FAIR ALLOCATION OF A WIRELESS FADING CHANNEL: AN AUCTION APPROACH

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Abstract. We study the use of auction algorithm in allocating a wireless fading channel among a set of non-cooperating users in both downlink and uplink communication scenarios. For the downlink case, we develop a novel auction-based algorithm to allow users to fairly compete for a wireless fading channel. We use the all-pay auction mechanism whereby user bid for the channel, during each time-slot, based on the fade state of the channel, and the user that makes the higher bid wins use of the channel. Under the assumption that each user has a limited budget for bidding, we show the existence of a unique Nash equilibrium strategy. We show that the strategy achieves a throughput allocation for each user that is proportional to the user's budget and establish that the aggregate throughput received by the users using the Nash equilibrium strategy is at least 3/4 of what can be obtained using an optimal centralized allocation scheme that does not take fairness into account. We also provide a distributed algorithm that enables user's bidding strategy to converge to the Nash equilibrium strategy.

For the uplink case, we present a game-theoretical model of a wireless communication system with multiple competing users sharing a multiaccess fading channel. With a specified capture rule and a limited amount of energy available, a user opportunistically adjusts its transmission power based on its own channel state to maximize the user's own individual throughput. We derive an explicit form of the Nash equilibrium power allocation strategy. Furthermore, this Nash equilibrium power allocation strategy is unique under certain capture rule. We also quantify the loss of efficiency in throughput due to user's selfish behavior. Moreover, as the number of users in the system increases, the total system throughput obtained by using a Nash equilibrium strategy approaches the maximum attainable throughput.

Key words. Stochastic processes, Mathematical programming/optimization.

AMS(MOS) subject classifications. Primary 91A80, 91A10, 93E03, 93A14.

1. Introduction.

The limited bandwidth and high demand in a communication network necessitate a systematic procedure in place for fair allocation. This is where the economic theory of pricing and auction can be applied in the field of communications and networks research, for pricing and auction are natural ways to allocate resources with limited supply. Recently, in the networks area, much work is done to address the allocation of a limited resource in a complex, large scaled system such as the internet. They approach the problem from a classical economic perspective where users have utility functions and cost functions, both measured in the same monetary unit. Pricing is used as a tool to balance users' demand for bandwidth.

Here, we are interested in solving a specific engineering problem of scheduling transmission among a set of users while achieving fairness in

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a specific wireless environment. We use game theoretical concepts such as Nash equilibrium as a tool for modelling the interaction among users. Both the objective and the constraint of the optimization problem that each user faces have physical meanings based on underlying system. Our focus in this paper will be on the use of the all-pay auction in allocating a wireless fading channel for both the uplink and the downlink.

A fundamental characteristic of a wireless network is that the channel over which communication takes place is often time-varying. This variation of the channel quality is due to constructive and destructive interference between multipaths and shadowing effects (fading). In a single cell with one transmitter (base station or satellite) and multiple users communicating through time-varying fading channels, the transmitter can send data at higher rates to users with better channels. In time slotted system such as the HDR system, time slots are allocated among users according to their channel qualities.

The problem of resource allocation in wireless networks has received much attention in recent years. In [1] the authors try to maximize the data throughput of an energy and time constrained transmitter sending over a fading channel. A dynamic programming formulation that leads to an optimal transmission schedule is presented. Other works address the similar problem, without consideration to fairness, include [7] and [8]. In [5], the authors consider scheduling policies for *maxmin fairness* allocation of bandwidth, which maximizes the allocation for the most poorly treated sessions while not wasting any network resources, in wireless ad-hoc networks. In [14], the authors designed a scheduling algorithm that achieves *proportional fairness*, a notion of fairness originally proposed by Kelly [6]. In [9], the authors present a slot allocation scheme that maximizes expected system performance subject to the constraint that each user gets a fixed fraction of time slots. The authors did not use a formal notion of fairness, but argue that their system can explicitly set the fraction of time assigned to each user. Hence, while each user may get to use the channel an equal fraction of the time, the resulting throughput obtained by each user may be vastly different.

The following simple example illustrates the different allocations that may result from the different notions of fairness. We consider the communication system with one transmitter and two users, A and B, and the allocations that use different notions of fairness discussed in the previous paragraph. We assume that the throughput is proportional to the the channel condition. The channel coefficient, which is a quantitative measure of the channel condition ranging from 0 to 1 with 1 as the best channel condition, for user A and user B in the two time slots are (0.1, 0.2) and (0.3, 0.9) respectively. The throughput result for each individual user and for total system under different notions of fairness constraint are given in Table I. When there is no fairness constraint, to maximize the total system throughput would require the transmitter to allocate both time slots to user

B. To achieve maxmin fair allocation, the transmitter would allocate slot one to user B and slot two to user A, thus resulting in a total throughput of 0.5. If the transmitter wants to maximize the total throughput subject to the constraint that each user gets one time slot (i.e., the approach of [9]), the resulting allocation, denoted as time fraction fair, is to give user A slot one and user B slot two. As a result, the total throughput is 1.0. In the

	Throughput for user A	Throughput for user B	Total throughput
No fair constraint	0	1.2	1.2
Maxmin fair	0.2	0.3	0.5
Time fraction	0.1	0.9	1.0

Table 1
Throughput results using different notions of fairness.

above example, the transmitter selects an allocation to ensure an artificially chosen notion of fairness. From Table I, we can see that from the user's perspective, no notion is truly fair as both users want slot two. In order to resolve this conflict, we use a new approach which allows users to compete for time slots. In this way, each user is responsible for its own action and its resulting throughput. We call the fraction of bandwidth received by each user *competitive fair*. Using this notion of competitive fairness, the resulting throughput obtained for each user can serve as a reference point for comparing various other allocations. Moreover, the competitive fair allocation scheme can provide fundamental insight into the design of a fair scheduler that make sense.

In our model, users compete for time-slots. For each time-slot, each user has a different valuation (i.e., its own channel condition). And each user is only interested in getting a higher throughput for itself. Naturally, these characteristics give rise to an auction. In this paper we consider the all-pay auction mechanism. Using the all-pay auction mechanism, users submit a "bid" for the time-slot and the transmitter allocates the slot to the user that made the highest bid. Moreover, in the all-pay auction mechanism, the transmitter gets to keep the bids of all users (regardless of whether or not they win the auction). Each user is assumed to have an initial amount of money. The money possessed by each user can be viewed as fictitious money that serves as a mechanism to differentiate the QoS given to the various users. This fictitious money, in fact, could correspond to a certain QoS for which the user paid in real money. As for the solution of the slot auction game, we use the concept of Nash equilibrium, which is a set of strategies (one for each player) from which there are no profitable unilateral deviation.

In the downlink communication scenario, we consider a communication system with one transmitter and two users. For each time slot, channel states are independent and identically distributed with known probability distribution. Each user wants to maximize its own *expected* total throughput subject to an average money constraint.

We have the following main results for the downlink case:

- We find a unique Nash equilibrium when both channel states are uniformly distributed over $[0, 1]$.
- We show that the Nash equilibrium strategy pair provides an allocation scheme that is fair in the sense that the price per unit of throughput is the same for both users.
- We show that the Nash equilibrium strategy of this auction leads to an allocations at which total throughput is no worse than $3/4$ of the throughput obtained by an algorithm that attempts to maximize total system throughput without a fairness constraint.
- We provide an estimation algorithm that enables users to accurately estimate the amount of money possessed by their opponent so that users do not need prior knowledge of each other's money.

The all-pay auction can be used to model the uplink power allocation as well. In the second part of this paper, we present a distributed uplink power allocation scheme that based on the all-pay auction. Specifically, we consider a communication system consisting of multiple users competing to access a satellite, or a base-station. Each user has an average power constraint. Time is slotted. During each time slot, each user chooses a power level for transmission based on the channel state of current slot, which is only known to itself. Depending on the capture model and the received power of that user's signal, a transmitted packet may be captured even if multiple users are transmitting at the same slot. If the objective of each user in the system is to find a power allocation strategy that maximizes its probability of getting captured based its average power constraint, we have a power allocation game that resembles the all-pay auction. Comparing with the all-pay auction, the average power constraint in the power allocation game corresponds to the average money constraint and transmission power corresponds to money. Both power and money is taken away once a bidding or a transmission is taken place. In this uplink scenario, using the technique to solve for Nash equilibrium in the all-pay auction, we get a similar Nash equilibrium strategy in the uplink game.

The game theoretical formulation of the uplink power allocation problem stems from the desire for a distributive algorithm in a wireless uplink. Due to the variation of channel quality in a fading channel, one can exploit the channel variation opportunistically by allowing the user with best channel condition to transmit, which require the presence of a centralized scheduler who knows each user's channel condition. As the number of users in the network increases, the delay in conveying user's channel conditions to the scheduler will limit the system's performance. Hence, a distributive multi-access scheme with no centralized scheduler becomes an attractive alternative. However, in a distributive environment, users may want to change their communication protocols in order to improve their own performance, making it impossible to ensure a particular algorithm will be adopted by all users in the network. Rather than following some mandated algorithm, in this paper users are assumed to act selfishly (i.e., choose their own power allocation strategies) to further their own individual interests.

With each user wants to maximize its own expected throughput, we obtain a Nash equilibrium power allocation strategy which determines the optimal transmission power control strategy for each user. The obtained optimal power control strategy specifies how much power a user needs to use to maximize its own throughput for any possible channel state. Users get different average throughput based on their average power constraint. Hence, this transmission scheme can be viewed as mechanism for providing quality of service (QoS) differentiation; whereby users are given different energy for transmission. The obtained Nash equilibrium power allocation strategy is unique under certain capture rule. When all users have the same energy constraint, we obtained a symmetric Nash equilibrium.

Due to the selfish behavior of individual users, the overall system throughput will be less than that of a system where users employ the same mandated algorithm. This loss in efficiency is also quantified. In the multiple users' case, as the number of user in the system increases, the symmetric Nash equilibrium strategy approaches the optimal algorithm specified by a system designer (i.e., algorithm that results in the largest total system throughput). In this case, there is no loss of efficiency when users employ the symmetric Nash equilibrium.

Game theoretical approaches to resource allocation problems have been explored by many researchers recently (e.g., [2][19]). In [2], the authors consider a resource allocation problem for a wireless channel, without fading, where users have different utility values for the channel. They show the existence of an equilibrium pricing scheme where the transmitter attempts to maximize its revenue and the users attempt to maximize their individual utilities. In [19], the authors explore the properties of a congestion game where users of a congested resource anticipate the effect of their action on the price of the resource. Again, the work of [19] focuses on a wireline channel without the notion of wireless fading. Our work attempts to apply game theory to the allocation of a wireless fading channel. In particular, we show that auction algorithms are well suited for achieving fair allocation in this environment. Other papers dealing with the application of game theory to resource allocation problems include [3][23][24].

