Diverse Routing in Networks With Probabilistic Failures

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***Abstract*—We develop diverse routing schemes for dealing with multiple, possibly correlated, failures. While disjoint path protection can effectively deal with isolated single link failures, recovering from multiple failures is not guaranteed. In particular, events such as natural disasters or intentional attacks can lead to multiple correlated failures, for which recovery mechanisms are not well understood. We take a probabilistic view of network failures where multiple failure events can occur simultaneously, and develop algorithms for finding diverse routes with minimum joint failure probability. Moreover, we develop a novel *Probabilistic Shared Risk Link Group* (PSRLG) framework for modeling correlated failures. In this context, we formulate the problem of finding two paths with minimum joint failure probability as an integer nonlinear program (INLP) and develop approximations and linear relaxations that can find nearly optimal solutions in most cases.**

***Index Terms*—Correlated failures, disjoint paths, path protection, probabilistic shared risk link group (SRLG), random link failures.**

I. INTRODUCTION

THIS paper deals with protection in communication networks with correlated probabilistic link failures. The objective of protection is to provide reliable communication in the event of failure of network components such as nodes or links. Such protection mechanisms are classified as *link protection* and *path protection*. Link protection precomputes an alternate detour for each link, and recovers from a link failure by rerouting the traffic along its predetermined detour. In contrast, path protection assigns two paths, a primary and a backup, to each connection, and the traffic is switched onto the backup path in case of a primary path failure. Therefore, the primary and backup paths need to be *disjoint* since otherwise the two paths will fail simultaneously if a link or node shared by the two paths fails. In this paper, we focus on path protection.

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The disjoint-path-based protection effectively addresses the case of a single point failure, but if more than one failure occurs at the same time, protection is not guaranteed since both paths may fail simultaneously. There are several factors that can cause multiple failures. First, modern communication networks are deployed over an optical fiber network, and so multiple communication links can share the same fiber in the optical layer. Consequently, any fiber cut can lead to the failure of all the (upper-layer) communication links sharing that fiber. Second, multiple link failures can occur if the second link fails before the first was repaired. Third, natural disasters or attacks can destroy several links (which do not necessarily share a fiber) in the vicinity of such events.

The concept of Shared Risk Link Group (SRLG) has been proposed in order to address multiple correlated link failures systematically [1]. An SRLG is a set of links sharing a common physical resource (cable, conduit, etc.) and thus a risk of failure. In this context, Bhandari first studied the so-called Physically Disjoint Paths (PDP) problem in [2] and proposed a shortest PDP algorithm for particular topologies. Since this pioneering work, there has been a large body of work [3]–[14] dealing with multiple failures in the context of SRLGs. In [15], Hu showed the NP-completeness of the *SRLG-Disjoint Paths* Problem (SDPP) where SRLG-disjoint paths are two paths touching no common SRLG.

All of the previous SRLG works assume that once an SRLG failure event occurs, all of its associated links fail simultaneously. Here, we generalize the notion of an SRLG to account for probabilistic link failures. This generalized notion allows us to model correlated failures that may result from a natural or man-made disaster. For example, in the event of a natural disaster, some, but not necessarily all, of the links in the vicinity of the disaster may be affected. Such failures cannot be described using a *deterministic failure model*, and this raises the need for a systematic approach to dealing with *correlated probabilistic link failures*. We address this issue by modeling SRLG events probabilistically so that upon an SRLG failure event, links belonging to that SRLG fail with some probability (not necessarily one). Our probabilistic SRLG model is applicable to a number of real-world failure scenarios. Some examples include: 1) *WDM Networks* where the lightpaths traversing a fiber form an SRLG and fail (with probability 1) in the event of a fiber cut; 2) *Satellite/wireless communication links* where links are subject to outage in the event of bad weather. In this case, the satellite links affected by the weather event form an SRLG and may fail with some probability; 3) *ElectroMagnetic Pulse (EMP) attack*: EMP is an intense energy field that can instantly overload or disrupt numerous electrical circuits at a distance [16]. In the event of an EMP attack, the fiber links in the vicinity of the attack may have a high probability of failure, and those distant from the attack would fail with low probability due to signal attenuation; and 4) *Natural/man-made disasters* such as earthquakes or floods where communication links in the vicinity of the disaster may fail. For example, an undersea cable was cut during the Taiwan earthquake of 2006 [17], disrupting most communications out of Taiwan. Similarly, during the Baltimore tunnel fire in 2001 [18], the fire melted away the fiber along the tunnel, leading again to a large number of correlated failures.

There are a number of papers dealing with probabilistic link failures [19]–[21]. Typically, they consider the *availability* (i.e., probability) that a connection is in the operating state and seek to find a path pair satisfying minimum availability requirement [19], [20] or a path pair with maximum availability [21]. While the above works assume independent link failures, there have been efforts to deal with correlated failures. In [22]–[24], the link failure probability is extended and defined as a function of SRLG parameters to account for correlated failures. In particular, in [24], the path failure probability is defined as the ratio of the number of touched SRLGs to the total number of SRLGs. Under this model, [24] considers the problem of finding a pair of primary and backup paths that satisfy joint reliability requirement. This model generalizes the traditional concept of SRLG-disjointness such that if the joint reliability of primary and backup paths is q, then it means they are disjoint with respect to q fraction of SRLGs. However, under this model, link failures are deterministic, given an SRLG failure. Hence, this model cannot be directly applied to the case of correlated failures with uncertainty that may occur due to disasters and attacks. In [25], a primary/backup path allocation problem is defined to find a pair of paths having minimum joint failure probability. They adopt a *correlated link failure probability model* where the correlation between the links is represented by their joint failure probability. This correlation model requires exponential number of conditional probabilities in general, prohibiting a simple formulation. Due to this difficulty, they take into account only the first-order correlation, i.e., the correlation between pairs of links.

In this paper, we consider finding a pair of paths with minimum joint failure probability in a network where the link failures occur randomly and are possibly correlated. We propose an alternative model that enables a simple formulation and captures the essence of correlated link failures. Our model assumes that once an SRLG failure event occurs, its associated links fail with some probabilities. Thus, the correlation exists among the links only when they belong to the same SRLG. Clearly, this model can be viewed as a generalization of the traditional (deterministic) SRLG model.

Our contributions can be summarized as follows.

- We generalize the SRLG framework to a probabilistic SRLG (PSRLG). This new framework enables us to effectively model correlated link failures and develop efficient formulations to otherwise intractable problems involving correlated link failures.
- We develop mathematical formulations for the problem of finding a pair of paths with minimum joint failure probability. This new approach enables the generalization of disjoint-path protection schemes to the case of multiple (probabilistic) failures.
- We develop heuristic algorithms for finding a pair of paths with minimum joint failure probability. Our algorithms are based on linear approximations and Lagrangian relaxations and are shown to find nearly optimal solutions.

While the deterministic SRLG model has been widely used in the literature, there are many scenarios where the deterministic model is not applicable. For example, the Working Group on California Earthquake Probabilities (WGCEP) has been developing the earthquake rate models for California. In particular, they compute the probabilities of all possible damaging earthquakes (according to their magnitudes) throughout a region and over a specified time span [26]. Such disasters lead to correlated failures that are well addressed by our new PSRLG model; namely, the earthquake probabilities correspond to SRLG failure probabilities, and the link failure probabilities can be computed based on their magnitudes. The network providers could use this PSRLG data and our formulation in order to protect a path in the presence of earthquakes.

The probabilistic SRLG model can also be applied to deal with the inaccuracy in the SRLG database. Typically, the SRLG data is the mapping between the IP layer components and the underlying physical components. This information is used for failure diagnosis[1] as well as survivable routing. However, it often contains errors due to traffic engineering and recovery mechanism [28]. Recently, in [30], failure diagnosis mechanisms were studied assuming probabilistic errors in the SRLG data where the association of an IP-layer component to a physical component is probabilistic. Note that with this erroneous SRLG data, the survivable routing problem is best handled probabilistically as in our PSRLG model.

The rest of the paper is organized as follows. In Section II, we present our new probabilistic SRLG model and describe the generalized path protection problems. In Section III, we study the case of independent link failures, which provides fundamental insights to the study of correlated failures. In Section IV, we formulate the path protection problems using the PSRLG model and develop algorithms for finding paths with minimum joint failure probability. Finally, in Section V, we analyze the performance of our algorithms via simulations.

II. Model and Problem Description

Consider a directed network graph $G=(V,E)$, where V is a set of nodes and E is a set of links. Any link in E will be denoted by (i,j) for $i,j \in V$, meaning that the link starts from node i and ends at node j. There is a set R of SRLG events that can incur link failures. Each SRLG event $r \in R$ occurs with probability π_r, and once an SRLG event r occurs, link (i,j) will fail with probability $p_{ij}^r \in [0,1]$. For example, if link (i,j) is never affected by event r, then we define $p_{ij}^r = 0$. On the other hand, if the event r is a cable cut and link (i,j) traverses that cable, then we will have $p_{ij}^r = 1$. In the following, we generalize the traditional notion of an SRLG to include probabilistic correlated failures.

