1.1 Plane wave reflection and transmission

1.1.1 Introduction

When we consider the propagating waves are plane waves, we can find a coordinate system which has $\partial u_i / \partial x_2 = 0$. From equation , if we choose these axes, we obtain

This means that we consider wave propagation on a plane, which is perpendicular to the x_2 axis.

$$\rho \begin{pmatrix} \frac{\partial^2 u_1}{\partial t^2} \\ \frac{\partial^2 u_2}{\partial t^2} \\ \frac{\partial^2 u_3}{\partial t^2} \end{pmatrix} = (\lambda + 2\mu) \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \\ 0 \\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \end{pmatrix} - \mu \begin{pmatrix} -\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial x_3}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1}\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \end{pmatrix}$$
$$= (\lambda + 2\mu) \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \\ 0 \\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}\right) \end{pmatrix} + \mu \begin{pmatrix} \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{\partial^2 u_2}{\partial x_3^2} + \frac{\partial^2 u_2}{\partial x_1^2} \\ -\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \end{pmatrix}. \tag{1.1}$$

The displacement on the x_2 direction is independent from x_1 and x_3, and only contain S waves, which are called SH waves. The waves described by u_1 and u_3 are called P-SV waves.

1.1.2 SH wave

From equation 1.1 with replacing u_2 to v, ρ/μ as $1/\beta^2$, and $x_1 x_2 x_3$ to xyz, we obtain

$$\frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2}, \tag{1.2}$$

which is a 2D scaler wave equation. The waves represented by v are called SH wave. We consider a plane-wave solution of equation 1.2 as

$$v = e^{-i\omega(t - px - \eta z)}, \tag{1.3}$$

where p is the ray parameter (and p is the horizontal slowness and η the vertical slowness). With p and β, η is

Slownesses and wavenumbers are also related.

$$k_x = p\omega,\ k_z = \eta\omega$$

$$\eta^2 = \frac{1}{\beta^2} - p^2. \tag{1.4}$$

Based on the incident angle of the wave ϕ (angle from the z axis), horizontal and vertical slownesses are

$$p = \frac{\sin\phi}{\beta},\ \eta = \frac{\cos\phi}{\beta}. \tag{1.5}$$

1.1.3 *Reflection and transmission of SH wave*

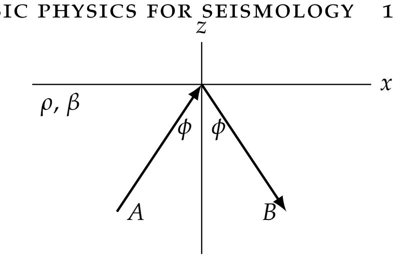

Figure 1.1: Reflection at the free surface.

Let us consider the reflection at the free surface (Figure 1.1). The general solution of SH waves reflected at the free surface is given by

$$v = \underbrace{Ae^{-i\omega(t-px-\eta z)}}_{incoming} + \underbrace{Be^{-i\omega(t-px+\eta z)}}_{reflection}, \tag{1.6}$$

where A and B are constants. As a boundary condition at the free surface, stresses σ_{zx}, σ_{yz}, and σ_{zz} are zero (because we are considering only the y direction, we use only the condition of σ_{yz}); therefore at $z = 0$,

$$\sigma_{yz} = \sigma_{zy} = \mu \frac{\partial v}{\partial z}, \tag{1.7}$$

where the first equation naturally satisfies by our coordinate system. From the second equation, we obtain the relationship that

$$\begin{aligned} (A - B)e^{-i\omega(t-px)} &= 0 \\ \frac{B}{A} &= 1, \end{aligned} \tag{1.8}$$

which is the reflection coefficient for SH waves at the free surface. SH waves bounce at the free surface with the same amplitude. From equation 1.8, the displacement at the free surface is $v(z = 0) = 2Aexp(-i\omega(t - px))$, which means twice as large as the incoming wave (and the reflected wave).

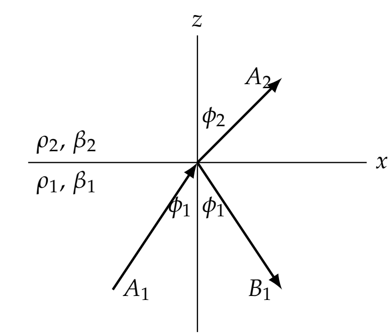

Figure 1.2: Reflection and transmission at a boundary.

why is p in equation 1.9 common for media 1 and 2?