This paper is organized as follows. In Section 2, we describe the donwlink communication system and the Nash equilibrium bidding strategy. Section 2.1 presents the problem formulation for the downlink case. In Section 2.2, the unique Nash equilibrium strategy pair and the resulting throughput for each user are provided for the case that each user can use only one bidding function. In Section 2.3, we show the unique Nash equilibrium strategy pair for the case that each user can use multiple bidding functions. In Section 2.4, we compare the throughput results of the Nash equilibrium strategy with two other centralized allocation algorithms. In Section 2.5, an estimation algorithm that enables the users to estimate the amount of money possessed by their opponent is developed. Section 3 presents the Nash equilibrium power allocation function for a uplink random access system. Section 3.1 describe the uplink communication scenario. In Section 3.2, the Nash equilibrium power allocation strategy is obtained for the two users case. In Section 3.3, we present a symmetric Nash equilibrium power allocation function for multiple users with the same average power constraint. Finally, Section 4 concludes the paper.

2. Downlink Transmission.

2.1. Downlink Problem Formulation. We consider a communication environment with a single transmitter sending data to two users over two different fading channels. We assume that there is always data to be sent to the users. Time is assumed to be discrete, and the channel state for a given channel changes according to a known probabilistic model independently over time. The two channels are also assumed to be independent of each other. The transmitter can transmit to only one user during a particular slot with a constant power P. The channel fade state thus determines the throughput that can be obtained.

For a given power level, we assume for simplicity that the throughput is a linear function of the channel state. This can be justified by the Shannon capacity at low signal-to-noise ratio, or by using a fixed modulation scheme [1]. For general throughput function, the method used in this paper applies as well. Let X_i be a random variable denoting the channel state for the channel between the transmitter and user i, $i = 1, 2$. When transmitting to user i, the throughput will then be $P \cdot X_i$. Without loss of generality, we assume $P = 1$ throughout this paper.

We now describe the all-pay auction rule used in this paper. Let α and β be the *average* amount of money available to user 1 and user 2 respectively during each time slot. We assume that the values of α and β are known to both users. Both users know the distribution of X_1 and X_2. We also assume that the exact value of the channel state X_i is revealed to user i only at the beginning of each time slot. During each time slot, the following actions take place:

1. Each user submits a bid according to the channel condition revealed to it.
2. The transmitter chooses the one with higher bid to transmit.
3. Once a bid is submitted by the user, it is taken by the transmitter regardless of whether the user gets the slot or not, i.e., no refund for the one who loses the bid.

The formulation of our auction is different from the type of auction used in economic theory in several ways. First, we look at a case where the number of object in the auction goes to infinity. While in the current auction research, the number of object is finite [20][21][22]. Second, in our auction formulation, the money used for bidding does not have a direct connection with the value of the time slot. Money is merely a tool for users to compete for time slots, and it has no value after the auction. Therefore, it is desirable for each user to spend all of its money. However, in auction theory, an object's value is measured in the same unit as the money used in the bidding process, hence their objective is to maximize the difference between the object's value and its cost. Lastly, in our formulation, the valuation of each commodity (time-slot) changes due to the fading channel model; a notion that is not common in economic theory.

Besides the all-pay auction, *first-price* auction and *second-price* auction are two other commonly used auction mechanisms. In the first-price auction, each bidder submits a single bid without seeing the others' bids, and the object is sold to the bidder who makes the highest bid. The winner pays its bid. In the second price auction, each user independently submits a single bid without seeing the others' bids, and the object is sold to the bidder who makes the highest bid. However, the price it pays is the *second*-highest bidder's bid [20]. We choose to use the all-pay auction in this paper to illustrate the auction approach to resource allocation in wireless networks. We believe that other auction mechanisms can be similarly applied and their application to the wireless channel allocation problem is a direction for future research.

The objective for each user is to design a bidding strategy, which specifies how a user will act in every possible distinguishable circumstance, to maximize its *own* expected throughput per time slot subject to the expected or average money constraint. Once a user, say user 1, chooses a function, say $f_1^{(i)}$, for its strategy in the ith slot, it bids an amount of money equal to $f_1^{(i)}(x)$ when it sees its channel condition in the ith slot is $X_1 = x$.

Formally, let F_1 and F_2 be the set of continuous and bounded real-valued functions with finite first and second derivative over the support of X_1 and X_2 respectively. Then, the strategy space for user 1, say S_1, and user 2, say S_2, are defined as follows:

$$\begin{aligned} S_1 &= \Big\{ f_1^{(1)}, \cdots, f_1^{(n)} \in F_1 \;\Big|\; \frac{1}{n} \sum_{i=1}^{n} E[f_1^{(i)}(X_1)] = \alpha \Big\} \\ S_2 &= \Big\{ f_2^{(1)}, \cdots, f_2^{(n)} \in F_2 \;\Big|\; \frac{1}{n} \sum_{i=1}^{n} E[f_2^{(i)}(X_2)] = \beta \Big\} \end{aligned} \tag{2.1}$$

For each user, a strategy is a sequence of bidding functions $f^{(1)}, \cdots, f^{(n)}$. Without loss of generality, we restrict each user to have n different bidding functions, where n can be chosen as an arbitrarily large number. Note that users choose a strategy for a block of n time slots instead of just for a single time slot, one bidding function for each slot. In order to maximize the overall throughput (over infinite horizon), each user chooses bidding functions to maximize the expected total throughput over this block of n slots. The term $E[f_1^{(i)}(X_1)]$ denotes the expected amount of money spent by user 1 if it uses bidding function $f_1^{(i)}$ for the ith slot in the block.

We first consider a special class of strategies in which each user can use only a single bidding function. More specifically, by setting $f_1 = f_1^{(1)} = \cdots = f_1^{(n)}$ and $f_2 = f_2^{(1)} = \cdots = f_2^{(n)}$, we have the following:

$$\begin{aligned} \bar{S}_1 &= \Big\{ f_1 \in F_1 \;\Big|\; E[f_1(X_1)] = \alpha \Big\} \\ \bar{S}_2 &= \Big\{ f_2 \in F_2 \;\Big|\; E[f_2(X_2)] = \beta \Big\} \end{aligned} \tag{2.2}$$

By considering first the set of strategies in $\bar{S}_1$ and $\bar{S}_2$, we are able to find the Nash equilibrium strategy pair within the set S_1 and S_2.

Given a strategy pair (f_1, f_2), where $f_1 \in \bar{S}_1$ and $f_2 \in \bar{S}_2$, the expected throughput or payoff function for user 1 is defined as the following assuming the constant power $P = 1$:

$$G_1(\alpha, \beta) = E_{X_1,X_2}[X_1 \cdot 1_{f_1(X_1) \geq f_2(X_2)}] \tag{2.3}$$

where

$$1_{f_1(X_1) \geq f_2(X_2)} = \begin{cases} 1 & \text{if} \quad f_1(X_1) \geq f_2(X_2) \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the throughput function for user 2 assuming $P = 1$:

$$G_2(\alpha, \beta) = E_{X_1,X_2}[X_2 \cdot 1_{f_2(X_2) > f_1(X_1)}] \tag{2.4}$$

Throughout this paper, for simplicity, we let the channel state X_i be uniformly distributed over $[0, 1]$. However, our approach can be extended to the case where the channel state has a general distribution. Due to space limitations, we omit the more complex analysis for general channel state distribution.

2.2. Unique Nash equilibrium strategy with a single bidding function. We present in this section a unique Nash equilibrium strategy pair (f_1^*, f_2^*). A strategy pair (f_1^*, f_2^*) is said to be in Nash equilibrium if f_1^* is the best response for user 1 to user 2's strategy f_2^*, and f_2^* is the best response for user 2 to user 1's strategy f_1^*. We consider here the case where both users choose their strategies from the strategy space $\bar{S}_1$ and $\bar{S}_2$ (i.e., the single bidding function strategy) and the value of α and β are known to both users.

To get the Nash equilibrium strategy pair, we first argue that an equilibrium bidding function must be nondecreasing. To see this, consider an arbitrary bidding function f such that $f(a) > f(b)$ for some $a < b$. If user 1 chooses f as its bidding function, user 1 will be better off if it bids $f(b)$ when the channel state is a and $f(a)$ when the channel state is b. This way, its odds of winning the slot when the channel state is b, which is more valuable to it, will be higher than before, and it has an incentive to change its strategy (i.e., f is not an equilibrium strategy). Hence, we conclude that, for each user, an equilibrium bidding function must be nondecreasing.

We further restrict users' bidding functions to be *strictly* increasing for technical reason which will be explained later. There is no loss of generality in this assumption because any continuous, bounded, nondecreasing function can be approximated by a *strictly* increasing function arbitrarily closely.

Next, we show some useful properties associated with the equilibrium strategy pair (f_1^*, f_2^*).

Lemma 1. *If* (f_1^*, f_2^*) *is a Nash equilibrium strategy pair,* $f_1^*(1) = f_2^*(1)$.

Proof. Suppose $f_1^*(1) \neq f_2^*(1)$. Without loss of generality, let assume that $f_1^*(1) > f_2^*(1)$. Since both f_1^* and f_2^* are continuous, there exists $\delta > 0$ such that $f_1^*(x) > f_2^*(1) + \frac{f_1^*(1) - f_2^*(1)}{2}$ $\forall x \in [1-\delta, 1]$. User 1 can devise a new bidding strategy, say $\bar{f}_1$, by moving a small amount of money, say $\delta \cdot \frac{f_1^*(1) - f_2^*(1)}{2}$, away from the interval $[1-\delta, 1]$ to some other interval, thus resulting in an increase in user 1's throughput. Therefore, when $f_1^*(1) > f_2^*(1)$, the bidding strategy pair (f_1^*, f_2^*) cannot be in equilibrium since the strategy pair $(\bar{f}_1, f_2^*)$ gives a higher throughput for user 1. Similar result holds for the case $f_2^*(1) > f_1^*(1)$. Thus, we must have $f_1^*(1) = f_2^*(1)$ if (f_1^*, f_2^*) is an equilibrium strategy pair. □

We have just established that $f_1^*(1) = f_2^*(1)$ is a necessary condition for (f_1^*, f_2^*) to be an equilibrium strategy pair. We also find that $f_1^*(0) = f_2^*(0) = 0$ since it does not make sense to bid for a slot with zero channel state. Thus, from now on, to find the Nash equilibrium strategy pair (f_1^*, f_2^*), we will consider only the function pair $f_1 \in \bar{S}_1$ and $f_2 \in \bar{S}_2$ that are strictly increasing and satisfying the above two boundary conditions (i.e., $f_1(1) = f_2(1)$ and $f_1(0) = f_2(0) = 0$).

These two boundary conditions, together with strictly increasing property of $f_1 \in \bar{S}_1$ and $f_2 \in \bar{S}_2$, make the inverse of f_1 and f_2 well defined. Thus, we are able to define the following terms. With user 2's strategy f_2 fixed, let $g_{f_2}^{(1)} : (x_1, b) \to \mathcal{R}$ denote user 1's expected throughput of a slot conditioning on the following events:

- User 1's channel state is $X_1 = x_1$.
- User 1's bid is b.

Specifically, we can the write the equation:

$$g_{f_2}^{(1)}(x_1, b) = x_1 \cdot P(f_2(X_2) \leq b) \tag{2.5}$$

where $P(f_2(X_2) \leq b)$ is the probability that user 1 wins the time slot.

Consequently, using a strategy f_1, user 1's throughput is given by:

$$G_1(\alpha,\beta) = \int_0^1 g_{f_2}^{(1)}(x_1, f_1(x_1)) \cdot p_{X_1}(x_1)\, dx_1 = \int_0^1 g_{f_2}^{(1)}(x_1, f_1(x_1))\, dx_1. \tag{2.6}$$

where the last equality results from the uniform distribution assumption.

