# Optimal Adaptive Data Transmission over a Fading Channel with Deadline and Power Constraints

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***Abstract*— We consider optimal rate control for energy-efficient data transmission over time-varying (fading) channels with strict deadline constraints. Specifically, the scenario consists of a transmitter with $B$ units of data that must be transmitted by deadline $T$ over a wireless channel. The transmitter can control the transmission rate over time by varying the transmission power subject to expected short-term power limits. The expended power depends on both the (chosen) transmission rate and the present channel condition and the objective is to adapt the rate over time and in response to the changing channel conditions so that the total energy cost is minimized. We present a novel continuous-time formulation of the problem; using stochastic control theory and lagrangian duality, we obtain explicitly the optimal rate control policy. We then present an illustrative simulation example comparing the energy costs of the optimal and the full power policies.**

## I. Introduction

Data services in modern communication systems are evolving from traditional email and web data transfers to more enhanced services such as video and real-time multimedia streaming [1], delay constrained data/file transfers, high throughput web access, and, Voice-over-IP (VoIP) which brings voice communication into the realm of data services (eg. 1xEV-DO, WiMAX). All these advancements require enhanced Quality of Service (QoS) which translates into stricter delay and throughput requirements on communication.

Communication over wireless channels adds another dimension of complexity associated with time-varying channel conditions and scarcity of resources. Among other resource limitations, energy consumption is an important concern in system design and is an active area of research in wireless networks [2]. Energy efficiency has numerous advantages in efficient battery utilization of mobile devices, increased lifetime of sensor and ad-hoc networks, and superior utilization of limited energy sources in satellites. As transmission energy constitutes the bulk of the communication energy expenditure, it is imperative to minimize this cost to achieve significant energy savings, henceforth, our focus in this paper will be on transmission energy cost. In this work, we address the above two issues under a specific setting and obtain optimal policies to transmit data with delay constraints over a fading channel.

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Modern wireless devices are equipped with channel measurement and rate adaptive capabilities [3]. Channel measurement allows the transmitter-receiver pair to measure the fade state using a pre-determined pilot signal while rate control capability allows the transmitter to adjust the reliable transmission rate over time. Such a control can be achieved in various ways that include adjusting the power level, symbol rate, coding rate/scheme, constellation size and any combination of these approaches; further, the receiver can detect these changes directly from the received data without the need for an explicit rate change control information [4]. With present technology, transmission rate can be adapted very rapidly in time over millisecond duration time-slots [3]. These capabilities, thus, provide a unique opportunity to utilize dynamic rate control algorithms to optimize system performance.

For a transmitter-receiver pair, the power-rate function defines the relationship that governs the amount of transmission power required to reliably transmit at a certain rate. Two fundamental aspects of this function, which are exhibited by most encoding/communication schemes and hence are common assumptions in the literature [5]–[10], are as follows. First, for a fixed bit error probability and channel state, the required transmission power is a convex function of the communication rate as shown in Figure 1(a). This implies (from a straightforward application of Jensen's inequality) that transmitting data at low rates over longer duration is more energy efficient as compared to high rate transmissions. Second, the wireless channel is time-varying which shifts the convex power-rate curves as a function of the channel state as shown in Figure 1(b). As good channel conditions require less transmission power, exploiting this variability over time by adapting the rate in response to the channel conditions leads to reduced energy cost. Thus, utilizing rate control capabilities and the above two aspects of power-rate curves, we can minimize energy cost while also satisfying delay constraints.

We consider a transmitter with $B$ units of data that must be transmitted by deadline $T$ over a wireless channel. The channel state (fading) is stochastic and modelled as a general Markov process. The transmitter can control the transmission rate over time by varying the transmission power subject to short-term power limits. The expended power depends on both the chosen transmission rate and the present channel condition

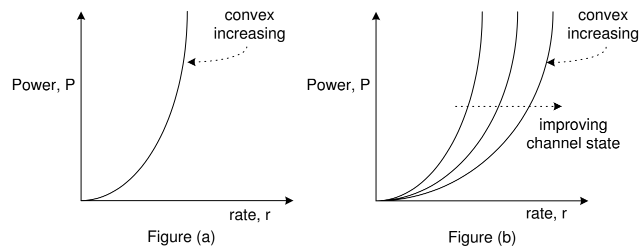


Fig. 1. Transmission power as a function of the rate and the channel state; (a) fixed channel state, (b) variable channel state.