Definition 1: A *probabilistic SRLG* (PSRLG) is a set of links with positive failure probability in the event of an SRLG failure. Namely, link (i,j) belongs to SRLG r if $p_{ij}^r > 0$, and SRLG $r = \{(i,j) \in E : p_{ij}^r > 0\}$.

[1]Typically, in failure diagnosis, the underlying physical-layer failures are inferred from the IP-layer failures by using the SRLG data, i.e., the mapping between physical-layer components and IP-layer components [27]–[31].

We say that links (i,j) and (k,l) are *correlated* if there exists an SRLG r such that $p_{ij}^r, p_{kl}^r > 0$. Clearly, this model is a generalization of the traditional SRLG model and enables us to deal with correlated probabilistic link failures.

We consider a single source–destination pair. Let $s \in V$ and $t \in V$ be source and destination nodes, respectively. Our objective is to find a pair of primary and backup paths from s to t with minimum joint failure probability. This problem will be considered using two different models: 1) independent link failure; and 2) SRLG-based correlated link failure. In each case, we seek to find a pair of paths with minimum joint failure probability.

Let $x_{ij} = 1$ if the primary path traverses link (i,j), and 0 otherwise. Similarly, define another binary variable y_{ij} for the backup path. We will just drop the index of a variable to denote its vector version, e.g., x represents the vector $[x_{ij}, (i,j) \in E]$. The set of n-dimensional binary vectors will be defined as B_n, i.e., $B_n = \{0,1\}^n$. Our problems will be formulated as integer programs (IPs).

III. Independent Link Failure Model

In order to gain insights into the problem, we start by first considering the independent link failure model. Moreover, we begin by considering the simple case of finding a single path with minimum failure probability. We then use the insights gained in order to formulate the problem of finding a pair of paths with minimum joint failure probability. In Section IV, we will further generalize our formulations to deal with correlated (SRLG) failures.

A. Single-Path Problem

First, consider the problem of finding a single path having minimum failure probability. Let p_{ij} be the probability that link (i,j) fails, then link (i,j) will survive with probability $1-p_{ij}$. Consequently, the survivability probability of path x is given in a product form by $\prod_{(i,j)}(1-p_{ij}x_{ij})$. The problem is formulated as

$$(\mathbf{P1.1}): \quad \min_{x \in B_{|E|}} 1 - \prod_{(i,j)} (1 - p_{ij}x_{ij})$$
$$\text{subject to} \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} 1, & i = s \\ -1, & i = t \\ 0, & \text{o.w.} \end{cases} \quad \forall i \in V \qquad (1)$$

where $|E|$ is the cardinality of E. The constraints in (1) require that the set of links selected by x forms a path from node s to node t. For simplicity, we will denote this constraint by $CC(x)$, representing the connectivity constraint on x for a path from s to t. As all the variables in this work are binary, we will sometimes omit the binary constraint for convenience. The problem (P1.1) is an integer nonlinear program (INLP), which generally is very difficult to solve. However, using the following theorem, we are able to reformulate (P1.1) as an integer linear program (ILP).

Theorem 1: Assume $p_{ij} \in [0,1), \forall (i,j)$, then the problem (P1.1) is equivalent to the following ILP:

$$(\mathbf{P1.1L}): \quad \min_{x \in B_{|E|}} -\sum_{(i,j)} x_{ij} \log(1 - p_{ij})$$
$$\text{subject to} \quad CC(x)$$

where $CC(x)$ is the connectivity constraint as given in (1).

Proof: First, the objective in (P1.1) can be equivalently written as $\max \prod_{(i,j)}(1-p_{ij}x_{ij})$. Taking logarithm over the entire function gives $\max \sum_{(i,j)} \log(1-p_{ij}x_{ij})$ without affecting the optimal solution. The proof is completed by applying the identity $\log(1-p_{ij}x_{ij}) = x_{ij}\log(1-p_{ij})$ for binary variable x_{ij} and noting that $\max f(x)$ is the same as $\min -f(x)$. ■

Observation 1: Theorem 1 shows that the path with minimum failure probability is the *shortest path* under link weights $-\log(1-p_{ij}), \forall (i,j)$. It immediately follows that if the failure probability is sufficiently small (i.e., $p_{ij} \ll 1, \forall (i,j)$), then the probability-wise shortest path has the minimum failure probability because $-\log(1-p_{ij}) \approx p_{ij}$ for small p_{ij}. Furthermore, with uniform failure probability, i.e., $p_{ij} = q, \forall (i,j) \in E$, the shortest-hop path has the lowest failure probability. This result will be used in developing heuristic algorithms.

B. Path-Pair Problem With Disjointness Constraint

Let $F_1(p,x)$ be the objective function of problem (P1.1), i.e., $F_1(p,x)$ is the failure probability of path x for given link failure probability vector p. Suppose that the two paths x and y are link-disjoint, then their failures are mutually independent because the link failures are independent and further the paths do not share any link. Then, the joint failure probability of two disjoint paths x and y is given by $F_1(p,x) \cdot F_1(p,y)$. The path-pair problem with disjointness constraint (DC) is thus formulated as

$$(\mathbf{P1.2}): \quad \min_{x,y} F_1(p,x)F_1(p,y)$$
$$\text{subject to} \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} 1, & i = s \\ -1, & i = t \\ 0, & \text{o.w.} \end{cases} \quad \forall i \in V$$
$$\sum_{j:(i,j)\in E} y_{ij} - \sum_{j:(j,i)\in E} y_{ji} = \begin{cases} 1, & i = s \\ -1, & i = t \\ 0, & \text{o.w.} \end{cases} \quad \forall i \in V$$
$$x_{ij} + y_{ij} \leq 1 \quad \forall (i,j) \in E.$$

Again, the first and second constraints are the connectivity constraints requiring that x and y are paths from s to t. The last constraints require that x and y cannot share any link, i.e., they are link-disjoint. Hence, x and y satisfying all the constraints in (P1.2) will form a pair of disjoint paths from s to t. For brevity, throughout this paper, we will denote by $CC(x)$ the connectivity constraint on any binary link selection vector x, and $DC(x,y)$ the disjointness constraint (DC) on paths x and y. The problem (P1.2) is an INLP and has a special structure that is difficult to solve in general. Let c be the length of the shortest path from s to t.

Lemma 1: Assume the uniform failure probability $p_{ij} = p, \forall (i,j)$, where $1-2^{-(1/c)} \leq p < 1$. Then, the problem (P1.2) is a concave minimization.

Proof: See Appendix A. ■

Therefore, the problem (P1.2) contains a concave minimization as a special case. Generally, a concave minimization problem is NP-hard [32], and hence the above lemma implies that the problem (P1.2) may not be easy. In the following, we consider heuristic algorithms to solve the problem.

1) Greedy Algorithm: First, we analyze the properties of the optimal solution to (P1.2). This motivates a greedy algorithm. Our analysis is based on the theory of majorization, whose introduction follows.

Definition 2 (Majorization): Given an n-tuple vector a, let $a_{[i]}$ be the ith largest of n coordinate. An n-tuple vector α is said to be *majorized* by β if

$$\sum_{i=1}^{k} \alpha_{[i]} \leq \sum_{i=1}^{k} \beta_{[i]}, k = 1, \ldots, n-1, \text{ and } \sum_{i=1}^{n} \alpha_{[i]} = \sum_{i=1}^{n} \beta_{[i]}.$$

This relationship is denoted by $\alpha \prec \beta$.

The majorization formalizes how evenly distributed the elements of a vector are. For example, $\alpha = [2,2,2]$ is the most evenly distributed vector over all the 3-tuple vectors summing up to 6, and majorized by any other such vector, e.g., $\beta = [1,1,4]$.

Definition 3 (Schur Convexity): A function $f : A \subseteq R^n \rightarrow R$ is *Schur convex* if $f(\alpha) \leq f(\beta)$ for any $\alpha, \beta \in A$ such that $\alpha \prec \beta$. It is called *Schur concave* if $f(\alpha) \geq f(\beta)$ for $\alpha \prec \beta$.

It is clear from this definition that a Schur convex function $f : A \rightarrow R$ is minimized at evenly distributed points, whereas if $f(x)$ is Schur concave, it is minimized at unevenly distributed points. We will show the Schur concavity of the objective function in (P1.2), and use this property to develop a greedy path selection algorithm.

Assume the uniform failure probability, i.e., $p_{ij} = p, \forall (i,j)$. Then, the objective function in (P1.2) can be written as

$$f(X,Y) = \left(1 - (1-p)^X\right)\left(1 - (1-p)^Y\right) \tag{2}$$

where $X = \sum_{(i,j)} x_{ij}$ and $Y = \sum_{(i,j)} y_{ij}$. Note that X and Y are the number of hops in primary and backup paths, respectively. Hence, for fixed p, $f(X,Y)$ is a function of the numbers of hops in the primary and back paths. The problem (P1.2) can be restated as: minimize $f(X,Y)$ subject to the same constraints as in (P1.2) with the additional constraints $X = \sum_{(i,j)} x_{ij}$ and $Y = \sum_{(i,j)} y_{ij}$. The objective function $f(X,Y)$ in this problem can be shown to be Schur concave.