With user 1's strategy f_1 fixed, similar terms for user 2 can be defined.

$$g_{f_1}^{(2)}(x_2, b) = x_2 \cdot P(f_1(X_1) \le b)$$

Then, user 2's throughput is given by:

$$G_2(\alpha,\beta) = \int_0^1 g_{f_1}^{(2)}(x_2, f_2(x_2)) \cdot p_{X_2}(x_2)\, dx_2 = \int_0^1 g_{f_1}^{(2)}(x_2, f_2(x_2))\, dx_2. \tag{2.7}$$

Due to the uniformly distributed channel state, $P(f_2(X_2) \le b)$ is given by

$$P(f_2(X_2) \le b) = P(X_2 \le f_2^{-1}(b)) = f_2^{-1}(b)$$

where f_2^{-1} is well defined. Thus, we can rewrite Eq. (3.4) as

$$g_{f_2}^{(1)}(x_1, b) = x_1 \cdot f_2^{-1}(b).$$

Hence we have,

$$G_1(\alpha,\beta) = \int_0^1 x_1 \cdot f_2^{-1}(f_1(x_1))\, dx_1 \tag{2.8}$$

$$G_2(\alpha,\beta) = \int_0^1 x_2 \cdot f_1^{-1}(f_2(x_2))\, dx_2 \tag{2.9}$$

The following lemma gives a necessary and sufficient condition of a Nash equilibrium strategy pair. For convenience, we denote $\frac{\partial g_{f_2}^{(1)}(x_1,b)}{\partial b}\big|_{b=b^*}$ (i.e., the marginal gain at $b=b^*$) as $Dg_{f_2}^{(1)}(x_1,b^*)$.

Lemma 2. *A strategy pair (f_1^*, f_2^*) is a Nash equilibrium strategy pair if and only if $Dg_{f_2^*}^{(1)}(x_1, f_1^*(x_1)) = c_1$ and $Dg_{f_1^*}^{(2)}(x_2, f_2^*(x_2)) = c_2$, for some constants c_1 and c_2, for all $x_1 \in [0,1]$ and all $x_2 \in [0,1]$.*

To understand the lemma intuitively, suppose there exists $x \ne \tilde{x}$ such that $Dg_{f_2^*}^{(1)}(x, f_1^*(x)) > Dg_{f_2^*}^{(1)}(\tilde{x}, f_1^*(\tilde{x}))$. Reducing the bid at $\tilde{x}$ to $f_1^*(\tilde{x}) - \delta$ and increasing the bid at x to $f_1^*(x) + \delta$ will result in an increase in the throughput by $(Dg_{f_2^*}^{(1)}(x, f_1^*(x)) - Dg_{f_2^*}^{(1)}(\tilde{x}, f_1^*(\tilde{x}))) \cdot \delta$. Thus, user 1 has an incentive to change its bidding function, and (f_1^*, f_2^*) cannot be a Nash equilibrium strategy pair in this case.

Proof. The complete proof is given in the Appendix. □

With Lemma 2, we are able to find the unique Nash equilibrium strategy pair. The exact form of the equilibrium bidding strategies are presented in the following Theorem.

Theorem 1. *Under the assumption of a single bidding function, the following is a unique Nash equilibrium strategy pair for the auction:*

$$f_1^*(x) = c \cdot x^{\gamma+1} \tag{2.10}$$

$$f_2^*(x) = c \cdot x^{\frac{1}{\gamma}+1} \tag{2.11}$$

where the constant γ *and* c *are chosen such that*

$$\int_0^1 c \cdot x^{\gamma+1}\, dx = \alpha \tag{2.12}$$

$$\int_0^1 c \cdot x^{\frac{1}{\gamma}+1}\, dx = \beta \tag{2.13}$$

Equations (3.11) and (3.12) impose the average money constraints. Fig. 1 shows an example of the Nash equilibrium bidding strategy pair when $\alpha = 1$ and $\beta = 2$. Since user 1 has less money than user 2, user 1 concentrates its bidding on time slots with very good channel state.

Proof. We show here that $f_1^*(x) = c \cdot x^{\gamma+1}$ and $f_2^*(x) = c \cdot x^{\frac{1}{\gamma}+1}$ is indeed a Nash equilibrium strategy pair by using the sufficiency condition of Lemma 2, and we leave the uniqueness part to the appendix. It is easy to check that both the condition $f_1^*(1) = f_2^*(1)$ and $f_1^*(0) = f_2^*(0)$ are satisfied. Since both functions are strictly increasing, we can write $g^{(1)}_{f_2^*}(x, b) = x \cdot f_2^{*-1}(b)$ and $g^{(2)}_{f_1^*}(x, b) = x \cdot f_1^{*-1}(b)$. Also, since both f_1^* and f_2^* are differentiable, we have $g^{(1)}_{f_2^*}(x, b)$ and $g^{(2)}_{f_1^*}(x, b)$ both differentiable with respect to b. Therefore,

$$\frac{\partial g^{(1)}_{f_2^*}(x, b)}{\partial b}\bigg|_{b=f_1^*(x)} = \frac{x}{f_2^{*\prime}(f_2^{*-1}(f_1^*(x)))} = \frac{x}{f_2^{*\prime}(x^\gamma)} = \frac{\gamma}{c(1+\gamma)}.$$

Similarly,

$$\frac{\partial g^{(2)}_{f_1^*}(x, b)}{\partial b}\bigg|_{b=f_2^*(x)} = \frac{x}{f_1^{*\prime}(f_1^{*-1}(f_2^*(x)))} = \frac{x}{f_1^{*\prime}(x^{1/\gamma})} = \frac{1}{c(1+\gamma)}.$$

From Lemma 2, we see that (f_1^*, f_2^*) is indeed a Nash equilibrium strategy pair because both $Dg^{(1)}_{f_2^*}(x, f_1^*(x))$ and $Dg^{(2)}_{f_1^*}(x, f_2^*(x))$ are constants.

The proof of uniqueness of (f_1^*, f_2^*) is given in the appendix. □

Fig. 2 shows the resulting allocation scheme when both users employ the Nash equilibrium strategy shown in Fig. 1. Above the curve, time slots will be allocated to user 2 since user 2's bid is higher than user 1's in this region. Similarly, user 1 gets the slots below the curve. Here, user 2 is allocated more slots than user 1 since it has more money.

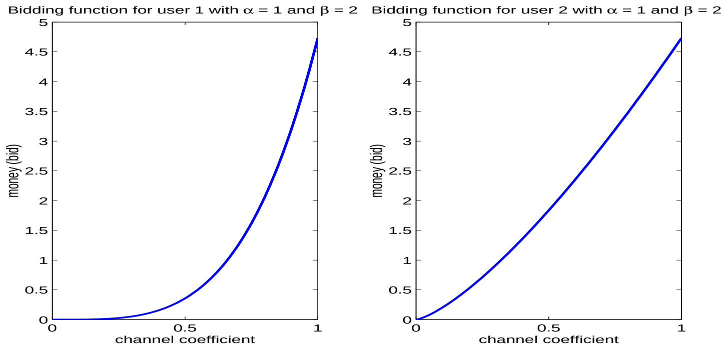

FIG. 1. *An example of Nash equilibrium strategy pair for* $\alpha = 1$ *and* $\beta = 2$.

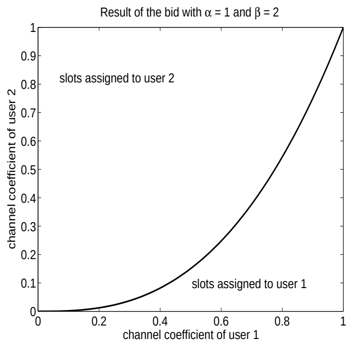

FIG. 2. *Allocation scheme from Nash equilibrium strategy pair for* $\alpha = 1$ *and* $\beta = 2$.

If both players use Nash equilibrium strategies, the expected throughput obtained are given by:

$$G_1(\alpha, \beta) = \frac{\alpha}{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \alpha\beta}} \tag{2.14}$$

$$G_2(\alpha, \beta) = \frac{\beta}{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \alpha\beta}} \tag{2.15}$$

As can be seen, the ratio of the throughput obtained $\frac{G_1(\alpha,\beta)}{G_2(\alpha,\beta)}$ is equal to $\frac{\alpha}{\beta}$ which is the ratio of the money each user had initially. Thus, the Nash equilibrium strategy pair provides an allocation scheme that is fair in the sense that the price per unit of throughput is the same for both users.

2.3. Unique Nash Equilibrium Strategy with multiple bidding functions. In the previous section, we restricted the strategy space of each user to be a single bidding function (i.e., $\bar{S}_1$ and $\bar{S}_2$) instead of a sequence of bidding functions (i.e., S_1 and S_2). However, the money constraint imposed upon each user is a long term average money constraint. A natural question to ask is the following: Is it profitable for an individual user to change its bidding functions over time while satisfying the long term average money constraint? Therefore, in this section, we allow the users to use a strategy within a broader class of strategy space, S_1 and S_2, and explore whether there is an incentive for a user to do so (i.e., whether there exists a Nash equilibrium strategy so that it can increase its throughput).

To choose a strategy (i.e., a sequence of bidding functions) from the strategy space S_1 or S_2, a user encounters two problems. First, it must decide how to allocate its money among these n bidding functions so that the average money constraint is still satisfied. Second, once the money allocated to the ith bidding function is specified, a user has to choose a bidding function for the ith slot. The second problem is already solved in the previous section (see Theorem 1). In this section, we will focus on the first problem that a user encounters, specifically, the problem of how to allocate money between the bidding functions while satisfying the following condition: The total expected amount of money for the *sequence* of n bidding functions is $n \cdot \alpha$ for user 1 and $n \cdot \beta$ for user 2.

More precisely, the strategy space or possible actions that can be taken by users are the following:

$$\hat{S}_1 = \{\alpha_1, \cdots, \alpha_n \mid \alpha_1 + \cdots + \alpha_n = n \cdot \alpha\}$$
$$\hat{S}_2 = \{\beta_1, \cdots, \beta_n \mid \beta_1 + \cdots + \beta_n = n \cdot \beta\}$$

The objective of each user is still to maximize its own throughput. When user 1 and user 2 allocate α_i and β_i for their ith bidding function which is given in Theorem 1, the payoff functions are $G_1(\alpha_i, \beta_i)$ for user 1 and $G_2(\alpha_i, \beta_i)$ for user 2.

The following lemma gives us a Nash equilibrium strategy pair for the auction game described in this section.

