\omega$ changes. Therefore, the maximum and minimum $T$ should be on the real or imaginary axes. On the real axis ($\sin \omega\eta_2 H = 0$ and $\cos \omega\eta_2 H = \pm 1$),

$$|T| = 1, \tag{2.108}$$

and on the imaginary axis ($\sin \omega\eta_2 H = \pm 1$ and $\cos \omega\eta_2 H = 0$),

$$|T| = \frac{\mu_1\eta_1}{\mu_2\eta_2} = \frac{\rho_1\beta_1 \cos\phi_1}{\rho_2\beta_2 \cos\phi_2}. \tag{2.109}$$

When we consider the vertical incident wave ($\phi_1 = \phi_2 = 0$), the maximum $|T|$ is on the real axis (equation 2.108) when the surface layer is harder than below ($\rho_1\beta_1 < \rho_2\beta_2$). On the other hand, when the surface layer is softer ($\rho_1\beta_1 > \rho_2\beta_2$), the maximum $|T|$ is on the imaginary axis (equation 2.109) and $|T| > 1$, which is the reason of amplification at the soft structure (e.g., figure 2.10). The frequency at the maximum amplification satisfies $\cos \omega\eta_2 H = 0 \rightarrow \omega\eta_2 H = (2n+1)\pi/2$.

The $T$ and $R$ (equation 2.106) include all reverberations (p101-102, Saito).

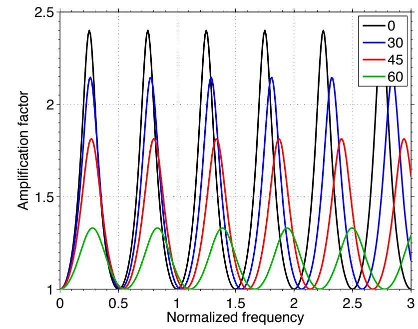


Figure 2.10: Site amplification caused by a soft surface layer for SH waves for different incident angles (line colors). The normalized frequency is $fH/\beta_2$ and the vertical axis $|T|$. In this example, I use $\rho_1/\rho_2 = 1.2$ and $\beta_1/\beta_2 = 2$.

### 2.6.4 P-SV waves

## 2.7 *Surface waves*

Surface and body waves are not very easy to distinguish because they are related. We consider that surface waves are propagating around the surface of media and the energy of them concentrate near the surface. Generally, the main features of surface waves compared with body waves are traveling slower, less amplitude decay, and velocities are frequency dependent.

### 2.7.1 *Dispersion*

One important feature is that surface waves are dispersive (in contrast to body waves), which means that the depth sensitivity of surface waves depends on frequencies of waves, and hence we can obtain vertical heterogeneity of subsurface from surface waves.

The simplest example of dispersion may be the sum of two harmonic waves with slightly different frequency and wavenumber (Figure 2.11):

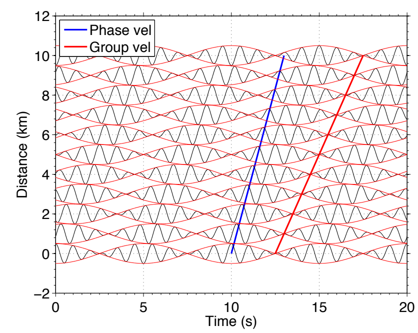


Figure 2.11: Superimposed cosine waves. Here, $\omega = 1 \times 2 \times \pi$ (1/s), $k = 0.3 \times 2 \times \pi$ (1/km), $\delta\omega = 0.1$ (1/s), and $\delta k = 0.05$ (1/km).

$$u(x,t) = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x), \tag{2.110}$$

where $\omega_1 = \omega - \delta\omega$, $\omega_2 = \omega + \delta\omega$, $k_1 = k - \delta k$, and $k_2 = k + \delta k$. Therefore,

$$\begin{aligned} u(x,t) &= \cos\{(\omega t - kx) - (\delta\omega t - \delta kx)\} + \cos\{(\omega t - kx) + (\delta\omega t - \delta kx)\} \\ &= 2\cos(\omega t - kx)\cos(\delta\omega t - \delta kx). \end{aligned} \tag{2.111}$$