Next, we consider the reflections at a boundary (Figure 1.2). This derivation is similar to the string case (1D scaler wave equation). We simply extend it to the 2D case. Now, we set $z = 0$ as a boundary, and medium 1 (ρ_1, β_1) is at $z < 0$ and medium 2 (ρ_2, β_2) $z > 0$. When the incoming wave propagation from medium 1, plane-wave solutions are

$$\begin{aligned} v_1 &= A_1 e^{-i\omega(t-px-\eta_1 z)} + B_1 e^{-i\omega(t-px+\eta_1 z)}, \ (z < 0) \\ v_2 &= A_2 e^{-i\omega(t-px-\eta_2 z)}, \ (z > 0) \end{aligned} \tag{1.9}$$

where the first term in v_1 is the incoming wave, the second term in v_1 the reflected wave, and v_2 the refracted wave. Define ϕ_1 and ϕ_2 are the angle of the incident and refracted waves, respectively, slownesses are

$$p = \frac{\sin\phi_1}{\beta_1} = \frac{\sin\phi_2}{\beta_2}, \ \eta_1 = \frac{\cos\phi_1}{\beta_1}, \ \eta_2 = \frac{\cos\phi_2}{\beta_2} \tag{1.10}$$

At $z = 0$, the displacement satisfies a boundary condition, in which displacements and stresses at the boundary are continuous:

$$v_1 = v_2, \ \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}. \tag{1.11}$$

From these conditions, we obtain

$$A_1 + B_1 = A_2,\ \mu_1\eta_1(A_1 - B_1) = \mu_2\eta_2 A_2 \tag{1.12}$$

and reflection and transmission coefficients are

$\mu/\rho = \beta^2$, $\eta_i = \cos\phi_i/\beta_i$

$$\begin{aligned} R_{12} &= \frac{B_1}{A_1} = \frac{\mu_1\eta_1 - \mu_2\eta_2}{\mu_1\eta_1 + \mu_2\eta_2} = \frac{\rho_1\beta_1\cos\phi_1 - \rho_2\beta_2\cos\phi_2}{\rho_1\beta_1\cos\phi_1 + \rho_2\beta_2\cos\phi_2} \\ T_{12} &= \frac{A_2}{A_1} = \frac{2\mu_1\eta_1}{\mu_1\eta_1 + \mu_2\eta_2} = \frac{2\rho_1\beta_1\cos\phi_1}{\rho_1\beta_1\cos\phi_1 + \rho_2\beta_2\cos\phi_2}. \end{aligned} \tag{1.13}$$

The impedance for SH waves at media 1 and 2 are $\rho_1\beta_1$ and $\rho_2\beta_2$, respectively.

Now, we show the energy is preserved during these reflection and transmission. The energy at a unit volume (at steady state) can be written by

$$E = \rho\omega^2 X^2, \tag{1.14}$$

where X is the amplitude of waves. When the plane wave propagating with velocity β, the energy flux at a unit area (perpendicular to the propagation) is

$$F = \beta E = \rho\beta\omega^2 X^2. \tag{1.15}$$

We apply this relationship to the reflection and transmission of SH waves. The energy of the incoming wave at are S is $S\rho_1\beta_1\omega^2\cos\phi_1$ and the sum of the reflection and transmission waves are $S|R_{12}|^2\rho_1\beta_1\omega^2\cos\phi_1 + S|T_{12}|^2\rho_2\beta_2\cos\phi_2$, and these energy should be equal:

$$\begin{aligned} S\rho_1\beta_1\omega^2\cos\phi_1 &= S|R_{12}|^2\rho_1\beta_1\omega^2\cos\phi_1 + S|T_{12}|^2\rho_2\beta_2\cos\phi_2 \\ 1 &= |R_{12}|^2 + \frac{\rho_2\beta_2\cos\phi_2}{\rho_1\beta_1\cos\phi_1}|T_{12}|^2, \end{aligned} \tag{1.16}$$

where equation 1.13 satisfies equation 1.16.

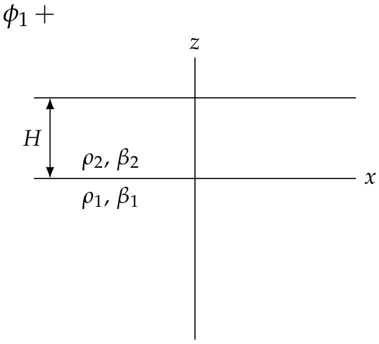

Figure 1.3: Reflection and transmission at a medium which has the free surface and a finite layer.