Lemma 3. *Given that user 2's strategy is to allocate its money evenly among its bidding functions (i.e., $\beta_i = \beta, i = 1 \cdots n$), user 1's best response is to allocate its money evenly as well (i.e., $\alpha_i = \alpha, i = 1 \cdots n$); and vice versa. Therefore, a Nash equilibrium strategy pair for this auction is for both users to allocate their money evenly.*

Proof. Without loss of generality, we consider the case that $n = 2$ where each user's strategy can consist of two different bidding functions. Suppose that user 2 allocates β for both bidding functions $f_2^{(1)}$ and $f_2^{(2)}$, and user 1 allocates α_1 for bidding function $f_1^{(1)}$ and α_2 for bidding function $f_1^{(2)}$ where $\alpha_1 + \alpha_2 = 2\alpha$ and $\alpha_1 \neq \alpha_2$. We now show that the throughput for user 1, $G_1(\alpha_1, \beta) + G_1(\alpha_2, \beta)$, is maximized when $\alpha_1 = \alpha_2 = \alpha$. Consider the function $G_1(\alpha_1, \beta)$ with β fixed. The equation

$$G_1(\alpha_1, \beta) = \frac{\alpha_1}{\alpha_1 + \beta + \sqrt{(\alpha_1 - \beta)^2 + \alpha_1 \beta}}$$

becomes

$$F(t) = \frac{t}{1 + t + \sqrt{(1-t)^2 + t}}$$

where $t = \frac{\alpha_1}{\beta}$. $F(t)$ is concave for $t \geq 0$. Thus, we have $G_1(\alpha_1, \beta) + G_1(\alpha_2, \beta)$ maximized when $\alpha_1 = \alpha_2 = \alpha$. □

We have already obtained a Nash equilibrium strategy pair from the above Lemma. The following theorem states that this Nash equilibrium strategy pair is in fact unique within the strategy space considered.

Theorem 2. *For the auction in this section, a unique Nash equilibrium strategy for both users is to allocate their money evenly among the bidding functions.*

Proof. The complete proof is in the Appendix. □

In this section, users are given more freedom in choosing their strategies (i.e., they can choose n different bidding functions). However, as Theorem 2 shows, the unique Nash equilibrium strategy pair is for each user to use a single bidding function from its strategy space. Thus, the throughput result obtained in this broader strategy space-S_1 and S_2–is the same as the throughput result from previous section. Therefore, there is no incentive for a user to use different bidding functions.

2.4. Comparison with Other Allocation Schemes. To this end, we have a unique Nash equilibrium strategy pair and the resulting throughput when both players choose to use the Nash equilibrium strategy. Inevitably, due to the fairness constraint, total system throughput will decrease as compared to the maximum throughput attainable without any fairness constraint. Hence we would like to compare the total throughput of the Nash equilibrium strategy to that of an unconstrained strategy. We address this question by first considering an allocation scheme that maximizes total throughput subject to no constraint. Then, we investigate the throughput of another centralized allocation scheme that maximize the total throughput subject to the constraint that the resulting throughput of individual user is kept at certain ratio.

2.4.1. Maximizing Throughput with No Constraint. To maximize throughput without any constraints, the transmitter sends data to the user with a better channel state during each time slot. Then the expected throughput is $E[\max\{X_1, X_2\}]$. Since X_1 and X_2 are independent uniformly distributed in $[0, 1]$, we have $E[\max\{X_1, X_2\}] = \frac{2}{3}$. Using the Nash equilibrium playing strategy, the total expected system throughput, $G_1(\alpha, \beta) + G_2(\alpha, \beta)$, is $\frac{1}{2}$ in the worst case (i.e., one users gets all of the time slots while the other user is starving). *Thus, the channel allocation scheme proposed in this paper can achieve at least* 75 *percent of the maximum attainable throughput.* This gives us a lower bound of the throughput performance of the allocation scheme derived from the Nash equilibrium pair.

2.4.2. Maximizing Throughput with A Constant Throughput Ratio Constraint. Now, we investigate an allocation scheme with a fairness constraint that requires the resulting throughput of the users to be kept at a constant ratio. Specifically, let G_1 and G_2 denote the expected throughput for user 1 and user 2 respectively. We have the following optimization problem:

$$\max \quad G_1 + G_2 \quad \text{subj.} \quad \frac{G_1}{G_2} = a \tag{2.16}$$

where a is a positive real number.

The resulting optimal allocation scheme for the above problem is of the form shown in Fig. 3. The space spanned by X_1 and X_2 is divided into two regions by the separation line $X_2 = c \cdot X_1$, where c is some positive real number. Above the line (i.e., $X_2 > c \cdot X_1$), the transmitter will assign the slot to user 2. Below the line (i.e., $X_2 < c \cdot X_1$), the transmitter will assign the slot to user 1.

To prove the above, we use a method that is similar to the one in [9]. Specifically, let $A : (X_1, X_2) \to \{1, 2\}$ be an allocation scheme that maps a slot, in which channel states are X_1 and X_2 to either user 1 or user 2. By using an allocation scheme A, the resulting throughput for user 1 and user 2 are $G_1^A = E[X_1 \cdot 1_{A(X_1,X_2)=1}]$ and $G_2^A = E[X_2 \cdot 1_{A(X_1,X_2)=2}]$ respectively. Now, we define an allocation scheme as follows:

$$A^*(X_1, X_2) = \begin{cases} 1 & \text{if} \quad X_1(1+\lambda^*) \geq X_2(1 - a \cdot \lambda^*) \\ 2 & \text{otherwise} \end{cases}$$

where λ^* is chosen such that $G_1^{A^*}/G_2^{A^*} = a$ is satisfied. It is straightforward to verify that such λ^* exists.

Consider an arbitrary allocation scheme A that satisfies $G_1^A/G_2^A = a$. We have

$$\begin{aligned}
&E[X_1 \cdot 1_{A(X_1,X_2)=1}] + E[X_2 \cdot 1_{A(X_1,X_2)=2}] \\
&= E[X_1 \cdot 1_{A(X_1,X_2)=1}] + E[X_2 \cdot 1_{A(X_1,X_2)=2}] \\
&\quad + \lambda^*(E[X_1 \cdot 1_{A(X_1,X_2)=1}] - aE[X_2 \cdot 1_{A(X_1,X_2)=2}]) \\
&= E[(X_1 + \lambda^* X_1) \cdot 1_{A(X_1,X_2)=1}] + E[(X_2 - a\lambda^* X_2) \cdot 1_{A(X_1,X_2)=2}] \\
&\leq E[(X_1 + \lambda^* X_1) \cdot 1_{A^*(X_1,X_2)=1}] + E[(X_2 - a\lambda^* X_2) \cdot 1_{A^*(X_1,X_2)=2}] \\
&= E[X_1 \cdot 1_{A^*(X_1,X_2)=1}] + E[X_2 \cdot 1_{A^*(X_1,X_2)=2}] \\
&\quad + \lambda^*(E[X_1 \cdot 1_{A^*(X_1,X_2)=1}] - aE[X_2 \cdot 1_{A^*(X_1,X_2)=2}]) \\
&= E[X_1 \cdot 1_{A^*(X_1,X_2)=1}] + E[X_2 \cdot 1_{A^*(X_1,X_2)=2}]
\end{aligned} \tag{2.17}$$

The inequality in the middle is from the definition of A^*. Specifically, if we were asked to choose an allocation scheme A to maximize $E[(X_1 + \lambda^* X_1) \cdot 1_{A(X_1,X_2)=1}] + E[(X_2 - a\lambda^* X_2) \cdot 1_{A(X_1,X_2)=2}]$. Then, A^* will be

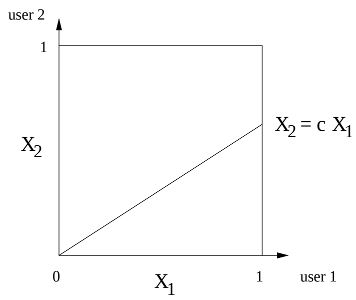

FIG. 3. *The optimal allocation scheme to achieve constant throughput ratio fairness.*

an optimal scheme from its definition. Thus, we are able to show that $A^*(X_1, X_2)$ is an optimal solution to the optimization problem in (2.16).

To find the slope c in Fig.3, we first write the throughput for each user:

$$G_1^{A^*} = \int_0^1 \int_0^{cx_1} x_1\, dx_1\, dx_2 = \frac{1}{3}c \tag{2.18}$$

and

$$G_2^{A^*} = \int_0^c \int_0^{\frac{1}{c}x_2} x_2\, dx_1 dx_2 + \int_c^1 x_2\, dx_2 = \frac{1}{2} - \frac{1}{6}c^2 \tag{2.19}$$

Since $G_1^A/G_2^A = a$, we get $c = \frac{-1+\sqrt{1+3a^2}}{a}$.

Using the Nash equilibrium strategy pair, the ratio of the resulting throughput pair $\frac{G_1(\alpha,\beta)}{G_2(\alpha,\beta)}$ is the same as the ratio of money individual user possess ($\frac{\alpha}{\beta}$). For the optimization problem described in (2.16), by setting $a = \alpha/\beta$, we compare the resulting throughput with the throughput obtained when both users employ the Nash equilibrium strategy. Fig. **??** and Fig. **??** show the comparison. For both users, the Nash equilibrium throughput result is very close to the throughput obtained by solving the constrained optimization problem (within 97 percent to be precise).

3. Uplink Transmission.

3.1. Uplink Problem Formulation. The uplink communication environment that we consider here consists of multiple users who are sending data to a single base station or satellite over multiple fading channels. We assume that each user always has data to be sent to the base station. Time is assumed to be discrete, and the channel state for a given user changes according to a known probabilistic model independently over time. The channels between the users and the base station are assumed to be independent of each other. Let X_i be a random variable denoting the channel state for the channel between user i and the base station.

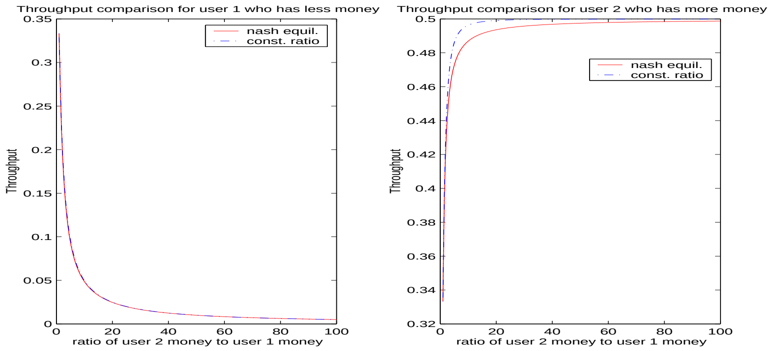

FIG. 4. *Throughput result comparison for both users.*

When multiple users are transmitting during the same time slot, it is still possible for the receiver to capture one (or more) user's data. The capture model can be described as a mapping from the received power of the transmitting users to the set $\{1, \cdots, n, 0\}$, where 0 indicates no packet is successfully received. In this paper, we are going to investigate two capture models which will be presented in the later sections.

We assume that each individual user is energy constrained. Specifically, each user i has an *average* amount of energy e_i available to itself during each time slot. We assume that the e_i values are known to all users, and that users know the distribution of X_i's. However, the exact value of the channel state X_i is known to user i only at the beginning of each time slot.

With a given capture model and the energy constraint, the objective for each user is to design a power allocation strategy to maximize its *own* expected throughput (or probability of success) per time slot subject to the expected or average power constraint. The power allocation strategy will specify how a user will allocate power in every time slot upon observing its channel state. Under power allocation strategy $g_i(\cdot)$, user i transmits a packet with power equal to $g_i(x)$ when it sees its channel condition in this time slot is $X_i = x$. The received power at the base station is denoted as $f_i(x) = x \cdot g_i(x)$.