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

The waveform of $u(x,t)$ consists of a cosine curve with frequency $\omega$ (*carrier*) with a superimposed cosine curve with frequency $\delta\omega$ (*envelope*). From equation 2.111, the velocities for short (carrier) and long (envelope) period waves are

$$c = \frac{\omega}{k},\ U = \frac{d\omega}{dk}, \tag{2.112}$$

respectively. In equation 2.112, we assume $\delta\omega$ and $\delta k$ approach to zero. We call $c$ as *phase velocity* and $U$ as *group velocity*. The group velocity $U$ can be written as

$$U = \frac{d\omega}{dk} = c + k\frac{dc}{dk} = c\left(1 - k\frac{dc}{d\omega}\right)^{-1}. \tag{2.113}$$

$$\begin{aligned} d\omega &= \omega - \omega_1 = ck - c_1 k_1 = ck - (c - dc)(k - dk) \\ &\approx cdk + kdc \\ dk &= k - k_1 = \frac{\omega}{c} - \frac{\omega_1}{c_1} = \frac{\omega}{c} - \frac{\omega - d\omega}{c - dc} \\ &\approx \frac{\omega}{c} - \frac{\omega - d\omega}{c} - \frac{\omega dc - dcd\omega}{c^2} \approx \frac{d\omega}{c} - \frac{\omega dc}{c^2} \\ \frac{1}{U} &= \frac{dk}{d\omega} = \frac{d\omega/c - \omega dc/c^2}{d\omega} = \frac{1}{c}\left(1 - k\frac{dc}{d\omega}\right) \end{aligned}$$

Usually, because the phase velocity $c$ of Love and Rayleigh waves increase with period (i.e., velocity increasing with depth), $dc/d\omega$ is negative. Therefore, the group velocity is slower than the phase velocity $U < c$.

### 2.7.2 Love waves

We consider the medium shown in Figure 2.12, which contains a finite thickness layer on top of a halfspace medium. Note that we need a layer to obtain Love waves. The Love-wave problem can be considered as that whether waves, which horizontally propagate with velocity $c$ and amplitude zero at $z \to \infty$, exist or not.

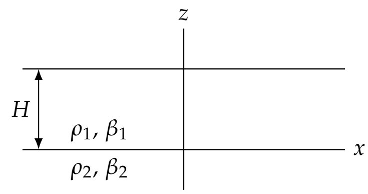

Figure 2.12: Two-layer model. I should follow the subscripts with Figure 2.9.

Love waves within a homogeneous layer can result from constructive interference between postcritical reflected SH waves.

$$\eta_1 = \sqrt{\beta_1^{-2} - p^2},\ c = 1/p$$

When we consider the condition $\beta_1 < c < \beta_2$ (which is the condition that Love waves exist I will proof later.), a solution in the medium 1 is

$$v_1(z) = \cos \omega\eta_1(z - H)e^{-i\omega(t-px)}, \tag{2.114}$$

which is equal to equation 2.104 with $A = 1/2$. Based on equation 2.93, a solution in the medium 2 is

$$v_2 = A_2 e^{-i\omega(t-px-\eta_2 z)} + B_2 e^{-i\omega(t-px+\eta_2 z)}, \tag{2.115}$$

where $\eta_2^2 < 0$ when $c < \beta_2$. When we choose $\Im(\eta_2) > 0$ $(\omega > 0)$, the first and second terms on the right-hand side of equation 2.115 are diverse and converse to zero at $z \to -\infty$, respectively. By considering the condition of amplitudes, we can write a solution in the medium 2 as

$$e^{i\omega\eta_2 z} = e^{i\omega(\Re(\eta_2)+i\Im(\eta_2))z} = \underbrace{e^{i\omega\Re(\eta_2)z}}_{oscillation}\ \underbrace{e^{-\omega\Im(\eta_2)z}}_{divergence(z=\infty)}$$

$$v_2 = B_2 e^{-i\omega(t-px+\eta_2 z)} = B_2 e^{-i\omega(t-px)} e^{\omega\hat{\eta}_2 z}, \tag{2.116}$$

where $\hat{\eta}_2 = \sqrt{p^2 - \beta_2^{-2}} > 0$.

