2.4 Seismic waves

With components, the 3D isotropic wave equation can be written as

$$\rho\begin{pmatrix}\frac{\partial^2 u_1}{\partial t^2}\\ \frac{\partial^2 u_2}{\partial t^2}\\ \frac{\partial^2 u_3}{\partial t^2}\end{pmatrix}=(\lambda+2\mu)\begin{pmatrix}\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\\ \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)\end{pmatrix}-\mu\begin{pmatrix}\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)-\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)\\ \frac{\partial}{\partial x_3}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)-\frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)\\ \frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3}\right)\end{pmatrix} \tag{2.67}$$

2.4.1 P- and S-wave velocities

We can separate equation 2.65 into solutions for P and S waves by calculating the divergence and curl, respectively.

When we compute the divergence of equation 2.65, we obtain

Equation 2.65:

$\rho\ddot{\mathbf{u}}=(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\times\nabla\times\mathbf{u},$

$$\begin{aligned}\rho\frac{\partial^2(\nabla\cdot\mathbf{u})}{\partial t^2}&=(\lambda+2\mu)\nabla^2(\nabla\cdot\mathbf{u})\\ \nabla^2(\nabla\cdot\mathbf{u})-\frac{1}{\alpha^2}\frac{\partial^2(\nabla\cdot\mathbf{u})}{\partial t^2}&=0,\end{aligned} \tag{2.68}$$

where α is the P-wave velocity:

$$\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}}. \tag{2.69}$$

By computing the curl of equation 2.65, we obtain

$$\begin{aligned}\nabla\times(\nabla\phi)&=0\\ \nabla\cdot(\nabla\times\gamma)&=0\\ \nabla\times\nabla\times\mathbf{u}&=\nabla\nabla\cdot\mathbf{u}-\nabla^2\mathbf{u}\end{aligned}$$

$$\begin{aligned}\rho\frac{\partial^2(\nabla\times\mathbf{u})}{\partial t^2}&=-\mu\nabla\times\nabla\times\nabla\times\mathbf{u}\\ \rho\frac{\partial^2(\nabla\times\mathbf{u})}{\partial t^2}&=\mu\nabla^2(\nabla\times\mathbf{u})\\ \nabla^2(\nabla\times\mathbf{u})-\frac{1}{\beta^2}\frac{\partial^2(\nabla\times\mathbf{u})}{\partial t^2}&=0,\end{aligned} \tag{2.70}$$

where β is the S-wave velocity:

$$\beta=\sqrt{\frac{\mu}{\rho}}. \tag{2.71}$$

Using α and β, we can rewrite equation 2.65 as

$$\ddot{\mathbf{u}}=\underbrace{\alpha^2\nabla(\nabla\cdot\mathbf{u})}_{P\ wave}-\underbrace{\beta^2\nabla\times(\nabla\times\mathbf{u})}_{S\ wave} \tag{2.72}$$

2.4.2 Potentials

A vector field can be represented as a sum of curl-free and divergence-free forms [1] (so called *Helmholtz decomposition*),

[1] Keiiti Aki and Paul G. Richards. *Quantitative Seismology*. Univ. Science Books, CA, USA, 2 edition, 2002

$$\mathbf{u} = \nabla\phi + \nabla \times \mathbf{\Psi}$$
$$\nabla \cdot \Phi = 0, \tag{2.73}$$

where ϕ is P-wave scalar potential and Ψ is S-wave vector potential. Therefore, we have

$$\nabla \cdot \mathbf{u} = \nabla^2\phi \tag{2.74}$$
$$\nabla \times \mathbf{u} = \nabla \times \nabla \times \mathbf{\Psi} = -\nabla^2\mathbf{\Psi} \tag{2.75}$$

Inserting equations 2.74 and 2.75 into equations 2.68 and 2.70, we obtain two equations for these potentials:

$$\nabla^2\phi - \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.76}$$
$$\nabla^2\mathbf{\Psi} - \frac{1}{\beta^2}\frac{\partial^2\mathbf{\Psi}}{\partial t^2} = 0, \tag{2.77}$$

and P- and S-wave displacements are given by gradient of ϕ and curl of $\mathbf{\Psi}$ in equation 2.76.

Equation 2.76 is exactly the same as the 3D scaler wave equation we expected from the 1D one (equation 2.23).

2.4.3 *Plane waves*

Because of the shape of wave equations (equations 2.70, 2.76, and 2.77), elastic wave equations also have plane waves as solutions. Plane-wave solution is a solution to the wave equation in which the displacement varies only in the direction of wave propagation and constant in the directions orthogonal to the wave propagation. The solution can be written as

$$\begin{aligned}\mathbf{u}(\mathbf{x}, t) &= \mathbf{f}(t - \hat{\mathbf{s}} \cdot \mathbf{x}/c) \\ &= \mathbf{f}(t - \mathbf{s} \cdot \mathbf{x}) \\ &= \mathbf{A}e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}\end{aligned} \tag{2.78}$$

where $\mathbf{s}$ is the slowness vector and c is the velocity. The slowness vector shows the direction of the wave propagation. $\mathbf{k} = \omega\mathbf{s}$ is the wavenumber vector.

2.4.4 *Spherical waves*

A spherical wave is also a solution for 3D scalar wave equation (equation 2.76). For convenience, we consider the spherical coordinates, and equation 2.76 becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) - \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{2.79}$$

$$\nabla^2\phi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r^2}\right)$$

For $r \neq 0$, a solution of equation 2.79 is

$$\phi(r, t) = \frac{f(t \pm r/\alpha)}{r}, \tag{2.80}$$

which indicates spherical waves.

2.4.5 *Polarizations of P and S waves*

Let us consider P plane waves propagating in x_1 direction. A plane-wave solution for equation 2.76 is

$$\phi(x_1, t) = Ae^{i(\omega t - kx_1)}, \tag{2.81}$$

and the displacement is

$$\mathbf{u}(x_1, t) = \nabla\phi(x_1, t) = (-ik, 0, 0)Ae^{i(\omega t - kx_1)}. \tag{2.82}$$

Because the compression caused by this displacement is nonzero ($\nabla \cdot \mathbf{u}(x_1, t) \neq 0$), the volume changes. From equation 2.82, the direction of wave propagation and the direction of displacements are the same (*longitudinal wave*).

For S waves, a plane-wave solution for equation 2.77 is a vector:

$$\boldsymbol{\Psi(x_1, t)} = (A_1, A_2, A_3)e^{i(\omega t - kx_1)}, \tag{2.83}$$

and the corresponding displacement is

$$\mathbf{u}(x_1, t) = \nabla \times \boldsymbol{\Psi}(x_1, t) = (0, -ikA_3, ikA_2)e^{i(\omega t - kx_1)}. \tag{2.84}$$

In contrast to P waves, S waves have no volumetric changes ($\nabla \cdot \mathbf{u}(x_1, t) = 0$) and the direction of displacements differ from the direction of wave propagation.