Formally, let F_i be the set of continuous and bounded real-valued functions with finite first and second derivative over the support of X_i. Then, the strategy space for user i (the set of all possible power allocation schemes), say S_i, is defined as follows:

$$S_i = \left\{ g_i \in F_i \;\middle|\; E[g_i(X_i)] \leq e_i \right\} \tag{3.1}$$

3.2. Two Users Case. We start by investigating users' strategies in a communication system consisting of exactly two users and one base station. The analytical method used in this section will help us in obtaining equilibrium power allocation scheme in the multiple users case. We begin our analysis with the assumption that channel state X_i is uniformly distributed over $[0, 1]$ for all i. The Nash equilibrium power allocation strategy with general channel state distribution is presented in the subsequent section.

Suppose user 1 and user 2 choose their power allocation strategies to be g_1 and g_2 respectively. Given a time slot with channel state realization (x_1, x_2), user 1 and user 2 will transmit their packets using power levels $g_1(x_1)$ and $g_2(x_2)$ respectively. The corresponding received power at the base station are $f_1(x_1) = x_1 \cdot g_1(x_1)$ and $f_2(x_2) = x_2 \cdot g_2(x_2)$. As in [12] and [13], the capture model used in this section is the following: if $[x_1 \cdot g_1(x_1)]/[x_2 \cdot g_2(x_2)] \geq K$ where $K \geq 1$, user 1's packet will be captured. Likewise, user 2's packet will be captured if $[x_2 \cdot g_2(x_2)]/[x_1 \cdot g_1(x_1)] \geq K$. Thus, given a power allocation strategy pair (g_1, g_2), where $g_1 \in S_1$ and $g_2 \in S_2$, the expected throughput for user 1 is defined as the following:

$$G_1(e_1, e_2) = E_{X_1,X_2}[1_{f_1(X_1) \geq K \cdot f_2(X_2)}] \tag{3.2}$$

where

$$1_{f_1(X_1) \geq f_2(X_2)} = \begin{cases} 1 & \text{if} \quad f_1(X_1) \geq K \cdot f_2(X_2) \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the throughput function for user 2:

$$G_2(e_1, e_2) = E_{X_1,X_2}[1_{f_2(X_2) > K \cdot f_1(X_1)}] \tag{3.3}$$

3.2.1. Nash equilibrium strategy. In this part, we present a Nash equilibrium power allocation strategy pair (g_1^*, g_2^*). The derivation of the Nash equilibrium is similar to the derivation of the Nash equilibrium in the all-pay auction part. We consider here the case where both users choose their strategies from the strategy space S_1 and S_2 and the value of e_1 and e_2 are known to both users.

To get the Nash equilibrium strategy pair, we first argue that at equilibrium the received power function $f_i^*(x_i)$ must be strictly increasing in x_i.

Lemma 4. *Given a Nash equilibrium power allocation strategy pair (g_1^*, g_2^*) and its corresponding received power function (f_1^*, f_2^*), the received power function $f_1^*(x_1)$ must be strictly increasing in x_1. Similarly, $f_2^*(x_2)$ must be strictly increasing in x_2.*

Proof. For an arbitrary received power function f which is not strictly increasing, we can always find another received power function that will result in a larger throughput gain. To see this, consider time slots with channel state in the small intervals $(a - \delta, a + \delta)$ and $(b - \delta, b + \delta)$ where $a < b$. When δ is small, the received power function is close to $f(a)$ for time slots in the interval $(a - \delta, a + \delta)$. Likewise, the received power function is close to $f(b)$ for time slots in the interval $(b - \delta, b + \delta)$.

For received power function f such that $f(a) = a \cdot g(a) > f(b) = b \cdot g(b)$ for some $a < b$. The total amount of transmission power used in time slots with channel state in the two intervals is given by:

$$[g(a) + g(b)]2\delta = [\frac{f(a)}{a} + \frac{f(b)}{b}]2\delta.$$

Now, if user 1 employs a new power allocation strategy $\bar{g}$ such that $\bar{g}(b) = \frac{f(a)}{b}$ and $\bar{g}(a) = \frac{f(b)}{a}$, user 1 will achieve the same expected throughput as before. However, the amount of power used $[\bar{g}(b) + \bar{g}(a)]2\delta$ is less than $[g(a) + g(b)]2\delta$, and the extra power can be used to get higher throughput. Hence, both equilibrium received power function $f_1^*(x_1)$ and $f_2^*(x_2)$ must be strictly increasing in x_1 and x_2 respectively. □

With one user's power allocation strategy, say g_2, fixed, we now seek the optimal power allocation scheme for user 1. From Lemma 4, we see that the inverse of f_1 and f_2 are well defined. With user 2's strategy g_2 fixed, let $u_{g_2}^{(1)} : (x_1, b) \to \mathcal{R}$ denote user 1's expected throughput of a slot conditioning on the following events:

- User 1's channel state is $X_1 = x_1$.
- User 1's allocated power is b.

For convenience, we will drop the term g_2 in the expression $u_{g_2}^{(1)}(x_1, b)$, and simply write it as $u_1(x_1, b)$. Specifically, we can the write the equation:

$$u_1(x_1, b) = P(f_2(X_2) \cdot K \leq x_1 \cdot b) \tag{3.4}$$

where $P(f_2(X_2) \cdot K \leq x_1 \cdot b)$ is the probability that user 1's packet gets captured in a time slot. Consequently, using a strategy g_1, user 1's throughput is given by:

$$G_1(e_1, e_2) = \int_0^1 u_1(x_1, g_1(x_1)) \cdot p_{X_1}(x_1)\, dx_1 = \int_0^1 u_1(x_1, g_1(x_1))\, dx_1 \tag{3.5}$$

where the last equality results from the uniform distribution assumption.

With user 1's strategy g_1 fixed, similar terms for user 2 can be defined.

$$u_2(x_2, b) = u_{g_1}^{(2)}(x_2, b) = P(f_1(X_1) \cdot K \leq x_2 \cdot b)$$

Then, user 2's throughput is given by:

$$G_2(e_1, e_2) = \int_0^1 u_2(x_2, g_2(x_2)) \cdot p_{X_2}(x_2)\, dx_2 = \int_0^1 u_2(x_2, g_2(x_2))\, dx_2 \tag{3.6}$$

Due to the uniformly distributed channel state, $P(f_2(X_2) \cdot K \leq x_1 \cdot b)$ is given by

$$P(f_2(X_2) \cdot K \leq x_1 \cdot b) = P(X_2 \leq f_2^{-1}(\frac{1}{K}x_1 \cdot b)) = f_2^{-1}(\frac{1}{K}x_1 \cdot b)$$

where f_2^{-1} is well defined. Thus, we can rewrite Eq. (3.4) as

$$u_1(x_1, b) = f_2^{-1}(\frac{1}{K}x_1 \cdot b).$$

Hence we have,

$$G_1(e_1, e_2) = \int_0^1 f_2^{-1}(\frac{1}{K}x_1 \cdot g_1(x_1))\, dx_1 \tag{3.7}$$

$$G_2(e_1, e_2) = \int_0^1 f_1^{-1}(\frac{1}{K}x_2 \cdot g_2(x_2))\, dx_2 \tag{3.8}$$

We begin our analysis of the Nash equilibrium strategy pair by first considering the power allocation on the boundary points 0 and 1. For a pair of power allocation functions (g_1^*, g_2^*) to be a Nash equilibrium, it is straightforward to see that $g_1^*(0) = g_2^*(0) = 0$ since it does not make sense to allocate power for a slot with zero channel state. Likewise, we must have $g_1^*(1) \leq K \cdot g_2^*(1)$ and $g_2^*(1) \leq K \cdot g_1^*(1)$ since allocating power $g_1(1) = Kg_2(1)$ or $g_1(1) = Kg_2(1) + \epsilon$, where $\epsilon > 0$, will result in the same throughput for user 1. We call these properties the boundary conditions of a Nash equilibrium strategy pair.

With the boundary conditions satisfied, the following lemma gives a necessary and sufficient condition for a pair of power allocation strategies to be a Nash equilibrium strategy pair. For convenience, we denote the marginal gain for user 1 when $X_1 = x_1$ and the allocated power $b = b^*$ as

$$\frac{\partial u_1(x_1, b)}{\partial b}\Big|_{b=b^*} \triangleq Du_1(x_1, b^*).$$

Lemma 5. *Given a power allocation strategy pair (g_1^*, g_2^*) that satisfies the boundary conditions, (g_1^*, g_2^*) is a Nash equilibrium strategy pair if and only if $Du_1(x_1, g_1^*(x_1)) = c_1$ and $Du_2(x_2, g_2^*(x_2)) = c_2$, for some constants c_1 and c_2, for all $x_1 \in [0, 1]$ and all $x_2 \in [0, 1]$.*

Note that the above lemma does not depend on the assumption of the uniformly distributed channel state. Thus, it is quite general and will be used in the subsequent section where channel states are not uniformly distributed. The proof is similar to the proof of Lemma 2.

With Lemma 5, we are able to find the Nash equilibrium strategy pair. The exact form of the equilibrium power allocation strategies are presented in the following Theorem.

Theorem 3. *Given the average power constraint e_1 and e_2, the Nash equilibrium power allocation strategy pair has the following form:*

$$g_1^*(x) = c_1 \cdot x^{\gamma} \tag{3.9}$$

$$g_2^*(x) = c_2 \cdot x^{\frac{1}{\gamma}} \tag{3.10}$$

where the constants c_1, c_2 and γ are chosen such that

$$\int_0^1 c_1 \cdot x^\gamma \, dx = e_1 \tag{3.11}$$

$$\int_0^1 c_2 \cdot x^{\frac{1}{\gamma}} \, dx = e_2 \tag{3.12}$$

Equations (3.11) and (3.12) impose the average power constraints.

The proof of the above theorem is similar to the proof of Theorem 1. From the above theorem, we see that equations (3.9) and (3.10) specify the Nash equilibrium power allocation strategy pair. Since there are two equations with three unknowns, the resulting Nash equilibrium may not be unique in general. However, if a packet with stronger received power can always be captured (i.e., $K = 1$), the Nash equilibrium power allocation strategy is unique.

Corollary 1. *For $K = 1$, the unique Nash equilibrium power allocation pair has the following form:*

$$g_1^*(x) = c \cdot x^\gamma, \quad g_2^*(x) = c \cdot x^{\frac{1}{\gamma}} \tag{3.13}$$

where the constants c and γ are chosen such that the average power constraints are satisfied.

Fig. 5 shows an example of the Nash equilibrium power allocation strategy pair when $e_1 = 1$ and $e_2 = 2$. Since user 1 has less average power than user 2, user 1 concentrates its power on time slots with very good channel state. Fig. 6 shows the capture result when both users employ the

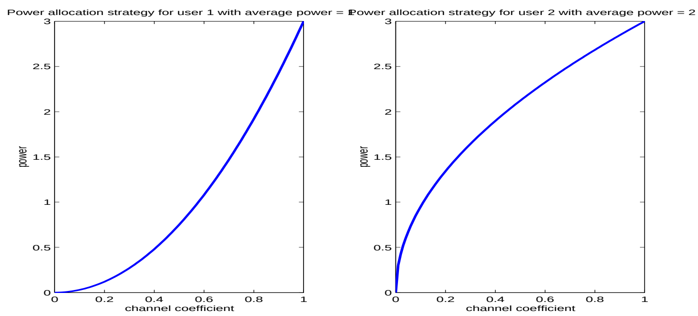

FIG. 5. *An example of Nash equilibrium strategy pair for $e_1 = 1$ and $e_2 = 2$.*

Nash equilibrium strategy shown in Fig. 5. For a time slot with channel state realization that fall into the region above the curve, user 2's packet will be successfully captured since user 2's received power is higher than that of user 1 in this region. Here, user 2 has more successful transmissions than user 1 since it has more power.