(Figure 1(b)). The objective at the transmitter is to dynamically adapt the rate over time and in response to the changing channel conditions such that the transmission energy cost is minimized and the deadline constraint is met. We formulate the problem in continuous-time; using the stochastic control theory and lagrangian duality, we obtain simple explicit/closed-form formulas for the optimal transmission rate as a function of the amount of data remaining, the present time relative to the deadline and the present channel state.

Transmission power/rate control is an active area of research in communication networks in various different contexts. Adaptive network control and scheduling has been studied in the context of network stability [13], average throughput [14], average delay [5], [10] and packet drop probability [15]. This literature considers "average metrics" that are measured over an infinite time horizon and hence do not directly apply for delay constrained/real-time data. Further, transmission adaptation simply based on steady state distributions does not suffice and to consider strict deadlines one needs to take into account the system dynamics over time, thus, introducing new challenges and complexity into the problem. Recent work in this direction includes [6]–[9], [11], [12]. The work in [6] studied offline formulations under non-causal knowledge of future channel states and devised heuristic online policies using the optimal offline solution. The works in [8], [9], [11] studied formulations without fading and in particular our work in [11] used a calculus approach to obtain minimum energy policies with general arrival curves and QoS constraints. This paper is an extension of the work in [12] where we considered the same problem without explicit expected power constraints. The additional complexity arising due to the power limit constraints is addressed using a lagrangian duality approach.

## II. Problem Setup

We consider a continuous-time model of the system. Clearly, such a model is an approximation of the actual system but the assumption is justified since in practice the communication slot durations (on the order of 1 msec) are much smaller than packet delay requirements (on the order of 100's of msec); thus, one can view the system as virtually operating in continuous-time. Such a model makes the problem mathematically tractable and yields simple solutions. To apply the results obtained here, one would then simply discretize the solution as done for the simulations in Section IV.

### A. Transmission Model

Let $h_t$ denote the channel gain, $P(t)$ the transmitted signal power and $P^{rcd}(t)$ the received signal power at time $t$. We make the common assumption [5], [6], [8]–[10] that the required received signal power for reliable communication (with a certain fixed bit-error probability) is convex in the rate, i.e. $P^{rcd}(t) = g(r(t))$. Since the received signal power is given as $P^{rcd}(t) = |h_t|^2 P(t)$, the required transmission power to achieve rate $r(t)$ is given by,

$$P(t) = \frac{g(r(t))}{c(t)} \tag{1}$$

where $c(t) \triangleq |h_t|^2$ and $g(r)$ is a non-negative convex increasing function for $r \geq 0$. The quantity $c(t)$ is referred to as the *channel state* at time $t$. Its value at time $t$ is assumed known either through prediction or direct channel measurement but evolves stochastically in the future. It's worth emphasizing that the relationship in (1) includes much more generality than discussed above. For example, $c(t)$ could in fact represent a combination of stochastic variations in the system and (uncontrollable) interference from other transmitter-receiver pairs, as long as the power-rate relationship in (1) holds.

We further assume that $g(r)$ belongs to the class of monomial functions, namely, $g(r) = kr^n,\ n > 1, k > 0\ (n, k \in \mathbb{R})$. While this assumption restricts the theoretical generality of the problem, it serves several purposes. First, it leads to simple closed form solutions that can be applied in practice. Second, for most practical transmission schemes, $g(\cdot)$ is described numerically and its exact analytical form is unknown. In such situations, one can obtain the best approximation of that function to a monomial function and apply the results thus obtained. Third, monomials form the first step towards studying a polynomial $g(\cdot)$ which would then apply to a general $g(\cdot)$ using the polynomial expansion. Also, note that for values of $n$ close to 1, $g(r)$ models a linear power-rate curve which is a widely studied model. Finally, without loss of generality throughout the paper we take the constant $k = 1$, since, as will be evident, any other value of $k$ simply scales the problem without affecting the results.

### B. Channel Model

We consider a general continuous time discrete state space Markov model for the channel state process. Markov processes constitute a large class of stochastic processes that exhaustively model a wide set of fading scenarios and there is substantial literature on these models [16], [17] and their applications to communication networks [17], [18]. Denote the channel state process as $C(t)$ and the state space as $\mathcal{C}$. Let $c(t)$ denote a sample path and $c = c(t), c \in \mathcal{C}$ be a particular realization at time $t$. Starting from state $c$, let $\mathcal{J}_c$ be the set of all states ($\neq c$) to which the channel can transition when the state changes. Let $\lambda_{c\tilde{c}}$ denote the channel transition rate from state $c$ to $\tilde{c}$, then, the sum transition rate at which the channel jumps out of state $c$ is, $\lambda_c = \sum_{\tilde{c} \in \mathcal{J}_c} \lambda_{c\tilde{c}}$. Clearly, the expected time that $C(t)$ spends in state $c$ is $1/\lambda_c$ and one can view $\frac{1}{\lambda_c}$ as the coherence time of the channel in state $c$.

