2.3.11 Principal stresses

For any stress tensor, we can always find a direction of $\hat{\mathbf{n}}$ that defines the plane of no shear stresses. This is important for earthquake source mechanisms.

To find the direction $\hat{\mathbf{n}}$ is an eigenvalue problem:

$$\begin{aligned}\underline{\sigma}\hat{\mathbf{n}} &= \lambda\hat{\mathbf{n}} \\ (\underline{\sigma} - \lambda\underline{\mathbf{I}})\hat{\mathbf{n}} &= 0,\end{aligned} \tag{2.57}$$

where λ is eigenvalues, not a Lamé constant. To find λ, we need to solve

$$\det[\underline{\sigma} - \lambda\underline{\mathbf{I}}] = 0, \tag{2.58}$$

and obtain three eigenvalues λ_1, λ_2, and λ_3 ($|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$), which are the *principal stresses* (σ_1, σ_2, and σ_3, respectively). Corresponding eigenvectors for each eigenvalue define the *principal stress axes* ($\hat{\mathbf{n}}^{(1)}$, $\hat{\mathbf{n}}^{(2)}$, and $\hat{\mathbf{n}}^{(3)}$).

Relationship between the original stress tensor $\underline{\sigma}$ and principle stresses.

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \underline{\mathbf{R}}^T \underline{\sigma} \underline{\mathbf{R}}$$

where $\underline{\mathbf{R}}$ is the rotational matrix based on the eigenvectors:

$$\underline{\mathbf{R}} = \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix}$$

2.3.12 Traction on a fault

The traction at an arbitrary plane of orientation (σ) is obtained by multiplying the stress tensor by σ:

$$\mathbf{T}(\hat{\mathbf{n}}) = \underline{\sigma}\hat{\mathbf{n}}. \tag{2.59}$$

Using this relationship, we can compute a traction on a fault.

In the 2D case, the stress tensor is

$$\underline{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \tag{2.60}$$

When the fault is oriented θ (clockwise) from the x_1 axis, the normal vector is

$$\hat{\mathbf{n}} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{2.61}$$

Therefore, from equation 2.59, the traction on the fault is

$$\mathbf{T}(\hat{\mathbf{n}}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \tag{2.62}$$

which indicates the direction and strength of the traction on the fault. We can decompose the traction into normal ($\mathbf{T}_N$) and shear $\mathbf{T}_S$ tractions on the fault:

$$\hat{\mathbf{f}} = \mathbf{R}\hat{\mathbf{n}}$$

where

$$\mathbf{R} = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{T}_N = \mathbf{T}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$\mathbf{T}_S = \mathbf{T}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{f}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \qquad (2.63)$$

where $\hat{\mathbf{f}}$ is the unit vector parallel to the fault direction.

2.3.13 Deviatoric stresses

Because in the deep Earth, compressive stresses are dominant, only considering the deviatoric stresses is useful for many applications. For example, the deviatoric stresses result from tectonic forces and cause earthquake faulting.

When the mean normal stress is given by $M = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$, the deviatoric stress is

$$\underline{\sigma_D} = \underline{\sigma} - M\underline{\mathbf{I}} \qquad (2.64)$$