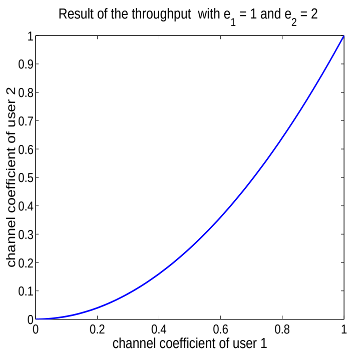

FIG. 6. *Results obtained when using the Nash equilibrium strategy pair for $e_1 = 1$ and $e_2 = 2$.*

3.2.2. General Channel State Distribution. In this section, we specify the conditions that a general channel state distribution has to satisfy in order for a Nash equilibrium strategy pair to exist.

From Lemma 4, one can see that f_1 and f_2 have to be increasing functions regardless of the distribution of the X_i's. Let $p_{X_i}(\cdot)$ denote the probability density function of X_i with the support over an interval starting at zero. Assuming $K = 1$, the probability that user 1's packet will be captured in a time slot with $X_1 = x_1$ and $g_1(x_1) = b$ can be written as the following:

$$\begin{aligned} u_1(x_1, b) &= P(f_2(X_2) \leq x_1 \cdot b) = P(X_2 \leq f_2^{-1}(x_1 \cdot b)) \\ &= \int_0^{f_2^{-1}(x_1 \cdot b)} p_{X_2}(x_2)\, dx_2 \end{aligned} \tag{3.14}$$

From the optimality condition stated in Lemma 2, we have $Du_1(x_1, b) = c_1$ where c_1 is some constant. This condition can be expanded as follows:

$$\frac{\partial u_1(x_1, b)}{\partial b} = p_{X_2}(f_2^{-1}(x_1 \cdot b)) \frac{x_1}{f_2'(f_2^{-1}(x_1 b))} = c_1 \tag{3.15}$$

Now, let's focus on finding a symmetric Nash equilibrium power allocation strategy. Substituting $b = g_1(x_1)$, the term $f_2^{-1}(x_1 \cdot b)$ is equal to $f_2^{-1}(f_1(x_1)) = x_1$ since $f_1 = f_2$. Thus, Eq.(3.15) can be reduced to the following:

$$p_{X_2}(x_1) \frac{x_1}{f_2'(x_1)} = c_1 \Rightarrow f_2'(x_1) = \frac{1}{c_1} x_1 \cdot p_{X_2}(x_1) \tag{3.16}$$

The above equation provides a condition on the distribution of the X_i such that there exists a Nash equilibrium power allocation scheme. The condition can be restated as the following:

$$x_1 \cdot g_1(x_1) = \int \frac{1}{c_1} x_1 \cdot p_{X_2}(x_1)\, dx_1 \tag{3.17}$$

From the above condition, for example, we see that if $p_{X_2}(\cdot)$ is a strictly increasing polynomial, there exist a Nash equilibrium power allocation strategy.

3.3. Multiple Users Equilibrium Strategies. In this section, we explore the Nash equilibrium power allocation strategies when n users are competing to access the single base station. User i's power allocation function is denoted as $g_i(\cdot)$. Given a time slot with channel state realization $\vec{x} = (x_1, \cdots, x_n)$, the transmitting power for each user is $g_i(x_i)$. The corresponding received power at the base station is again denoted as $f_i(x_i) = x_i \cdot g_i(x_i)$. The new capture rule used in this section is given as the following: a packet from user 1 will be successfully received if the following holds:

$$f_1(x_1) \geq (1 + \Delta) \max(f_2(x_2), \cdots, f_n(x_n))$$

Similar capture model can be found in [15] (i.e., protocol model). The quantity Δ models situations where a guard zone is specified to prevent interference. Note also that the capture rule used in the two users' case can be viewed as a special case the above capture rule.

We start with each user facing the same average power constraint (i.e., $e_1 = e_2 = \cdots = e_n$). Since users are identical, it is reasonable to seek a symmetric Nash equilibrium power allocation strategy. Specifically, the set of strategies $(g_1 = g, \cdots, g_n = g)$ is said to be a symmetric Nash equilibrium strategies if $g_i = g$ is the best power allocation strategy for user i when all other users are also employing the power allocation strategy g. For a power allocation function g to be a symmetric Nash equilibrium strategy, $f(x) = xg(x)$ must be a strictly increasing function using a similar argument as in the two users case. The following theorem shows the existence and the form of a symmetric Nash equilibrium power allocation strategy.

Theorem 4. *Given that each user has the same average power constraint, there exists a symmetric Nash equilibrium power allocation strategy with the following form:*

$$g_i(x_i) = c \cdot x_i^{n-1} \quad \forall\, i \in \{1, \cdots, n\} \tag{3.18}$$

where c is chosen such that the average power constraint is satisfied.

Proof. The complete proof is given in the Appendix. □

With the symmetric Nash equilibrium power allocation strategy given in Eq.(3.18), the expected throughput for each user is given by:

$$\begin{aligned} &P(f(X_1) \geq (1 + \Delta) \max(f(X_2), \cdots, f(X_n))) \\ &= P(X_1^n \geq (1 + \Delta) \max(X_2^n, \cdots, X_n^n)) \\ &= P(X_1 \geq (1 + \Delta)^{\frac{1}{n}} \max(X_2, \cdots, X_n)) \end{aligned} \tag{3.19}$$

To quantify the loss of efficiency due to users' selfish behavior, we consider a system where all users implement the same power allocation policy provided by a system designer such that the overall system throughput is maximized. To find such scheme, we solve the following optimization problem as in the two users' case:

$$\max_{v \in S_1} P(X_1 v(X_1) \geq (1+\Delta) \cdot \max(X_2 v(X_2), \cdots, X_n v(X_n))$$

By symmetry, we have the following upper bound for the above probability:

$$P(X_1 v(X_1) \geq (1+\Delta) \cdot \max(X_2 v(X_2), \cdots, X_n v(X_n)) < \frac{1}{n}$$

As in the two users' case, we consider a series of functions, $v_m(x) = x^m$ for $m \geq 1$. As $m \to \infty$, we have

$$\begin{aligned} &P(X_1^{m+1} \geq (1+\Delta) \cdot \max(X_2^{m+1}, \cdots, X_n^{m+1})) \\ &= P(X_1 \geq (1+\Delta)^{\frac{1}{m+1}} \max(X_2, \cdots, X_n)) \to \frac{1}{n} \end{aligned}$$

Thus, there indeed exists a power allocation scheme that will achieve the maximum possible throughput. In other words, it is possible to have a packet successfully captured in every time slot. Now, when users behave selfishly, the expected throughput for each user is given as follows from Eq.(3.19):

$$P(X_1 \geq (1+\Delta)^{\frac{1}{n}} \max(X_2, \cdots, X_n)) \tag{3.20}$$

As n increases, the above equation goes to $1/n$ which is the maximum attainable throughput. Therefore, as the number of users becomes large, the symmetric Nash equilibrium power allocation scheme is optimal in the sense that the throughput obtained approaches the maximum attainable throughput.

For the special case where $\Delta = 0$, the capture rule becomes that the user with the largest received power get captured. With this simple rule, a Nash equilibrium strategy can be derived with general channel state distribution (i.e., X_i has probability density function $p_{X_i}(\cdot)$). From Eq.(4.23), we have

$$\begin{aligned} &p_Z(f^{-1}(x_1 \cdot b)) \frac{x_1}{f'(f^{-1}(x_1 \cdot b))} = c \\ &f'(x_1) = \frac{1}{c} x_1 p_Z(x_1) \end{aligned} \tag{3.21}$$

where

$$p_Z(z) = (n-1) p_{X_1}(z) [\int_0^z p_{X_1}(x)\, dx]^{n-2}.$$

Hence, we can write the received power function as the following:

$$f(x) = \frac{1}{c}\int x p_Z(x)\, dx$$

From the above equation, one can get the optimal power allocation function by using $g(x) = \frac{f(x)}{x}$.

4. Conclusion. We apply an auction algorithm to the problem of fair allocation of a wireless fading channel. Using the all-pay auction mechanism, we are able to obtain a unique Nash equilibrium strategy. Our strategy allocated bandwidth to the users in accordance with the amount of money that they possess. Hence, this scheme can be viewed as a mechanism for providing quality of service (QoS) differentiation; whereby users are given fictitious money that they can use to bid for the channel. By allocating users different amounts of money, the resulting QoS differentiation can be achieved.

We also show that the Nash equilibrium strategy of this auction leads to an allocation at which total throughput is no worse than 3/4 the maximum possible throughput when fairness constraints are not imposed (i.e., slots are allocated to the user with the better channel). In this paper, we focused on finding a Nash equilibrium strategy when both channels are uniformly distributed. However, as we mentioned earlier, our analysis can be extended to channel state with general distribution. An interesting extension could be to find the exact form of a Nash equilibrium with general channel state distribution.

In the uplink communication scenario, we consider a communication system with multiple users competing, in a non-cooperative manner, for the access of a single satellite, or base station. With a specified capture rule and an average power constraint, users opportunistically adjust their transmission power based on their channel state to maximize their throughput. A Nash equilibrium power allocation strategy is characterized, and the resulting throughput efficiency loss, due to selfish behavior, is quantified. As the number of users increases, the Nash equilibrium power allocation strategy approaches the optimal power allocation strategy that can be achieved in a cooperative environment.

Appendix.

Proof of Lemmma 2. Proof: We first show that if (f_1^*, f_2^*) is a Nash equilibrium strategy pair, $Dg_{f_2^*}^{(1)}(x_1, f_1^*(x_1))$ and $Dg_{f_1^*}^{(2)}(x_2, f_2^*(x_2))$ must be constants for all $x_1 \in [0,1]$ and $x_2 \in [0,1]$. From user 1's perspective with f_2^* fixed, consider a small variation of the function f_1^*. Specifically, let $f_\delta = f_1^* + \delta(\hat{f} - f_1^*)$ where $\hat{f}$ is an arbitrary function in $\bar{S}_1$. Since both $\hat{f}$ and f_1^* are in $\bar{S}_1$, they are both bounded (i.e., $|\hat{f}(x_1)| \leq B$ and $|f_1^*(x_1)| \leq B$ for all $x_1 \in [0,1]$). Therefore, we have $|f_\delta(x_1) - f_1^*(x_1)| \leq 2B\delta$ for all

$x_1 \in [0,1]$. Using the Lagrange's form of Taylor's theorem, we get for any $x_1 \in [0,1]$, there exists a real number $c_{[x_1]} \in [f_1^*(x_1), f_\delta(x_1)]$ such that

$$
\begin{aligned}
g^{(1)}_{f_2^*}(x_1, f_\delta(x_1)) &= g^{(1)}_{f_2^*}(x_1, f_1^*(x_1)) \\
&+ \delta(\hat{f}(x_1) - f_1^*(x_1)) \frac{\partial g^{(1)}_{f_2^*}(x_1, b)}{\partial b}\Big|_{b=f_1^*(x_1)} \\
&+ \frac{1}{2}\delta^2(\hat{f}(x_1) - f_1^*(x_1))^2 \frac{\partial^2 g^{(1)}_{f_2^*}(x_1, b)}{\partial b^2}\Big|_{b=c_{[x_1]}}
\end{aligned} \tag{4.1}
$$

The last term is bounded by $K \cdot \delta^2$ for some K since both $\hat{f}$ and f_1^* are bounded, and $g^{(1)}_{f_2^*}(x_1, b)$ has finite second derivative. Therefore, for small enough δ, it is negligible comparing with the other terms.