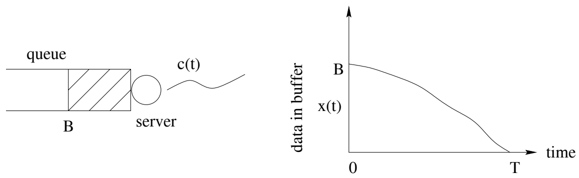


Fig. 2. Schematic description of the system.

Now, define $\lambda \triangleq \sup_c \lambda_c$ and a random variable, $Z(c)$, as,

$$Z(c) \triangleq \begin{cases} \tilde{c}/c, & \text{with prob. } \lambda_{c\tilde{c}}/\lambda,\ \tilde{c} \in \mathcal{J}_c \\ 1, & \text{with prob. } 1-\lambda_c/\lambda \end{cases} \quad (2)$$

With this definition, we obtain a compact description of the process evolution as follows. *Given a channel state c, there is an Exponentially distributed time duration with rate $\lambda$ after which the channel state changes. The new state is a random variable which is given as $C = Z(c)c$.* Clearly, from (2) the transition rate to state $\tilde{c} \in \mathcal{J}_c$ is unchanged at $\lambda_{c\tilde{c}}$, whereas with rate $\lambda - \lambda_c$ there are indistinguishable self-transitions. Note that there is no generality lost with this new description as it yields a stochastically identical scenario and the self-transitions are indistinguishable over any sample path. The representation simply helps in notational convenience

*Example:* Consider the standard Gilbert-Elliott channel model [17] that has two states $b$ and $g$ denoting the "bad" and the "good" channel conditions respectively. The two states correspond to a two level quantization of the channel gain. If the measured channel gain is below some value, the channel is labelled as "bad" and $c(t)$ is assigned an average value $c_b$, otherwise $c(t) = c_g$ for the good condition. Let the transition rate from the good to the bad state be $\lambda_{gb}$ and from the bad to the good state be $\lambda_{bg}$. Let $\gamma = c_b/c_g$, and using the earlier notation, $\lambda = \max(\lambda_{bg}, \lambda_{gb})$. For state $c_g$ we have,

$$Z(c_g) = \begin{cases} \gamma, & \text{with prob. } \lambda_{gb}/\lambda \\ 1, & \text{with prob. } 1-\lambda_{gb}/\lambda \end{cases} \quad (3)$$

To obtain $Z(c_b)$, replace $\gamma$ with $1/\gamma$ and $\lambda_{gb}$ with $\lambda_{bg}$ in (3).

### C. Problem Formulation

As mentioned earlier, the transmitter has $B$ units of data and a deadline $T$ by which the data needs to be sent. Let $x(t)$ denote the amount of data left in the queue and $c(t)$ be the channel state at time $t$. The system state can be described as $(x, c, t)$, where the notation means that at time $t$, we have $x(t) = x$ and $c(t) = c$. Let $r(x, c, t)$ denote the chosen transmission rate for the corresponding system state $(x, c, t)$. Since the underlying process is Markov, it's sufficient to restrict attention to transmission policies that depend only on the present system state [21]. Clearly then, $(x, c, t)$ is a Markov process. The system is depicted in Figure 2.

Given a policy $r(x, c, t)$, the system evolves in time as a Piecewise-Deterministic-Process (PDP) as follows. We are given $x(0) = B$ and $c(0) = c_0$. Until $\tau_1$, where $\tau_1$ is the first time instant after $t = 0$ at which the channel changes, the buffer is reduced at the rate $r(x(t), c_0, t)$. Hence, over the interval $[0, \tau_1]$, $x(t)$ satisfies the ordinary differential equation,