Lemma 2: The function $f(X,Y)$ in (2) is Schur concave for $0 \leq p < 1$.

Proof: See Appendix B. ■

As mentioned above, a Schur concave function is minimized at unevenly distributed points rather than evenly distributed ones. So, for example, we have $f(2,2) \geq f(1,3)$ since $(2,2) \prec (1,3)$. Accordingly, Lemma 2 implies that a pair with unbalanced (in terms of the number of hops) paths is preferred because its joint failure probability may be lower than that of a balanced pair. Consider an example in Fig. 1, where there are two pairs of s–t paths: one with $X = 3, Y = 3$ and the other with $X = 2, Y = 4$. Fig. 1(b) plots the joint path failure probabilities of the two pairs and shows that the unbalanced pair (i.e., the one with $X = 2, Y = 4$) is more reliable than the balanced pair for all values of p.

It should be noted that a similar observation can also be made as follows for the nonuniform failure probabilities where p_{ij}'s can be different.

Observation 2: Consider an example topology in Fig. 2 where the number on each link is its failure probability. We want to find a pair of disjoint paths with minimum joint failure probability. It is easy to see that there are only two pairs shown in the figure. The pair in the top has individual path failure probabilities of (0.1,0.1), and thus its joint failure probability is 0.01. On the other hand, the pair in the bottom has individual path failure probabilities of (0,0.19), leading to zero joint failure probability. This reasserts that a good–bad path pair might be better than a medium–medium pair.

The above observations suggest that it is important to include the best path (i.e., path having minimum failure probability) in the pair. Furthermore, it motivates a greedy algorithm that selects the best path first and then selects the next best disjoint path (See Algorithm 1). Note that according to Observation 1, the best path is obtained by the probability-wise shortest path. Hence, Algorithm 1 only needs to run a shortest path algorithm twice whose complexity is $O(|V|^2)$.

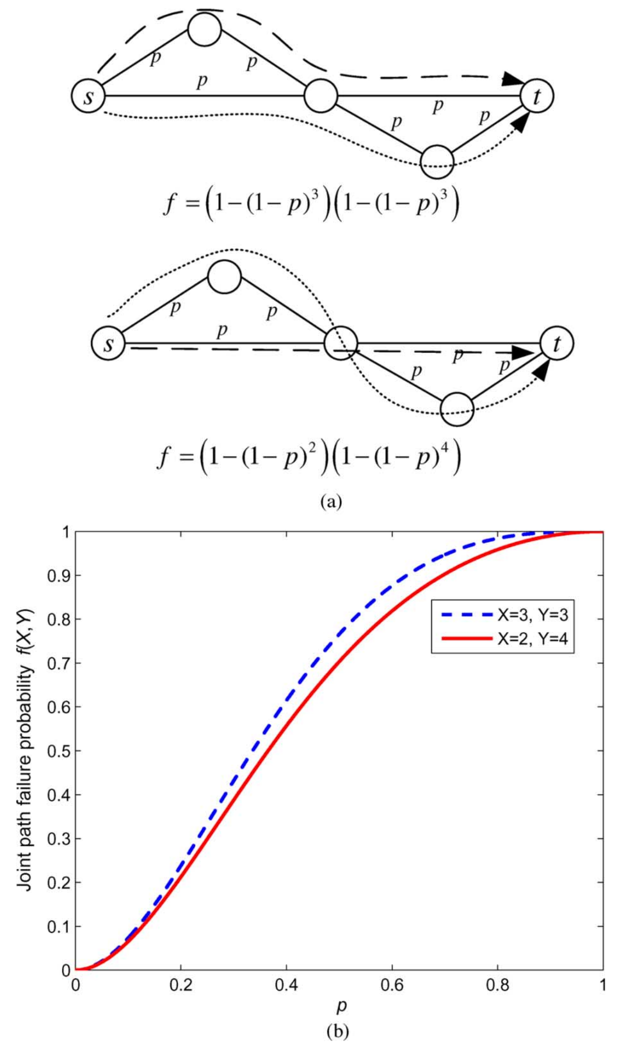


Fig. 1. Two pairs of disjoint paths and their failure probabilities under uniform link failure probability. The unbalanced pair is always more reliable than the balanced one. (a) Two pairs of disjoint paths: (top) $X = 3, Y = 3$ and (bottom) $X = 2, Y = 4$. (b) Plot of joint path failure probabilities.

Algorithm 1 Greedy: IND w/ DC

1: Set link weight $w_{ij} = p_{ij}, \forall (i,j) \in E$

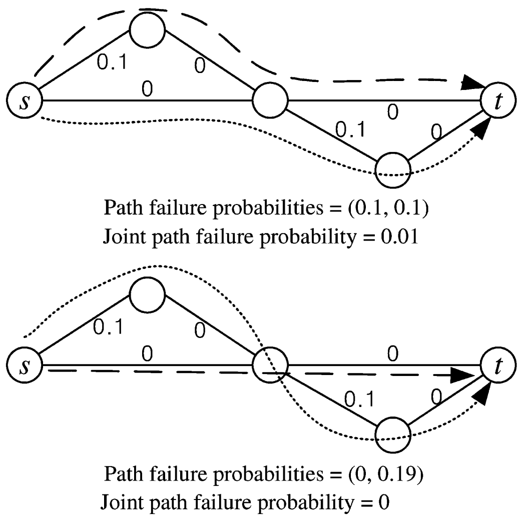


Fig. 2. Example motivating greedy algorithm. Including the most reliable path is also important in the case of nonuniform failure probabilities.

2: Find shortest path x
3: Remove all the directed edges used by x
4: Find shortest path y

Note that the greedy algorithm can possibly run out of s–t paths after the first path is found. This is a so-called trap problem. In the following, we show that such a case does not happen under mild assumptions. First, notice that the paths found by the greedy algorithm are simple (i.e., do not contain cycles) or can contain cycles of zero length, in which case they can be removed without affecting the path failure probability. Hence, we may assume that the paths x and y found by the greedy algorithm are simple.

Lemma 3: Consider a k-connected[2] bidirectional graph G where $k \geq 2$. Removing a simple s–t path in G does not disconnect s and t.

Proof: See Appendix C. ■

Therefore, the greedy algorithm does not run out of s–t paths after the first path is found, provided that the graph is bidirectional and k-connected with $k \geq 2$. We note that this assumption is very common, as most practical networks use bidirectional links.

2) ILP Approximation of (P1.2) and Its Lagrangian Relaxation: We develop another heuristic algorithm based on the ILP approximation of the problem (P1.2). First, the objective function in (P1.2) can be expanded as

$$F_1(p,x)F_1(p,y) = 1 - \prod_{(i,j)}(1-p_{ij}x_{ij}) - \prod_{(i,j)}(1-p_{ij}y_{ij}) + \prod_{(i,j)}(1-p_{ij}x_{ij})\prod_{(i,j)}(1-p_{ij}y_{ij}). \quad (3)$$

Further expanding the product terms and canceling out common terms yields

$$F_1(p,x)F_1(p,y) = \sum_{(i,j)}\sum_{(k,l)} p_{ij}p_{kl}x_{ij}y_{kl} + HOT \quad (4)$$

where HOT stands for high-order terms—namely, terms involving the product of 3 or more failure probabilities. In the low-failure-probability regime, i.e., $p_{ij} \ll 1, \forall (i,j)$, the HOTs can be neglected, and the ILP can be formulated as follows:

$$\begin{aligned} (\mathbf{P1.2L}): \quad & \min_{x,y,z} \sum_{(i,j)}\sum_{(k,l)} p_{ij}p_{kl}z_{ij}^{kl} \\ & \text{subject to } CC(x), CC(y), DC(x,y) \\ & (C1)\ \ z_{ij}^{kl} \geq x_{ij} + y_{kl} - 1, \forall (i,j),(k,l) \in E \end{aligned}$$

where we have introduced the binary variables $z_{ij}^{kl}, \forall (i,j),(k,l)$ such that $z_{ij}^{kl} = 1$ only if both of x_{ij} and y_{kl} are 1. That is, link (i,j) is used by the primary and (k,l) by the backup path. This enables us to use z_{ij}^{kl} instead of $x_{ij}y_{kl}$ in the objective, hence resulting in a linear formulation. Consequently, the objective function represents the joint failure probability based on the pair-wise (one from x and one from y) joint link failure.