Now we show that if $Dg^{(1)}_{f_2^*}(x_1, f_1^*(x_1))$ is not a constant for all $x_1 \in [0,1]$, we can find a strategy f_δ which gives user 1 a higher throughput than f_1^*. To do that, we can write the following equations:

$$
\begin{aligned}
&\int_0^1 g^{(1)}_{f_2^*}(x_1, f_\delta(x_1))\, dx_1 - \int_0^1 g^{(1)}_{f_2^*}(x_1, f_1^*(x_1))\, dx_1 \\
&= \delta \int_0^1 (\hat{f}(x_1) - f_1^*(x_1)) \frac{\partial g^{(1)}_{f_2^*}(x_1, b)}{\partial b}\Big|_{b=f_1^*(x_1)}\, dx_1 + o(\delta)
\end{aligned} \tag{4.2}
$$

Now, since $Dg^{(1)}_{f_2^*}(x_1, f_1^*(x_1))$ is not a constant for all $x_1 \in [0,1]$, we can find a $\hat{f}$ such that the above equation is positive which implies that there is an incentive for user 1 to use f_δ. Hence, (f_1^*, f_2^*) is not a Nash equilibrium strategy pair. Similarly, we can show that $Dg^{(2)}_{f_1^*}(x_2, f_2^*(x_2))$ is a constant for all $x_2 \in [0,1]$ if (f_1^*, f_2^*) is a Nash equilibrium strategy pair.

For the converse, consider again Eq.(4.2). Since $Dg^{(1)}_{f_2^*}(x_1, f_1^*(x_1)) = \frac{\partial g^{(1)}_{f_2^*}(x_1,b)}{\partial b}\Big|_{b=f_1^*(x_1)}$ equals to a constant c_1 for all $x_1 \in [0,1]$. We have

$$
\begin{aligned}
&\delta \int_0^1 (\hat{f}(x_1) - f_1^*(x_1)) \frac{\partial g^{(1)}_{f_2^*}(x_1, b)}{\partial b}\Big|_{b=f_1^*(x_1)}\, dx_1 \\
&= \delta c_1 \int_0^1 (\hat{f}(x_1) - f_1^*(x_1))\, dx_1 = 0
\end{aligned} \tag{4.3}
$$

for all $\hat{f} \in \bar{S}_1$ (i.e., $\int_0^1 \hat{f}(x_1)\, dx_1 = \alpha$). Thus, there is no incentive for user 1 to use strategy $\hat{f}$. Therefore, (f_1^*, f_2^*) is a Nash equilibrium strategy pair.

Proof of Theorem 1 (the Uniqueness). Consider any Nash equilibrium strategy pair (f_1, f_2) under the all-pay auction rule. From previous discussion, we know that the inverse functions, f_2^{-1} and f_1^{-1}, are well defined. With user 2's strategy f_2 fixed, we have

$$
g^{(1)}_{f_2}(x_1, b) = x_1 \cdot P(f_2(X_2) \le b) = x_1 \cdot f_2^{-1}(b)
$$

Similarly, with user1's strategy f_1 fixed, we get

$$g^{(2)}_{f_1}(x_2, b) = x_2 \cdot P(f_1(X_1) \le b) = x_2 \cdot f_1^{-1}(b)$$

From Lemma 2, we know that $Dg^{(1)}_{f_2}(x_1, f_1(x_1))$ and $Dg^{(2)}_{f_1}(x_2, f_2(x_2))$ are two constants for all $x_1 \in [0,1]$ and $x_2 \in [0,1]$ since (f_1, f_2) is a Nash equilibrium strategy pair.

Now, consider the set of channel state pair (x_1, x_2) such that $f_1(x_1) = f_2(x_2)$ (i.e., two users' bids are equal). It forms a separation line in space span by X_1 and X_2. Mathematically, this line can be defined as $h : [0,1] \to [0,1]$ such that $x_2 = h(x_1) = f_2^{-1}(f_1(x_1))$. By the *all-pay* auction rule, a slot with channel state (x_1, x_2') will be assigned to user 2 if (x_1, x_2') is above the line $x_2 = h(x_1)$ and to user 1 if (x_1, x_2') is below the separation line. Fig.2 shows an example of $h(x_1)$. The following lemma shows the uniqueness of $h(x_1)$. We then derive the uniqueness of the strategy pair (f_1, f_2) from the lemma.

Lemma 6. *If $Dg^{(1)}_{f_2}(x_1, f_1(x_1))$ and $Dg^{(2)}_{f_1}(x_2, f_2(x_2))$ are two constants, c_1 and c_2 respectively, for all $x_1 \in [0,1]$ and $x_2 \in [0,1]$, then $h(x_1) = x_1^{c_1/c_2}$.*

Proof. Since $Dg^{(1)}_{f_2}(x_1, f_1(x_1)) = c_1$, from $g^{(1)}_{f_2}(x_1, b) = x_1 \cdot f_2^{-1}(b)$, we have

$$\begin{aligned} Dg^{(1)}_{f_2}(x_1, f_1(x_1)) &= \frac{\partial g^{(1)}_{f_2}(x_1, b)}{\partial b}\Big|_{b=f_1(x_1)} = \frac{x_1}{f_2'(f_2^{-1}(f_1(x_1)))} = c_1 \\ f_2'(h(x_1)) &= \frac{x_1}{c_1} \end{aligned} \tag{4.4}$$

Similarly, for user 2, we get

$$\begin{aligned} Dg^{(2)}_{f_1}(x_2, f_2(x_2)) &= \frac{\partial g^{(2)}_{f_1}(x_2, b)}{\partial b}\Big|_{b=f_2(x_2)} = \frac{x_2}{f_1'(f_1^{-1}(f_2(x_2)))} = c_2 \\ f_1'(h^{-1}(x_2)) &= \frac{x_2}{c_2} \end{aligned} \tag{4.5}$$

We also know that $f_1(x_1) = f_2(h(x_1))$ and $f_1'(x_1) = f_2'(h(x_1)) \cdot h'(x_1)$. Thus, we have

$$\begin{aligned} f_1'(h^{-1}(x_2)) &= f_2'(h(h^{-1}(x_2))) \cdot h'(h^{-1}(x_2)) \\ &= f_2'(x_2) \cdot h'(x_1) = f_2'(h(x_1)) \cdot h'(x_1) \end{aligned} \tag{4.6}$$

By combining the equations $f_1'(h^{-1}(x_2)) = \frac{x_2}{c_2}$ and $f_1'(h^{-1}(x_2)) = f_2'(h(x_1)) \cdot h'(x_1)$, we get

$$\frac{x_2}{c_2} = f_2'(h(x_1)) \cdot h'(x_1).$$

Next we substitute Eq.(4.4) and $x_2 = h(x_1)$ in the above equation to obtain,

$$x_1 \cdot \frac{dh(x_1)}{dx_1} = \frac{c_1}{c_2} h(x_1) \Rightarrow \frac{dh(x_1)}{h(x_1)} = \frac{c_1}{c_2} \frac{dx_1}{x_1}$$
$$\ln |h(x_1)| = \frac{c_1}{c_2} \ln |x_1| + c_3 \Rightarrow h(x_1) = e^{c_3} \cdot x_1^{\frac{c_1}{c_2}}$$

Combined with fact that $h(1) = 1$, we get $h(x_1) = x_1^{\frac{c_1}{c_2}}$. □

Now, we are in a position to derive the exact form of the Nash equilibrium strategy pair. From the equations $f_1'(h^{-1}(x_2)) = \frac{x_2}{c_2}$ and $x_2 = h(x_1)$, we get $f_1'(x_1) = \frac{h(x_1)}{c_2} = x_1^{\frac{c_1}{c_2}}/c_2$. Combined with the condition that $f_1(0) = 0$, we have $f_1(x) = \frac{1}{c_1+c_2} x^{\frac{c_1}{c_2}+1}$. Following the similar method, we get $f_2(x) = \frac{1}{c_1+c_2} x^{\frac{c_2}{c_1}+1}$. Let $\gamma = \frac{c_1}{c_2}$ and $c = \frac{1}{c_1+c_2}$, the Nash equilibrium strategy pair for the all-pay auction must have the following form:

$$f_1^*(x_1) = c \cdot x_1^{\gamma+1}, \quad f_2^*(x_2) = c \cdot x_1^{\frac{1}{\gamma}+1} \tag{4.7}$$

The constant γ and c are chosen such that equations (3.11) and (3.12) are satisfied. The uniqueness of the above Nash equilibrium strategy comes from the fact that there is a unique pair of c and γ that satisfy equations (3.11) and (3.12).

Proof of Theorem 2. *Proof.* Again, we consider $n = 2$ case for simplicity. For $\alpha_1+\alpha_2 = 2\alpha$ and $\beta_1+\beta_2 = 2\beta$, this theorem stated that the pair (α_1, β_1) and (α_2, β_2) cannot be in equilibrium if $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. We will show this by contradiction. Suppose the pair (α_1, β_1) and (α_2, β_2) are in equilibrium for $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$. That is, for given β_1 and β_2, α_1 and α_2 are chosen such that user 1's throughput $G_1(\alpha_1, \beta_1)+G_1(\alpha_2, \beta_2)$ is the maximum. This implies the following:

$$\frac{\partial G_1(\alpha, \beta_1)}{\partial \alpha}\Big|_{\alpha=\alpha_1} = \frac{\partial G_1(\alpha, \beta_2)}{\partial \alpha}\Big|_{\alpha=\alpha_2}. \tag{4.8}$$

To see this, if $\frac{\partial G_1(\alpha,\beta_1)}{\partial \alpha}\Big|_{\alpha=\alpha_1} > \frac{\partial G_1(\alpha,\beta_2)}{\partial \alpha}\Big|_{\alpha=\alpha_2}$, we will have $G_1(\alpha_1+\delta, \beta_1)+ G_1(\alpha_2 - \delta, \beta_2) > G_1(\alpha_1, \beta_1) + G_1(\alpha_2, \beta_2)$ by first order expansion, thus contradicting the statement that $G_1(\alpha_1, \beta_1) + G_1(\alpha_2, \beta_2)$ is the maximum throughput for user 1 for given β_1 and β_2.