$$\frac{dx(t)}{dt} = -r(x(t), c_0, t) \quad (4)$$

Equivalently, $x(t) = x(0) - \int_0^t r(x(s), c_0, s)ds \ \ ,\ t \in [0, \tau_1]$. Now, starting from the new state $(x(\tau_1), c_1, \tau_1)$, the above procedure repeats until $t = T$ is reached. At time $T$, the data that missed the deadline (amount $x(T)$) is assigned a penalty cost of $\frac{\tau g(x(T)/\tau)}{c(T)}$ for some $\tau > 0$. This peculiar cost can be viewed in the following two ways. First, it simply represents a specific penalty function where $\tau$ can be adjusted and in particular made small enough[1] so that the data that misses the deadline is small. This will ensure that with good source-channel coding, the entire data can be recovered even if $x(T)$ misses the deadline. Second, note that $\frac{\tau g(x(T)/\tau)}{c(T)}$ is the amount of energy required to transmit $x(T)$ data in time $\tau$ with the channel state being $c(T)$. Thus, $\tau$ is the small time window in which the remaining data is completely transmitted out assuming that the channel state does not change over that period. In fact, viewing $T + \tau$ as the actual deadline, $\tau$ then models a small buffer window in which unlimited power can be used to meet the deadline, albeit at an associated cost.

Let the interval $[0, T]$ be partitioned into $L$ equal periods[2] and denote $P$ as the short-term expected power constraint at the transmitter. Then, over each partition the power constraint requires that the expected energy cost, $E\left[\int \frac{1}{c(s)} g(r(x(s), c(s), s))ds\right]$, is less than $P(T/L)$. Note that $T/L$ is the duration of each partition interval. Clearly, by varying $L$, the time scale of the partition can be varied and the power constraint can be made either more or less restrictive.

A transmission policy $r(x, c, t)$ must also satisfy the following additional requirements,

(a) $0 \le r(x, c, t) < \infty$, (non-negativity)

(b) $r(x, c, t) = 0$, if $x = 0$ (no data left to transmit)[3].

Let $\Phi$ denote the set of $r(x, c, t)$ that satisfy the above requirements. We say that a policy $r(x, c, t)$ is *admissible* if $r(x, c, t) \in \Phi$ and it satisfies the power constraint on all the partition intervals.

Denote the optimization problem as $(\mathcal{P})$; it can now be summarized as follows,

$$(\mathcal{P}) \quad \inf_{r(\cdot)\in\Phi} \ E\left[\int_0^T \frac{1}{c(s)} g(r(x(s), c(s), s))ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right]$$

$$\text{subject to} \quad E\left[\int_0^{\frac{T}{L}} \frac{1}{c(s)} g(r(x(s), c(s), s))ds\right] \le \frac{PT}{L}$$

$$\vdots$$

$$E\left[\int_{\frac{T(L-1)}{L}}^T \frac{1}{c(s)} g(r(x(s), c(s), s))ds\right] \le \frac{PT}{L}$$

[1] Since $g(\cdot)$ is strictly convex, making $\tau$ smaller increases the penalty cost.

[2] Extensions to arbitrary sized partitions is fairly straightforward and such a generality is omitted for mathematical simplicity.

[3] Additional technical requirements are, $r(\cdot)$ be locally lipschitz in $x$ ($x > 0$) and piecewise continuous in $t$, to ensure that (4) has a unique solution.

The expectations above are conditioned on $(x_0, c_0)$[4], the starting values at $t = 0$. For the analysis, we will keep the general notation $x_0$ but its value in our case is simply, $x_0 = B$.

## III. Optimal Policy

We consider a lagrangian duality approach to solve the problem in $(\mathcal{P})$. The basic steps involved in such an approach are to form the lagrangian, obtain the dual function that depends on the lagrange multipliers, maximize the dual function with respect to the lagrange multipliers and show that there is no duality gap; that is, maximizing the dual function gives the optimal cost for the constrained problem. However, there are important subtleties in problem $(\mathcal{P})$ which make it non-standard. First, the domain of the rate functions $r(\cdot)$ is a functional space which makes $(\mathcal{P})$ an infinite dimensional optimization, and, second, $(\mathcal{P})$ is a stochastic optimization and by this we mean that there is a probability space involved over which the expectation is taken. In this section, we delve into the solution details proceeding along the steps outlined above.