While generally ILPs are difficult to solve, in this case we also observe that the constraints $CC(x)$ and $CC(y)$ are totally unimodular[3] (TU) [33], hence the linear program (LP) relaxation has an integral optimal solution [33]. Furthermore, we can use Lagrangian relaxation on the constraints $DC(x,y)$ and (C_1) to further simplify the problem. In particular, define the Lagrangian function as $L(x,y,z,\mu,\nu) = \sum_{(i,j)}(\mu_{ij} + \sum_{(k,l)} \nu_{ij}^{kl})x_{ij} + \sum_{(i,j)}(\mu_{ij} + \sum_{(k,l)} \nu_{kl}^{ij})y_{ij} + \sum_{(i,j),(k,l)}(p_{ij}p_{kl} - \nu_{ij}^{kl})z_{ij}^{kl}$, where μ and ν are Lagrangian multiplier vectors associated with $DC(x,y)$ and $(C1)$, respectively. The (Lagrangian) relaxed problem is given by

$$\begin{aligned} (\mathbf{P1.2LR}): \quad & \min_{x,y,z} \quad L(x,y,z,\mu,\nu) \\ & \text{subject to} \quad CC(x), CC(y). \end{aligned}$$

The above problem is TU, and so it can be solved by LP relaxation, which is polynomial-time solvable. Moreover, for given μ and ν, the problem (P1.2LR) is completely separable with respect to x, y, and z. Namely, the optimal x and y are shortest paths, respectively, and optimal z_{ij}^{kl} is obtained as: $z_{ij}^{kl} = 1$ if $\nu_{ij}^{kl} > p_{ij}p_{kl}$, and 0 otherwise. Now, the above optimization can be solved using a simple primal-dual method as described in Algorithm 2, where γ_m is a positive diminishing step size, M is the maximum number of iterations, and w_{ij} is the weight of link (i,j). Note that steps 2.1–2.3 solve the relaxed problem (P1.2LR), and steps 2.4–2.5 are the subgradient-based update of Lagrangian multipliers. The algorithm keeps the best path pair all over the iterations (step 2.6), and takes it as the final solution. Such a Lagrangian relaxation method for IP does not guarantee an optimal solution due to the duality gap, but it has been very successful in solving many IPs [34].

Algorithm 2 Lagrangian Relaxation: IND w/ DC

1: Initialization: $m = 0$, $\mu_{ij}(0) = p_{ij}$ and $\nu_{ij}^{kl}(0) = p_{ij}p_{kl}, \forall (i,j),(k,l) \in E$, and $p_{rmbest} = \infty$

[2]A graph is said to be k-connected if it remains connected after up to $k-1$ link (or node) failures. Equivalently, if every source–destination pair has at least k disjoint paths, then it is k-connected.

[3]A matrix A is said to be *totally unimodular* if the determinant of each square submatrix of A is 0, 1, or -1 (the network flow conservation matrices are TU). If the constraint matrix of an ILP is TU, then its linear program relaxation has an integral optimal solution, that is equivalent to the optimal solution of the ILP. Hence, the optimal solution of such an ILP can be obtained by solving its linear program relaxation, which is polynomial time solvable.

2: While $m < M$
  2.1 Set $w_{ij} = \mu_{ij}(m) + \sum_{(k,l)} \nu_{ij}^{kl}(m), \forall(i,j) \in E$; Find shortest path $x(m)$
  2.2 Set $w_{ij} = \mu_{ij}(m) + \sum_{(k,l)} \nu_{kl}^{ij}(m), \forall(i,j) \in E$; Find shortest path $y(m)$
  2.3 $z_{ij}^{kl}(m) = \begin{cases} 1, & \text{if } \nu_{ij}^{kl}(m) > p_{ij}p_{kl} \\ 0, & \text{otherwise} \end{cases}, \forall(i,j),(k,l)$
  2.4 $\mu_{ij}(m+1) = [\mu_{ij}(m) + \gamma_m(x_{ji}(m) + y_{ij}(m) - 1)]^+$
  2.5 $\nu_{ij}^{kl}(m+1) = [\nu_{ij}^{kl}(m) + \gamma_m(x_{ji}(m) + y_{kl}(m) - z_{ij}^{kl}(m) - 1)]^+, \forall(i,j),(k,l) \in E$
  2.6 Set $(x^{\text{best}}, y^{\text{best}}) = (x(m), y(m))$ if $p(m) < p_{\text{best}}$ where $p(m)$ is the joint failure prob. of $(x(m), y(m))$
  2.7 $m = m + 1$

Although the linear approximation in (P1.2L) may not be accurate in the high-probability regime, it is an upper bound on the original (nonlinear) objective function.

Lemma 4: The objective function in (P1.2L) is an upper bound on the joint path failure probability $F_1(p,x)F_1(p,y)$.

See Appendix D. ■

Therefore, the linearization approach may work as well in other regimes including the high-probability regime. This is verified through simulations in Section V.

C. Path-Pair Problem Without Disjointness Constraint

The disjointness constraint is a necessary condition for surviving a single link failure in the traditional deterministic failure model. However, in a probabilistic model, a link may be shared if it is known that the link is very reliable. If link (i,j) is shared by x and y, then (i,j)'s failure leads to the simultaneous failure of both x and y. Hence, the probability that both x and y fail can be written as

$$F_{\text{S}}(p,x,y) + (1 - F_{\text{S}}(p,x,y))\, F_{\text{NS}}(p,x,y) \tag{5}$$

where $F_{\text{S}}(p,x,y)$ is the probability that both x and y fail due to a *shared link failure*, and $F_{\text{NS}}(p,x,y)$ is the probability that both x and y fail due to the *failure of nonshared links*. For path pair (x,y), let E_{xy} denote the set of links shared by x and y, i.e., $E_{xy} = \{(i,j) \in E : x_{ij} = 1, y_{ij} = 1\}$. Then, the probability $F_{\text{S}}(p,x,y)$ can be written as

$$\begin{aligned} F_{\text{S}}(p,x,y) &= 1 - \prod_{(i,j)\in E_{xy}} (1 - p_{ij}) \\ &= 1 - \prod_{(i,j)\in E} (1 - p_{ij}x_{ij}y_{ij}). \end{aligned} \tag{6}$$

For a binary vector v, define its complement as $\bar{v} = \vec{1} - v$ where $\vec{1}$ is a vector of 1's with appropriate dimension. Then, the vector $\bar{y}$ only includes the links which are not selected by y. Hence, the probability that x fails due to the failure of nonshared links is equivalent to the probability that both x and $\bar{y}$ fail due to the failure of the links shared by x and $\bar{y}$. This probability can be subsequently expressed as $F_{\text{S}}(p,x,\bar{y})$ following to the definition $F_{\text{S}}(p,x,y)$. Similarly, $F_{\text{S}}(p,\bar{x},y)$ denotes the probability that y fails due to the failure of nonshared links. The probability $F_{\text{NS}}(p,x,y)$ is then given by $F_{\text{NS}}(p,x,y) = F_{\text{S}}(p,x,\bar{y})F_{\text{S}}(p,\bar{x},y)$, leading to the following formulation:

$$\begin{aligned} (\mathbf{P1.3}): \quad & \min_{x,y} \; F_{\text{S}}(p,x,y) + (1 - F_{\text{S}}(p,x,y))\, F_{\text{NS}}(p,x,y) \\ & \text{subject to } \; CC(x), CC(y). \end{aligned}$$

The problem (P1.3) has an equivalent [if disjoint paths are optimal in (P1.3)] or better optimal solution compared to the problem (P1.2).

Under the low-failure-probability regime, we can approximate the problem (P1.3) by an ILP as follows:

$$\begin{aligned} (\mathbf{P1.3L}): \quad & \min_{x,y,z,\tilde{z}} \; \sum_{(i,j)} p_{ij}\tilde{z}_{ij} + \sum_{(i,j)}\sum_{(k,l)} p_{ij}p_{kl}z_{ij}^{kl} \\ & \text{subject to } \; CC(x), CC(y) \\ & \quad (C1) \quad \tilde{z}_{ij} \geq x_{ij} + y_{ij} - 1 \;\; \forall(i,j) \in E \\ & \quad (C2) \quad z_{ij}^{kl} \geq x_{ij} - y_{ij} + y_{kl} - x_{kl} - 1 \\ & \qquad\qquad \forall(i,j),(k,l). \end{aligned}$$

In constraint $(C1)$, $\tilde{z}_{ij} = 1$ only if $x_{ij} = y_{ij} = 1$, which means that link (i,j) is shared. Hence, the first term in the objective function is the joint failure probability due to the failure of shared links. In constraint $(C2)$, $\tilde{z}_{ij}^{kl} = 1$ only if $x_{ij} = y_{kl} = 1$ and $y_{ij} = x_{kl} = 0$, which means links (i,j) and (k,l) are respectively used by x and y, but neither of them are shared. Hence, the second term is the joint failure probability due to the failure of nonshared links.

The formulation (P1.3L) is a standard ILP that can be solved using an ILP solver such as CPLEX. However, it can take unacceptably long to run because it is NP-complete in general. Inspired by the approximation (P1.3L), we propose a simple greedy algorithm, as shown in Algorithm 3. Similar to Algorithm 1, step 1 finds a shortest path for x using link weights $w_{ij} = p_{ij}\ \forall(i,j)$, and this gives a primary path x with minimum failure probability. For the backup path y, the weight of each link (i,j) is set to the joint path failure probability due to the failure of link (i,j) and the links in x. Hence, two different cases have to be considered. First, if link (i,j) has not been selected by the primary path x, then its weight is set to the product of (i,j)'s failure probability p_{ij} and the (approximated) failure probability $\sum_{(k,l)} p_{kl}x_{kl}$ of path x. If (i,j) was selected by x, then (i,j)'s failure leads to joint path failure (provided that y also selects (i,j)), and so its weight is set to p_{ij}. The shortest path under these link weights will minimize the joint path failure probability and will be used as the backup path y. Note that if link (i,j) is to be shared, its weight is set to a first-order value, i.e., p_{ij}, which is obviously larger than the second-order weight in the nonshared case. Hence, the links with relatively low failure probability will be more likely to be shared.