Similarly, for given α_1 and α_2, if β_1 and β_2 maximize $G_2(\alpha_1, \beta_1) + G_2(\alpha_2, \beta_2)$ then,

$$\frac{\partial G_2(\alpha_1, \beta)}{\partial \beta}\Big|_{\beta=\beta_1} = \frac{\partial G_2(\alpha_2, \beta)}{\partial \beta}\Big|_{\beta=\beta_2}. \tag{4.9}$$

By taking the derivative of equations (2.14) and (2.15), we get the following:

$$\frac{\partial G_1(\alpha,\beta_1)}{\partial \alpha}\Big|_{\alpha=\alpha_1} = -\frac{\beta_1(-2\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2}+\alpha_1-2\beta_1)}{2(\alpha_1+\beta_1+\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2})^2\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2}} \tag{4.10}$$

$$\frac{\partial G_2(\alpha_1,\beta)}{\partial \beta}\Big|_{\beta=\beta_1} = -\frac{\alpha_1(-2\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2}+\beta_1-2\alpha_1)}{2(\alpha_1+\beta_1+\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2})^2\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2}} \tag{4.11}$$

Substituting Eq.(4.10) into Eq.(4.8) and Eq.(4.11) into Eq.(4.9), we then have the following after combining Eq.(4.8) and Eq.(4.9):

$$\begin{aligned}&\frac{\beta_1(-2\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2}+\alpha_1-2\beta_1)}{\beta_2(-2\sqrt{\alpha_2^2-\alpha_2\beta_2+\beta_2^2}+\alpha_2-2\beta_2)}\\ &=\frac{\alpha_1(-2\sqrt{\alpha_1^2-\alpha_1\beta_1+\beta_1^2}+\beta_1-2\alpha_1)}{\alpha_2(-2\sqrt{\alpha_2^2-\alpha_2\beta_2+\beta_2^2}+\beta_2-2\alpha_2)}\end{aligned} \tag{4.12}$$

To simplify the above equation, we multiply $\frac{\alpha_2^2}{\alpha_1^2}$ on both sides, and let $\gamma_1 = \frac{\beta_1}{\alpha_1}$, $\gamma_2 = \frac{\beta_2}{\alpha_2}$. We get

$$\frac{\gamma_1(-2\sqrt{1-\gamma_1+\gamma_1^2}+1-2\gamma_1)}{\gamma_2(-2\sqrt{1-\gamma_2+\gamma_2^2}+1-2\gamma_2)} = \frac{-2\sqrt{1-\gamma_1+\gamma_1^2}+\gamma_1-2}{-2\sqrt{1-\gamma_2+\gamma_2^2}+\gamma_2-2} \tag{4.13}$$

or, after rearranging terms, the following:

$$\frac{\gamma_1(-2\sqrt{1-\gamma_1+\gamma_1^2}+1-2\gamma_1)}{-2\sqrt{1-\gamma_1+\gamma_1^2}+\gamma_1-2} = \frac{\gamma_2(-2\sqrt{1-\gamma_2+\gamma_2^2}+1-2\gamma_2)}{-2\sqrt{1-\gamma_2+\gamma_2^2}+\gamma_2-2} \tag{4.14}$$

We define

$$f(\gamma) = \frac{\gamma(-2\sqrt{1-\gamma+\gamma^2}+1-2\gamma)}{-2\sqrt{1-\gamma+\gamma^2}+\gamma-2}.$$

Then Eq.(4.14) becomes $f(\gamma_1) = f(\gamma_2)$. Now we show that this implies $\gamma_1 = \gamma_2$ by observing that

$$\frac{\partial f(\gamma)}{\partial \gamma} = -\frac{(\gamma+1)(2\sqrt{1-\gamma+\gamma^2}-1+2\gamma)}{\sqrt{1-\gamma+\gamma^2}(-2\sqrt{1-\gamma+\gamma^2}+\gamma-2)},$$

and it is easy to check that $\frac{\partial f}{\partial \gamma} > 0 \ \forall \gamma \geq 0$. Now, we have $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2}$. We further show that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Observe that for fixed β_1 and β_2, we can write

$$G_1(\alpha,\beta_1) = \frac{\alpha}{\alpha+\beta_1+\sqrt{(\alpha-\beta_1)^2+\alpha\beta_1}} = \frac{\frac{\alpha}{\beta_1}}{1+\frac{\alpha}{\beta_1}+\sqrt{(1-\frac{\alpha}{\beta_1})^2+\frac{\alpha}{\beta_1}}} \triangleq F(\frac{\alpha}{\beta_1}) \tag{4.15}$$

where

$$F(\sigma) = \frac{\sigma}{1+\sigma+\sqrt{(1-\sigma)^2+\sigma}}.$$

Thus, we have

$$\frac{\partial G_1(\alpha,\beta_1)}{\partial\alpha}\Big|_{\alpha=\alpha_1} = \frac{1}{\beta_1}\frac{\partial F(\sigma)}{\partial\sigma}\Big|_{\sigma=\frac{\alpha_1}{\beta_1}} \tag{4.16}$$

$$\frac{\partial G_1(\alpha,\beta_2)}{\partial\alpha}\Big|_{\alpha=\alpha_2} = \frac{1}{\beta_2}\frac{\partial F(\sigma)}{\partial\sigma}\Big|_{\sigma=\frac{\alpha_2}{\beta_2}} \tag{4.17}$$

From Eq.(4.8), we have

$$\frac{1}{\beta_1}\frac{\partial F(\sigma)}{\partial\sigma}\Big|_{\sigma=\frac{\alpha_1}{\beta_1}} = \frac{1}{\beta_2}\frac{\partial F(\sigma)}{\partial\sigma}\Big|_{\sigma=\frac{\alpha_2}{\beta_2}} \tag{4.18}$$

It is easy to verify that $\frac{\partial F(\sigma)}{\partial\sigma} \neq 0\ \forall\sigma \geq 0$. Therefore, since $\frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2}$, the above equation implies that $\beta_1 = \beta_2$ which contradicts our original assumption of $\beta_1 \neq \beta_2$. Therefore, the pair (α_1,β_1) and (α_2,β_2) cannot be in equilibrium if $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$.
□

Proof of Theorem 4. *Proof.* With all users $i \neq 1$ using a fixed power allocation strategy g, we now explore the optimal power allocation strategy for user 1 which is denoted by g_1^*. Let $u_g^{(1)} : (x_1, b) \to \mathcal{R}$ denote user 1's expected throughput during a slot conditioning on the following events:

- User 1's channel state is $X_1 = x_1$.
- User 1's allocated power is b.

As before, we will drop the term g in the expression $u_g^{(1)}(x_1, b)$, and simply write it as $u_1(x_1, b)$. Specifically, we can the write the equation:

$$\begin{aligned} u_1(x_1,b) =& P((1+\Delta)\max(f_2(X_2),\cdots,f_n(X_n)) \leq x_1\cdot b) \\ =& P((1+\Delta)Y \leq x_1\cdot b) \end{aligned}$$

where $Y = \max(f_2(X_2),\cdots,f_n(X_n))$. Since all users $i \neq 1$ use the same strategy g, we have $Y = \max(f(X_2),\cdots,f(X_n))$ where $f(X_i) = X_i\cdot g(X_i)$ for all $i \neq 1$. Moreover, since f is strictly increasing, we can write:

$$Y = \max(f(X_2),\cdots,f(X_n)) = f(\max(X_2,\cdots,X_n))$$

Denoting $Z = \max(X_2, \cdots, X_n)$, we have the following:

$$\begin{aligned} u_1(x_1, b) &= P((1+\Delta)Y \le x_1 \cdot b) = P(Z \le f^{-1}(\frac{1}{1+\Delta} x_1 \cdot b)) \\ &= \int_0^{f^{-1}(\frac{1}{1+\Delta}x_1 \cdot b)} p_Z(z)\, dz \end{aligned} \tag{4.19}$$

where $p_Z(\cdot)$ denote the probability density function of the random variable Z. The optimization problem that user 1 faces can be written as the following:

$$\begin{aligned} \max G_1(e) &= \int_0^1 u_1(x_1, g_1(x_1)) \cdot p_{X_1}(x_1)\, dx_1 = \int_0^1 u_1(x_1, g_1(x_1))\, dx_1 \\ \text{subj.} &\int_0^1 g_1(x_1)\, dx_1 \le e \end{aligned} \tag{4.20}$$

Writing the Lagrangian function, we have

$$\begin{aligned} &\int_0^1 u_1(x_1, g_1(x_1))\, dx_1 - \lambda(\int_0^1 g_1(x_1)\, dx_1 - e) \\ =&\int_0^1 [u_1(x_1, g_1(x_1)) - \lambda g_1(x_1)]\, dx_1 + \lambda e \end{aligned} \tag{4.21}$$

Therefore, for each fixed x_1, we want to choose a $g_1(x_1)$ to maximize the term $u_1(x_1, g_1(x_1)) - \lambda g_1(x_1)$. For convenience, let $b = g_1(x_1)$. Then, we have

$$\max_b L(b) = \max_b u_1(x_1, b) - \lambda b = \max_b \int_0^{f^{-1}(\frac{1}{1+\Delta}x_1 \cdot b)} p_Z(z)\, dz - \lambda b \tag{4.22}$$

Maximizing $L(b)$ with respect to b yields the first order condition:

$$\frac{\partial L(b)}{\partial b} = p_Z(f^{-1}(\frac{1}{1+\Delta} x_1 \cdot b)) \frac{\frac{x_1}{1+\Delta}}{f'(f^{-1}(\frac{1}{1+\Delta} x_1 \cdot b))} - \lambda = 0 \tag{4.23}$$

Since $Z = \max(X_2, \cdots, X_n)$ and X_i's are i.i.d, we have

$$p_Z(z) = (n-1)z^{n-2}.$$

Now, consider $b = g_1(x_1) = c x_1^m$. Since we are seeking a symmetric Nash equilibrium power allocation strategy, user $i \ne 1$ will adopt the same strategy as user 1. Thus, we have $f(x) = x \cdot g(x) = x \cdot c x^m = c x^{m+1}$. The second term in Eq.(4.23) can be written as the following:

$$\begin{aligned} &f'(f^{-1}(\frac{1}{1+\Delta} x_1 \cdot b)) = f'(f^{-1}(\frac{c}{1+\Delta} x_1 \cdot x_1^m)) \\ =& f'((\frac{1}{1+\Delta} x_1 \cdot x_1^m)^{\frac{1}{m+1}}) = c(m+1)(\frac{1}{1+\Delta})^{\frac{m}{m+1}} x_1^m \end{aligned} \tag{4.24}$$

Similarly,

$$p_Z(f^{-1}(\frac{1}{1+\Delta}x_1 \cdot b)) = p_Z((\frac{1}{1+\Delta})^{\frac{1}{m+1}}x_1) = (n-1)(\frac{1}{1+\Delta})^{\frac{n-2}{m+1}}x_1^{n-2} \tag{4.25}$$

Eq.(4.23) can be re-written in the following form:

$$(n-1)(\frac{1}{1+\Delta})^{\frac{n-2}{m+1}}x_1^{n-2}\frac{\frac{x_1}{1+\Delta}}{c(m+1)(\frac{1}{1+\Delta})^{\frac{m}{m+1}}x_1^m} - \lambda = 0 \tag{4.26}$$

Since the above equality has to hold for all $x_1 \in [0,1]$, the following must be true

$$x_1^{n-2} \cdot x_1 \cdot x_1^{-m} = 1$$

Thus, we have $m = n-1$ and $g_i(x) = cx^{n-1}$ for all $i = 1, \cdots, n$. □

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