When medium 2 has a finite thickness (H) and the free surface exists on top of it, waves reverberate. The solution in medium 1 is the same as equation equation 1.9. Because we have another reflected waves from the boundary at $z = H$, the solution in medium 2 is

$$v_2 = A_2 e^{-i\omega(t - px - \eta_2(z-H))} + B_2 e^{-i\omega(t - px + \eta_2(z-H))}. \tag{1.17}$$

Because the stress σ_{yz} is 0 at the free surface $z = H$, we obtain $A_2 = B_2$. Therefore, equation 1.17 becomes

$$v_2 = 2A_2 e^{-i\omega(t - px - \eta_2(z-H))}. \tag{1.18}$$

The boundary condition at $z = 0$ is the same as equation 1.11 and we obtain

$$\begin{aligned} A_1 + B_1 &= 2A_2\cos\omega\eta_2 H \\ i\mu_1\eta_1(A_1 - B_1) &= 2\mu_2\eta_2 A_2 \sin\omega\eta_2 H. \end{aligned} \tag{1.19}$$

From equation 1.19, we can compute reflection and transmission coefficients:

$$T = \frac{A_2}{A_1} = \frac{\mu_1\eta_1}{\mu_1\eta_1 \cos\omega\eta_2 H - i\mu_2\eta_2 \sin\omega\eta_2 H},$$
$$R = \frac{B_1}{A_1} = \frac{\mu_1\eta_1 \cos\omega\eta_2 H + i\mu_2\eta_2 \sin\omega\eta_2 H}{\mu_1\eta_1 \cos\omega\eta_2 H - i\mu_2\eta_2 \sin\omega\eta_2 H}. \quad (1.20)$$

Different from equations 1.8 or 1.13, equation 1.20 is a function of the frequency. This is because the reflection and transmission depend on the thickness H.
Proof $|R| = 1$.

Waves are amplified because of the surface layer. The amplitude ratio between the incident wave and the wave represented by equation 1.17 is

$$\left|\frac{v_2(z=H)}{A_1}\right| = \left|\frac{2A_2}{A_1}\right| = 2|T|. \quad (1.21)$$

Compared with the ratio without the surface layer (2 due to equation 1.8), $|T|$ relates to the amplification of the waves.

If η_i is real, the denominator of T is following an ellipse on the real-imaginary domain with principal axes on the real and imaginary axes when ω changes. Therefore, the maximum and minimum T should be on the real or imaginary axes. On the real axis ($\sin\omega\eta_2 H = 0$ and $\cos\omega\eta_2 H = \pm 1$),

$$|T| = 1, \quad (1.22)$$

and on the imaginary axis ($\sin\omega\eta_2 H = \pm 1$ and $\cos\omega\eta_2 H = 0$),

$$|T| = \frac{\mu_1\eta_1}{\mu_2\eta_2} = \frac{\rho_1\beta_1\cos\phi_1}{\rho_2\beta_2\cos\phi_2}. \quad (1.23)$$

When we consider the vertical incident wave ($\phi_1 = \phi_2 = 0$), the maximum $|T|$ is on the real axis (equation 1.22) when the surface layer is harder than below ($\rho_1\beta_1 < \rho_2\beta_2$). On the other hand, when the surface layer is softer ($\rho_1\beta_1 > \rho_2\beta_2$), the maximum $|T|$ is on the imaginary axis (equation 1.23) and $|T| > 1$, which is the reason of amplification at the soft structure (e.g., figure 1.4). The frequency at the maximum amplification satisfies $\cos\omega\eta_2 H = 0 \rightarrow \omega\eta_2 H = (2n+1)\pi/2$.

The T and R (equation 1.20) include all reverberations (p101-102, Saito).

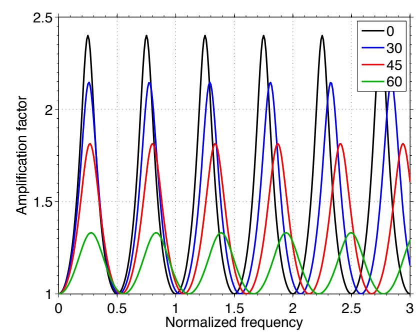

Figure 1.4: Site amplification caused by a soft surface layer for SH waves for different incident angles (line colors). The normalized frequency is fH/β_2 and the vertical axis $|T|$. In this example, I use $\rho_1/\rho_2 = 1.2$ and $\beta_1/\beta_2 = 2$.