Algorithm 3 Greedy: IND w/o DC

1: Set $w_{ij} = p_{ij}, \forall(i,j) \in E$; Find shortest path x
2: $w_{ij} = \begin{cases} p_{ij}\sum_{(k,l)} p_{kl}x_{kl}, & \text{if } x_{ij} = 0 \\ p_{ij}, & \text{if } x_{ij} = 1 \end{cases}, \forall(i,j) \in E$; Find shortest path y

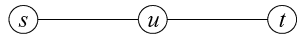


Fig. 3. Example topology.

D. Extension to Correlated Failures

As discussed in the introduction, many failure scenarios involve multiple links. Hence, link failure events may be correlated. In order to account for such correlation, the path failure probability expressions must include conditional probabilities for joint link failures. Let $p_{ij|kl}$ be the probability of link (i,j)'s failure given (k,l)'s failure, then the joint link failure probability is given by $p_{kl}p_{ij|kl}$ or $p_{ij}p_{kl|ij}$. Note that high value of conditional probability $p_{ij|kl}$ implies strong correlation between the failures of (i,j) and (k,l). Consider an example topology in Fig. 3, then the path s–t's failure probability can be expressed as

$$p_{su} + p_{ut} - p_{su}p_{ut|su}. \tag{7}$$

Generally, it can be written as

$$\begin{aligned} &\sum_{(i,j)\in E(s,t)} p_{ij} - \sum_{(i,j)\in E(s,t)} \sum_{\substack{(k,l)\in E(s,t),\\ (k,l)\neq(i,j)}} p_{ij}p_{kl|ij} \\ &\quad + \sum_{(i,j)\in E(s,t)} \sum_{\substack{(k,l)\in E(s,t),\\ (k,l)\neq(i,j)}} \sum_{\substack{(m,n)\in E(s,t),\\ (m,n)\neq(i,j)\\ (m,n)\neq(k,l)}} p_{ij}p_{kl|ij}p_{mn|ij,kl} \\ &\quad - \cdots + \text{high order terms} \end{aligned} \tag{8}$$

where $E(s,t)$ is the set of links on s–t path. These conditional probability expressions involve an exponential number of terms accounting for the joint failure probability of multiple links. Hence, formulating the above problems under correlated failures seems to be intractable. Due to this difficulty, [25] considers only the first-order correlation, i.e., conditional probability for every pair of links.

In order to better account for correlated failures, we propose a new model using *probabilistic SRLGs*. In our model, once an SRLG failure event occurs, its associated links fail with some probabilities. Thus, the link failures are correlated only if the links belong to the same SRLG (while in the link-wise model, the correlation is considered between every pair of links). Moreover, under the condition that an SRLG event occurs, its associated link failures are mutually independent, and thus the formulations developed in the independent model can be used. This enables a simple formulation for the path protection problems with correlated failures. More importantly, it can be used to model most correlated failure scenarios, as events leading to failures can be modeled as a probabilistic SRLG (PSRLG).

IV. PSRLG-Based Correlated Failure Model

We consider a single SRLG model where only one SRLG failure event can take place at a time. Let π_r be the probability that the failed SRLG is $r \in R$, then we will have $\sum_{r\in R}\pi_r = 1$. We refer to this model as the *mutually exclusive PSRLGs*. Note that the traditional deterministic SRLG model also assumes a single SRLG failure, and so our model of mutually exclusive PSRLGs is a probability-wise generalization of the traditional model.

A. Single-Path Problem

Again, we start by considering a single-path problem. Given that SRLG event r happens, each link (i,j) will fail with probability p_{ij}^r as if they are independent. Hence, the failure probability of path x is given by $1-\prod_{(i,j)}(1-p_{ij}^r x_{ij})$, and according to the definition of F_1 in Section III, this probability can be denoted by $F_1(p^r, x)$, where $p^r = [p_{ij}^r, \forall(i,j)\in E]$. The single-path problem can be simply written as

$$\begin{aligned} (\mathbf{P2.1}):\quad &\min_{x\in B_{|E|}} \sum_{r\in R}\pi_r F_1(p^r,x) \\ &\text{subject to}\quad CC(x). \end{aligned}$$

Note that the path failure probability is averaged over all SRLGs because they are mutually exclusive. As shown in Section III-A, the single-path problem under independent failures is an easy shortest-path problem. However, the same problem in the correlated failures case [i.e., (P2.1)] can be shown to be difficult.

Theorem 2: The single-path problem (P2.1) is NP-complete.

Proof: This is proved by showing that the problem (P2.1) contains as a special case a Minimum-Color Single-Path (MCSiP) Problem that is NP-complete [35]. The MCSiP problem is stated as follows. Given network graph $G=(V,E)$ and set of colors $C=\{c_1,c_2,\ldots,c_K\}$, each edge is colored with one of the colors in C. The problem is to find a path from s to t that touches the minimum number of colors. Note that it assumes monochromatic edges (i.e., single color to each edge), but this assumption can be easily relaxed by graph transformation. Namely, an edge with k colors is replaced by k serial edges (and $k-1$ intermediate nodes), each of which is associated with a different color from the set of k colors. Hence, the MCSiP problem without the monochromatic assumption is also NP-complete.

First, assume uniform failure probabilities, i.e., $\pi_r = (1/|R|), \forall r$ and $p_{ij}^r = p$ for all $(i,j)\in r$ such that $p_{ij}^r > 0$. Then, using the identity $\min f(x) \equiv \max -f(x)$, the objective in (P2.1) can be rewritten as

$$\begin{aligned} &\min_x \sum_{r\in R}\pi_r\left(1-\prod_{(i,j)\in E}\left(1-p_{ij}^r x_{ij}\right)\right) \\ &= \min_x \sum_{r\in R}\frac{1}{|R|}\left(1-\prod_{(i,j)\in r}(1-px_{ij})\right) \\ &\Leftrightarrow \max_x \sum_{r\in R}\prod_{(i,j)\in r}(1-px_{ij}). \end{aligned} \tag{9}$$

Furthermore, assume $p=1$. Note that this case corresponds to the traditional SRLG model that assumes deterministic failures. Under this assumption, the above objective function value represents the number of untouched SRLGs. In other words, it finds a path that touches the minimum number of SRLGs. If SRLG is replaced by color, it is a Minimum-Color Single-Path Problem (without monochromatic assumption). This shows that the problem (P2.1) contains an NP-complete problem, implying that it is NP-complete. ■

Due to this difficulty, its approximation is again considered. Under the low-failure-probability assumption (i.e., $p_{ij}^r \ll 1, \forall r, \forall (i,j)$), the objective function can be written as

$$\sum_{(i,j)} \left(\sum_{r \in R} \pi_r p_{ij}^r \right) x_{ij}. \tag{10}$$

We first begin by considering the simple case of uniform SRLG failure probability. Namely, let $\pi_r = 1/|R|, \forall r \in R$ and $p_{ij}^r = p, \forall r \in R(i,j), \forall (i,j) \in E$, where $R(i,j) = \{r \in R : p_{ij}^r > 0\}$.

Observation 3: Under the uniform failure probability, minimizing the objective function (10) is equivalent to minimizing $\sum_{(i,j)} |R(i,j)| x_{ij}$, where each term $|R(i,j)|$ in the summation is the number of SRLGs to which link (i,j) belongs. This shows that the path touching the minimum number of SRLGs has the lowest failure probability.

The traditional SRLG model falls into a special case of the uniform failure probability with $q = 1$, and hence, Observation 3 implies that in the traditional model, the number of SRLGs touched by a link is an important link-weight metric when finding a reliable path. In fact, some previous works have used this metric [10], and our result provides a theoretical basis for those works in traditional SRLG model.

B. Path-Pair Problem With Disjointness Constraint

The path-pair problem can also be formulated in a simple form as in the case of single path. Once an SRLG event r occurs, paths x and y will fail with probabilities $F_1(p^r, x)$ and $F_1(p^r, y)$, respectively. In this case, their joint failure probability is given by the product $F_1(p^r, x) F_1(p^r, y)$ because the link failures are independent, under the condition that SRLG event r has occurred. The problem can be formulated as follows:

$$(\mathbf{P2.2}): \quad \min_{x,y} \quad \sum_{r \in R} \pi_r F_1(p^r, x) F_1(p^r, y)$$
$$\text{subject to} \quad CC(x), CC(y), DC(x,y).$$

Our probabilistic SRLG model has enabled us to express the joint failure probability in a product form leading to a simple formulation. Namely, the objective function in (P2.2) is the combination of the objective functions in (P1.2) and (P2.1). That is, for given SRLG r, the joint path failure probability is equivalent to the joint path failure probability with link failure probability vector p^r in the independent model, and those joint failure probabilities are averaged over all SRLGs, as done in (P2.1). This is in sharp contrast with the link-wise correlated failure model where the path failure probability would include terms of the conditional probabilities involving all the combinations of link failures.