Because $\eta_2$ is complex number, the reflected waves from the medium 1 perfectly reflect at the boundary $z = 0$. Also from equation 2.121,

$$v_1 = e^{-i\omega(t-px)}\left[e^{i\omega\eta_1(z-H)} + e^{-i\omega\eta_1(z-H)}\right],$$

which is the summation of upgoing and downgoing plane waves (with propagating to the $+x$ direction. Therefore, we can consider Love waves are reverberation of SH waves.

Because the boundary condition at the free surface is already satisfied in equation 2.114, the boundary condition at $z = 0$ should be satisfied (displacements and stresses should be continuous):

$$v_1 = v_2,\ \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}$$
$$\cos \omega\eta_1 H = B_2,\ \mu_1(\omega\eta_1 \sin \omega\eta_1 H) = \mu_2(\omega\hat{\eta}_2 B_2) \tag{2.117}$$

where $\mu_i = \rho_i\beta_i^2$. Therefore, to exist Love waves, waves satisfy

$$\Delta_l(p,\omega) = \mu_2\hat{\eta}_2 \cos \omega\eta_1 H - \mu_1\eta_1 \sin \omega\eta_1 H = 0, \tag{2.118}$$

or

$$\tan \omega\eta_1 H = \frac{\mu_2\hat{\eta}_2}{\mu_1\eta_1}, \tag{2.119}$$

which are called the characteristic equation for Love waves. With equation 2.120, Love waves exist when $\eta_1$ and $\hat{\eta}_2$ are real positive number for an angular frequency.

**Mode** The equation defines the dispersion curve for Love wave propagation within the layer. On the plane of $p\omega$, for each $p$, we have multiple values of $\omega$ satisfies equation 2.120 due to the tangent function, and the smallest $\omega$ defines the fundamental mode, and the second smallest is the first higher mode, etc. Equation 2.120 cannot be solved analytically, but we can do numerically. When $\omega$ is small, we only have one solution, which is the fundamental mode (Saito, p149). Also in the fundamental mode, $c \to \beta_2$ ($\omega \to 0$) and $c \to \beta_1$ ($\omega \to \infty$).

The angular frequency of $n$th higher modes can be defined as

$$\frac{\omega_n H}{\beta_1} = \frac{n\pi}{\sqrt{1-(\beta_1/\beta_2)^2}}, \tag{2.120}$$

and called *cut-off angular frequency*.

**Depth variation of amplitude** From equations 2.114, 2.116, and 2.117, the displacements of Love waves are

$$\begin{aligned} v_1(z) &= \underbrace{\cos\omega\eta_1(z-H)}_{amplitude}\underbrace{e^{-i\omega(t-px)}}_{phase} \\ v_2(z) &= \underbrace{\cos\omega\eta_1(H)e^{\omega\hat{\eta}_2 z}}_{amplitude}\underbrace{e^{-i\omega(t-px)}}_{phase}. \end{aligned} \tag{2.121}$$

**Group velocity** We can estimate the group velocity of Love waves by computing equation 2.113. The $p(\omega)$ derivative of $\Delta_L(p,\omega) = 0$ is

$$\begin{aligned} \frac{\partial\Delta_L(p,\omega)}{\partial\omega} + \frac{\partial\Delta_L(p,\omega)}{\partial p}\frac{\partial p(\omega)}{\partial\omega} &= 0 \\ \frac{\partial p(\omega)}{\partial\omega} &= -\frac{\partial\Delta_L/\partial\omega}{\partial\Delta_L/\partial p} \end{aligned} \tag{2.122}$$

When $f(x,y) = 0$,

$$\begin{aligned} \frac{d}{dx}f(x,y(x)) &= 0 \\ \frac{df(x,y)}{dx} + \frac{df(x,y)}{dy}\frac{dy(x)}{dx} &= 0 \end{aligned}$$

For the two-layer case (equation 2.118),

$$\frac{c}{U} = 1 + \frac{\eta_1^2}{p^2}\left[1 + \frac{(\mu_2/\mu_1)(\beta_1^{-2}-\beta_2^{-2}}{\omega\hat{\eta}_2 H[\eta_1^2 + (\mu_2/\mu_1)^2\hat{\eta}_2^2]}\right]^{-1}$$

### 2.7.3 *Rayleigh waves*