It is obvious that the path-pair problem is harder than the single-path problem, and thus we can infer from Theorem 2 that the problem (P2.2) will also be difficult. In fact, we can show its NP-completeness as well.

Theorem 3: The path-pair problem (P2.2) is NP-complete.

Proof: First, note that the objective value of (P2.2) is nonnegative, and so if any path pair results in zero objective value, then it is optimal. The probability $F_1(p^r, x)$ in (P2.2) can be written as

$$\begin{aligned} F_1(p^r, x) &= 1 - \prod_{(i,j)} \left(1 - p_{ij}^r x_{ij}\right) \\ &= 1 - \prod_{(i,j) \in r} \left(1 - p_{ij}^r x_{ij}\right) \end{aligned} \tag{11}$$

because $p_{ij}^r = 0$ if link (i,j) does not belong to SRLG r, i.e., $(i,j) \notin r$. Consequently, the function $F_1(p^r, x)$ becomes zero for the path x, which does not touch SRLG r, i.e., $x_{ij} = 0, \forall (i,j) \in r$. Hence if x and y do not share any SRLG, then the product $F_1(p^r, x) F_1(p^r, y)$ will be zero for every $r \in R$, thereby leading to zero objective value. This implies that any pair of SRLG-disjoint paths is an optimal solution to the problem (P2.2). Subsequently, the problem (P2.2) becomes an SRLG-disjoint paths problem if one exists. Therefore, the problem (P2.2) is NP-complete because it includes (as a special case) the SRLG-disjoint paths problem, which is NP-complete [15]. ■

Again, it is easy to show that when the link failure probabilities are low, the objective function of (P2.2) can be expressed as

$$\sum_{(i,j)} \sum_{(k,l)} \left(\sum_{r \in R} \pi_r p_{ij}^r p_{kl}^r \right) x_{ij} y_{kl}. \tag{12}$$

Next, we observe that the problem is still NP-complete even after the approximation.

Observation 4: Under the uniform failure probability (i.e., $\pi_r = 1/|R|, \forall r \in R$ and $p_{ij}^r = q, \forall r \in R(i,j), \forall (i,j) \in E$), the objective function (12) becomes $\sum_{(i,j)} \sum_{(k,l)} |R(i,j) \bigcap R(k,l)| x_{ij} y_{kl}$, where each term in the summation represents the number of SRLGs shared by corresponding link pair in x and y. This obviously contains SRLG-disjoint paths problem as a special case (if there exist SRLG-disjoint paths), and so it is NP-complete (following to the proof of Theorem 3). Subsequently, the approximated problem (12) is also NP-complete.

As this approximation is still difficult to solve, we propose a heuristic in Algorithm 4 using the approximations (10) and (12). For primary path x, we set the link weights and find the shortest path according to (10). This will give a path having minimum failure probability. Then, all the links selected by x are removed for disjointness. Finally, the obtained primary x is substituted into (12), and the backup path y is computed by minimizing (12) for fixed x. Hence, the greedy algorithm takes into account the correlation so that the links highly correlated with the selected links can be avoided. One can also develop a Lagrangian relaxation based algorithm by linearizing the problem. However, this development is nearly identical to that in Section III-B2 and omitted for brevity.

Algorithm 4 Greedy: MES w/ DC

1: Set $w_{ij} = \sum_{r \in R} \pi_r p_{ij}^r, \forall (i,j) \in E$
2: Find shortest path x
3: Remove all the links used by x
4: Set $w_{ij} = \sum_{(k,l)} x_{kl} \sum_{r \in R} \pi_r p_{ij}^r p_{kl}^r, \forall (i,j) \in E$
5: Find shortest path y

The path-pair problem without DC was also studied, but omitted for brevity.

V. Performance Evaluation

In this section, we evaluate and compare the performance of the algorithms developed in this paper. In particular, we consider the following four algorithms:

- the brute-force solution to the ILP formulations using the CPLEX solver (denoted by CPLEX;
- the Lagrangian relaxation for the ILP (Algorithm 2; denoted by LR);
- the greedy algorithms that select the first path with minimum failure probability and the second path with adjusted link weights to reduce the joint path failure probability (i.e., Algorithms 1, 3, and 4; denoted by Greedy);
- the shortest-disjoint-paths algorithm that finds a pair of disjoint paths with minimum total weight, where the weight of a link is its failure probability (i.e., for each link (i,j), $w_{ij} = p_{ij}$ in the independent model or $w_{ij} = \sum_r \pi_r p_{ij}^r$ in the PSRLG model). This algorithm is a straightforward approach simply selecting a shortest path pair, and as mentioned in Observation 2, such a pair does not necessarily have minimum joint failure probability. Note that this joint-shortest-path approach is in contrast with our heuristic, which is one-by-one shortest-path approach. (denoted by SDP)

The protection quality (i.e., joint path failure probability) and run-time of the above algorithms will be compared.

First, we compare the ILP (P1.1L) and probability-wise shortest path (PSP) algorithm that finds a shortest path under link weights $w_{ij} = p_{ij}, \forall(i,j)$. As discussed in Theorem 1 and Observation 1, the ILP finds a path with minimum failure probability while the PSP algorithm approximates the optimal path in the low-failure-probability regime. Because the PSP algorithm is used in our heuristics (Algorithms 1, 3, and 4) to find a path with minimum failure probability, this comparison will demonstrate the suitability of the PSP algorithm in our heuristics. We generated 100 random graphs, each of which has 10 nodes and maximum node degree of 5, and is 3-connected[4] from s to t. To avoid the trap problem discussed in Section III, we assume that the graphs are bidirectional. In each graph, the failure probability of each link (i,j) is assigned as follows:

$$p_{ij} = \alpha\left(\beta + (1-\beta)u\right)$$

where α and β are constants in [0,1], and u is a random number uniformly distributed on the interval (0,1). Note that as β increases, the network approaches to the uniform-failure-probability regime (i.e., $p_{ij} = q, \forall(i,j)$). For example, if $\beta = 1$, it will be $p_{ij} = \alpha, \forall(i,j)$, which implies uniform-failure-probability regime. In contrast, if $\beta = 0$, p_{ij} will be a random number from $(0,\alpha)$. On the other hand, small α corresponds to the low-failure-probability regime and large α to the high-failure-probability regime.

Fig. 4 plots the path failure probability for each combination of (α,β), where each point is the average of the results of 100 random graphs. As expected in Observation 1, the PSP algorithm finds an optimal path in the uniform or low-failure-

[4]In our simulation, once a graph is generated, we examine the 3-connectedness of the graph, and if it is not 3-connected, it is discarded.

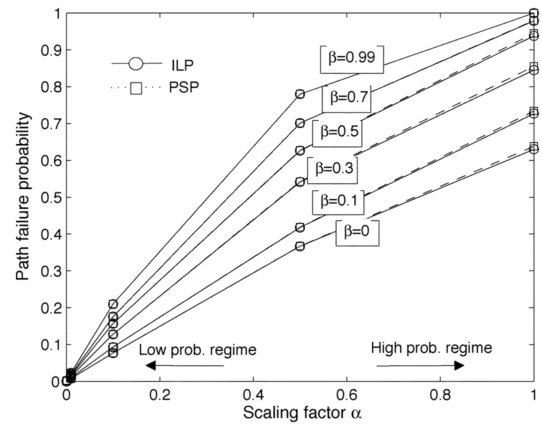


Fig. 4. Comparison of ILP and probability-wise shortest path (PSP).

probability regime (large β or small α, respectively). Furthermore, even in high-probability-regime (large α), the PSP approximates the ILP very well. When the network is nearly in the uniform-probability regime (large β), shortest-hop path would be optimal, and the PSP obviously finds this shortest path. With small β, the network is in a mixed regime having high and low failure probabilities. In this case, both the ILP and PSP would select only the links with low failure probability whenever feasible. Then, it is highly likely that the same path is optimal after the links with high failure probability are removed. This is equivalent to being in the low-probability regime, and therefore, the PSP performs comparably to the ILP. Overall, the PSP algorithm finds an optimal path in most cases, as desired.

Next, we consider the problem of finding the path pair with minimum joint failure probability. The proposed greedy and LR-based algorithms are compared with CPLEX and the SDP algorithm using various topologies. The CPLEX solves the ILP version of every path-pair problem. For LR-based iterative algorithms, the following parameters are used: maximum iteration number $M = 2 \times 10^4$ and step size $\gamma_m = 10^{-9}/\sqrt{m}$. The comparison is performed by changing the number of nodes, and 100 random geometric graphs are generated for each case. In a random geometric graph, N nodes are randomly located on the $\sqrt{N} \times \sqrt{N}$ plane. Two nodes are connected if the distance from one to the other is less than 0.5. As in the above, each graph is 3-connected (from s to t) with maximum node degree of 5. For each topology, we consider only a single s–t pair and average these values over 100 different topologies. For the independent model, the failure probability of each link is set to a random number uniformly distributed on the interval $(0, 10^{-3})$, hence $p_{ij} \in (0, 10^{-3}), \forall(i,j) \in E$. For the PSRLG model, 20 SRLGs are generated for every graph, and their failure event probabilities π_r's are set to uniformly distributed random numbers such that $\sum_r \pi_r = 1$. Each SRLG is associated with a circle whose center is randomly located on the plane and whose radius is a uniformly distributed random number in (1,1.5). An SRLG includes all the links touched by its circle. Once a link, say (i,j), is included in SRLG r, its failure probability p_{ij}^r is set to a uniformly distributed random number in $(0, 10^{-3})$ for low-probability regime and in (0.5,1) for high-probability regime.

Fig. 5 plots the joint path failure probability achieved by each algorithm in the independent model, with and without the disjointness constraints. Fig. 5(a) shows that with the DC,

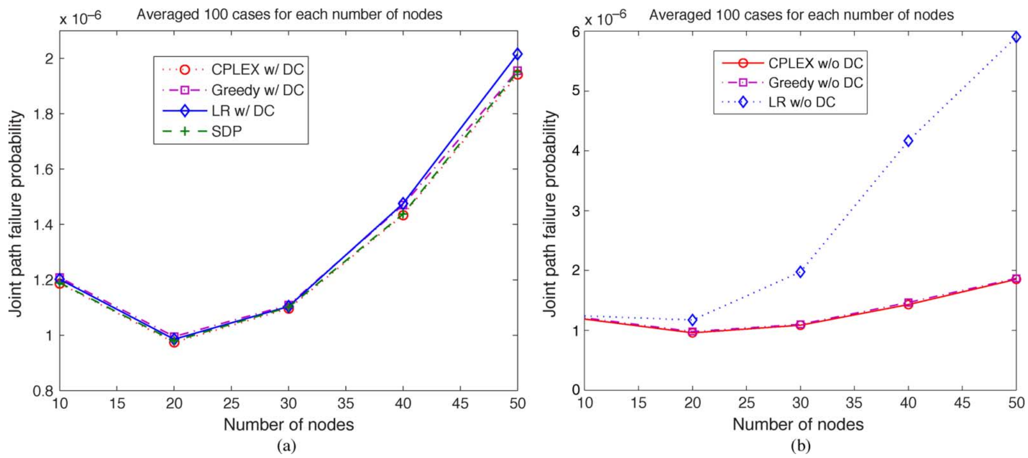


Fig. 5. Joint path failure probability in independent model. (a) With disjointness constraint (DC). (b) Without DC.

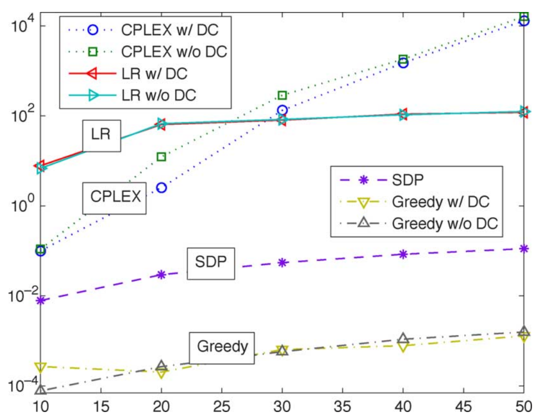


Fig. 6. Run-time (on DELL Workstation T7400) in independent model.

the CPLEX always finds the best path pair, while the other algorithms achieve nearly the same protection performance as the CPLEX. Without DC, the CPLEX still performs better than any other, whereas the performance of Lagrangian relaxation (LR)-based algorithm is substantially degraded as the number of nodes increases [see Fig. 5(b)]. It is remarkable that our greedy algorithm that does not explicitly attempt to solve the optimization problem performs as well as the CPLEX solution that finds a path pair by solving the optimization problem. This verifies our observation in Section III-B that for path protection against probabilistic failures, it is important to include the best path in the primary and backup path pair. Notice that the LR-based algorithm performs well with DC. However, without DC, the LR-based algorithm does not perform very well, especially in large networks [see Fig. 5(b)]. Clearly, the search space without DC is much larger than that with DC. This implies that in large networks, it requires many more iterations in order to find a better solution. This is part of the reason why the LR-based algorithm does not work well in large networks.

The run-time of each algorithm is shown in Fig. 6. As the number of nodes increases, the run-time of CPLEX increases exponentially. This shows that CPLEX takes a brute-force approach having exponential run-time and, hence, may be prohibitively complex. The LR algorithm also takes a long time, but its run-time increases much more slowly than CPLEX. On the other hand, the greedy and SDP find a path pair in minimal time, almost independent of the problem size. Therefore, both the greedy and the SDP algorithms find a fairly reliable pair of paths with short run-times.

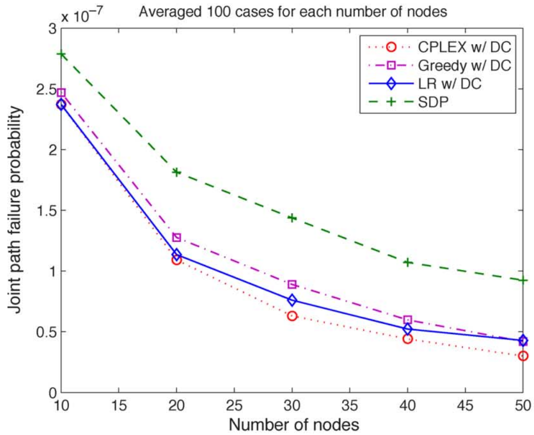
The joint path failure probabilities in the PSRLG model are shown in Fig. 7. As in the independent model, the CPLEX always finds the most reliable pair of paths in the low-failure-probability regime [see Fig. 7(a)]. Observe from Fig. 7(b) that the CPLEX performs as well in the high-probability regime. As discussed earlier, this is because the linear approximation solved by CPLEX is an upper bound on the joint path failure probability. Fig. 7 also shows that our greedy algorithm provides better protection than the SDP. This is due to the fact that our greedy algorithm adjusts the failure probability before selecting the second path in order to reduce the joint failure probability while the SDP algorithm fails to take correlation into account.

As mentioned in Section III-C, relaxing the DC should improve the protection quality (if nondisjoint path pair is optimal). Fig. 8 shows that the joint failure probability is decreased by relaxing the DC. Observe further that our greedy algorithm finds a more reliable path pair than SDP, again verifying our observation in Section III-B that it is important to include the best path in the pair.

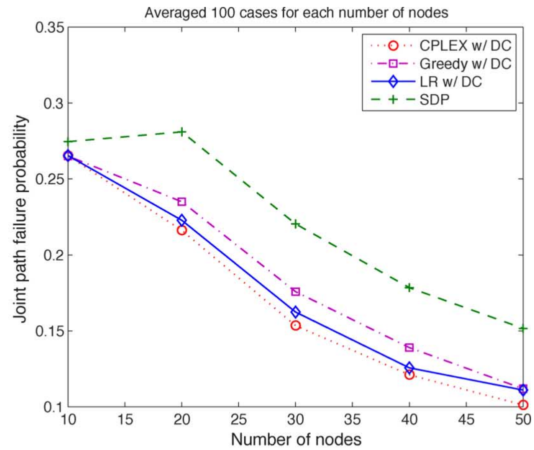
We further examine the algorithms using a U.S. IP network topology shown in Fig. 9. As in the previous simulations, the link failure probability p_{ij} under the independent model is uniformly distributed on the interval $(0, 10^{-3})$. For the PSRLG model, the nodes are located on the plane as seen in Fig. 9. As in the previous simulations, $|R|(= 5, 10, 20)$ SRLGs are randomly located on the plane and include the links touched by their circles of radius in (1,2). The link failure probability p_{ij}^r is uniformly distributed on the interval (0.5,1). Under this assumption, 100 realizations of the SRLGs and failure probabilities are generated. In each realization, 100 source–destination pairs are randomly selected to find their disjoint primary and backup paths. The joint failure probabilities are averaged over all realizations and source–destination pairs. Table I shows that the greedy algorithm works as well in the U.S. IP network topology.

VI. Conclusion

In this paper, we studied path protection problems in a network with multiple, possibly correlated, failures. In such



(a)



(b)

Fig. 7. Joint path failure probability in PSRLG model (with disjoint constraint). (a) Low-probability regime: $p_{ij}^r \in (0, 10^{-3})$. (b) High-probability regime: $p_{ij}^r \in (0.5, 1)$.

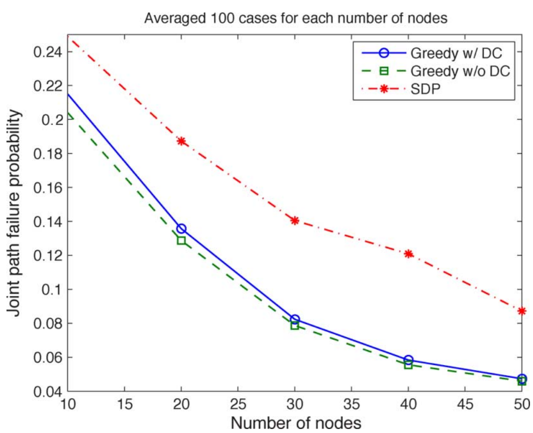


Fig. 8. Improvement of protection quality by relaxing disjointness constraint: PSRLG model with 20 SRLGs and $p_{ij}^r \in (0,1), \forall (i,j) \in r, \forall r \in R$. The average numbers of links in an SRLG are [6.8, 17.1, 26.9, 32.5, 42.3] for $N = 10, 20, 30, 40, 50$, respectively.

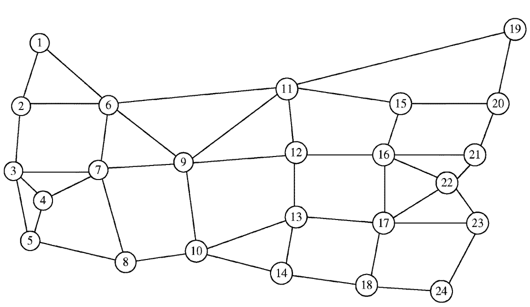


Fig. 9. U.S. IP network [36].

a network, protection cannot be guaranteed by simply providing disjoint paths, and thus we sought to find diverse routes that maximize reliability, i.e., have the minimum joint failure probability. To that end, we first developed a probabilistic SRLG (PSRLG) framework by generalizing the traditional notion of SRLG. Under this model, given an SRLG failure, links belonging to that SRLG fail independently; significantly simplifying the computation of the joint failure probability between two paths. This enables a simple formulation to the path protection problem under correlated failures that would be otherwise intractable. Using this model, we formulated the path protection problem of finding a pair of paths with minimum joint failure probability as an integer nonlinear program (INLP).

Furthermore, using linear approximations, we transformed the INLP to an ILP and developed algorithms for finding a pair of paths with minimum joint failure probability. Finally, we showed through simulations that our heuristic algorithms often find a better path pair and require less run-time than a brute-force solution using an ILP solver such as CPLEX.

TABLE I
JOINT PATH FAILURE PROBABILITIES IN A U.S. IP NETWORK: P_{inc} IS THE PROBABILITY THAT EACH LINK IS INCLUDED IN EACH SRLG

	Independent	PSRLG		
		$\lvert R\rvert = 5$	$\lvert R\rvert = 10$	$\lvert R\rvert = 20$
CPLEX	3.73×10^{-6}	0.0446	0.0377	0.0425
Greedy	3.81×10^{-6}	0.0454	0.0380	0.0456

APPENDIX A
PROOF OF LEMMA 1

Under the uniform failure probability, the objective function in (P1.2) is written as

$$f(X,Y) = 1 - (1-p)^X - (1-p)^Y + (1-p)^{X+Y} \quad (13)$$

where $X = \sum_{(i,j)\in E} x_{ij}$ and $Y = \sum_{(i,j)\in E} y_{ij}$. The lemma can be proved by showing the concavity of the above objective function. We use the following lemma for the proof.

Lemma 5 [37]: Let A be the Hessian of $f(X,Y)$. Then, $f(X,Y)$ is concave if $A_{11} \leq 0$, $A_{22} \leq 0$, and $A_{11}A_{22} - A_{12}A_{21} \geq 0$.

The Hessian of $f(X,Y)$ is given by

$$e^{(X+Y)\log(1-p)} \log^2(1-p) \times \begin{bmatrix} 1 - e^{-Y\log(1-p)} & 1 \\ 1 & 1 - e^{-X\log(1-p)} \end{bmatrix}.$$

Since $p < 1$, $X > 0$, and $Y > 0$, we have $1 - e^{-Y\log(1-p)} < 0$ and $1 - e^{-X\log(1-p)} < 0$, which satisfy the concavity condition on the diagonal components. The condition on the determinant can be written as $e^{X\log(1-p)} + e^{Y\log(1-p)} \leq 1$. This is also satisfied because

$$e^{X\log(1-p)} + e^{Y\log(1-p)} \leq 2e^{c\log(1-p)} \leq 1$$

where the first inequality is due to $X \geq c$ and $Y \geq c$, and the second inequality is due to the assumption $p \geq 1 - 2^{-(1/c)}$. By Lemma 5, the objective function $f(X,Y)$ is concave in X and Y. This shows that the original objective function

$$1 - (1-p)^{\sum_{(i,j)} x_{ij}} - (1-p)^{\sum_{(i,j)} y_{ij}} + (1-p)^{\sum_{(i,j)} x_{ij}+y_{ij}}$$

is also concave since the convexity and concavity are preserved under the composition with a linear mapping [38]. Therefore, the problem (P1.2) is a concave maximization.

Appendix B
Proof of Lemma 2

For the proof, we need the following lemma on the Schur concavity criterion.

Lemma 6 (Schur's Criterion): Given that the function $f : (a,b)^n \to R$ is continuously differentiable and symmetric, it is Schur concave on $(a,b)^n$ if and only if

$$(x_j - x_k)\left(\frac{\partial f(x)}{\partial x_j} - \frac{\partial f(x)}{\partial x_k}\right) \leq 0$$

for all $1 \leq j < k \leq n$ and all $x \in (a,b)^n$.

Obviously, the function $f(X,Y)$ is differentiable and also symmetric because the function value remains unchanged under the exchange of X and Y. To apply this criterion to the function $f(X,Y)$, we write

$$(X-Y)\left(\frac{\partial f(X,Y)}{\partial X} - \frac{\partial f(X,Y)}{\partial Y}\right) = -\log(1-p)(X-Y) \times \left((1-p)^X - (1-p)^Y\right) \leq 0.$$

According to Lemma 6, this shows that $f(X,Y)$ is Schur concave.

Appendix C
Proof of Lemma 3

Consider an s–t partition $S \subset V$ and $T = V \setminus S$ such that $s \in S$ and $t \in T$. Let $\delta^-(S)$ and $\delta^+(S)$ be the sets of links going out from S to T and T to S, respectively. That is

$$\delta^-(S) = \{(i,j) \in E : i \in S, j \in T\}$$
$$\delta^+(S) = \{(i,j) \in E : i \in T, j \in S\}.$$

Since the graph G is bidirectional, we have $|\delta^-(S)| = |\delta^+(S)|$. Consider a simple s–t path P. Then, it follows that

$$\left|\delta^-(S) \cap P\right| - \left|\delta^+(S) \cap P\right| = 1. \qquad (14)$$

Suppose $|\delta^-(S) \cap P| = l + 1$, that is, the path P touches $l+1$ links in $\delta^-(S)$. By (14), $|\delta^+(S) \cap P| = l$. This implies that at least l links in $\delta^-(S)$ are untouched by path P because it is simple and the graph is bidirectional. Consequently, if $l \geq 1$, at least one outgoing link remains after removing all the links in P. If $l = 0$, then $|\delta^-(S) \cap P| = 1$, i.e., only one outgoing link is removed. Since the graph is k-connected with $k \geq 2$, we have $|\delta^-(S)| \geq 2$. Consequently, in the case of $l = 0$, at least one outgoing remains untouched. Therefore, at least one link in $|\delta^-(S)|$ survives after removing a simple s–t path. This is true for any s–t partition S and T, and this completes the proof.

Appendix D
Proof of Lemma 4

For notational convenience, let x and y also denote the sets of links in the primary and backup paths, respectively. Let $\mathcal{E}_{ij}$ denote the event that link (i,j) fails. Since we are assuming independent failures, it follows that for $(i,j) \neq (k,l)$

$$\Pr[\mathcal{E}_{ij} \cap \mathcal{E}_{kl}] = \Pr[\mathcal{E}_{ij}]\Pr[\mathcal{E}_{kl}] = p_{ij}p_{kl}. \qquad (15)$$

Then, the joint failure probability can be written as

$$\begin{aligned} F_1(p,x)F_2(p,y) &= \Pr\left[\left(\bigcup_{(i,j)\in x} \mathcal{E}_{ij}\right) \bigcap \left(\bigcup_{(k,l)\in y} \mathcal{E}_{kl}\right)\right] \\ &= \Pr\left[\bigcup_{(i,j)\in x,(k,l)\in y} \left(\mathcal{E}_{ij} \bigcap \mathcal{E}_{kl}\right)\right] \\ &\leq \sum_{(i,j)\in x,(k,l)\in y} \Pr\left[\mathcal{E}_{ij} \bigcap \mathcal{E}_{kl}\right] \\ &= \sum_{(i,j)\in x,(k,l)\in y} p_{ij}p_{kl} \end{aligned}$$

where the inequality is due to the union bound, and the last equality is due to (15). This proves that the linear approximation is an upper bound on the joint failure probability.

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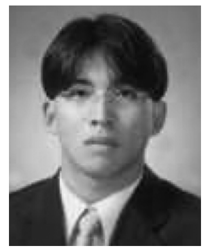

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