### A. Dual Function

The inequality constraints in $(\mathcal{P})$ can be written as,

$$E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(r(\cdot))}{c(s)} ds\right] - \frac{PT}{L} \leq 0, \quad k = 1, \ldots, L \tag{5}$$

Let $\bar{\nu} = (\nu_1, \ldots, \nu_L)$ be the lagrange multipliers for these power constraints corresponding to the $L$ partitions of $[0, T]$. Since these are inequality constraints, the lagrange multipliers must be non-negative, i.e. $\nu_1 \geq 0, \ldots, \nu_L \geq 0$. The *lagrangian* is then given as,

$$\begin{aligned}\mathcal{H}(r(\cdot), \bar{\nu}) =& E\left[\int_0^T \frac{g(r(\cdot))}{c(s)} ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \\ &+ \sum_{k=1}^{L} \nu_k \left(E\left[\int_{\frac{(k-1)T}{L}}^{\frac{kT}{L}} \frac{g(r(\cdot))}{c(s)} ds\right] - \frac{PT}{L}\right)\end{aligned} \tag{6}$$

Re-arranging the above equation, it can be written in the form,

$$\begin{aligned}\mathcal{H}(r(\cdot), \bar{\nu}) =& E\left[\int_0^T \frac{(1 + \nu(s))g(r(\cdot))}{c(s)} ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \\ &- (\nu_1 + \ldots + \nu_L)(PT)/L\end{aligned} \tag{7}$$

where $\nu(s)$ takes value $\nu_k$ over the $k^{th}$ partition interval, i.e. $\nu(s) = \nu_k, s \in [\frac{(k-1)T}{L}, \frac{kT}{L})$. As is the case in duality theory, the dual function is the infimum of $\mathcal{H}(r(\cdot), \bar{\nu})$ over $\Phi$. The point to note here is that the $r(\cdot)$ over which this minimization is considered do not have to satisfy the power constraints, though other requirements still apply. This is because the short term power constraints (violation) have been added as a cost in the objective function of the dual problem. Denoting the *dual function* as $\mathcal{L}(\bar{\nu})$, we thus have,

$$\mathcal{L}(\bar{\nu}) = \inf_{r(\cdot) \in \Phi} \mathcal{H}(r(\cdot), \bar{\nu}) \tag{8}$$

[4]To avoid being cumbersome on notation, we will, throughout, represent conditional expectation without an explicit notation but rather mention the conditioning parameter when there is ambiguity.

One of the interesting properties of the dual function is that it gives a lower bound to the optimal cost in $(\mathcal{P})$. This standard property is referred to as weak duality and it applies in our case as well. It's summarized in the lemma below; the proof is direct and omitted for brevity.

***Lemma 1:*** Let $(x_0, c_0)$ be the starting state at $t = 0$ and denote $J(x_0, c_0)$ as the optimal cost for $(\mathcal{P})$. Then, for all $\bar{\nu} \geq 0$, we have, $\mathcal{L}(\bar{\nu}) \leq J(x_0, c_0)$

Before we proceed to strong duality which involves maximizing $\mathcal{L}(\bar{\nu})$ over $\bar{\nu} \geq 0$, we solve the minimization in (8) and obtain the dual function.

*Evaluating the dual function*: The approach we adopt to evaluate the dual function is to view the problem in $L$ stages corresponding to the $L$ partitions and solve for the optimal rate functions in each of the partition interval with the necessary boundary conditions at the edges. An immediate observation from (7) shows that the effect of the lagrange multipliers is to multiply the instantaneous power function $\frac{g(r(\cdot))}{c(s)}$ with a time-varying function $(1 + \nu(s))$. Thus, the difference over the various intervals is in a different multiplicative factor to the cost function, which for the $k^{th}$ interval is, $1 + \nu(s) = \nu_k$.

Since (8) involves a minimization over $r(\cdot)$ for fixed lagrange multipliers $\bar{\nu}$, the second term in (7), i.e. $\frac{(\nu_1 + \ldots + \nu_L)PT}{L}$, is irrelevant for the minimization and we will neglect it for now. Define,

$$H_\nu^r(x, c, t) = E\left[\int_t^T \frac{(1 + \nu(s))g(r(\cdot))}{c(s)} ds + \frac{\tau g(\frac{x(T)}{\tau})}{c(T)}\right] \tag{9}$$

$$H_\nu(x, c, t) = \inf_{r(\cdot) \in \Phi} H_\nu^r(x, c, t) \tag{10}$$

where the expectation in (9) is conditioned on the state $(x, c, t)$. Stated simply, $H_\nu^r(x, c, t)$ is the cost-to-go function for policy $r(\cdot)$, starting from state $(x, c, t)$ and $H_\nu(x, c, t)$ is the optimal cost-to-go function starting from state $(x, c, t)$. Relating back to (7), $H_\nu^r(x_0, c_0, 0)$ is the expectation term in (7) and $H_\nu(x_0, c_0, 0)$ is the minimization of this term over $\Phi$. Clearly from (7) and (8), having solved for $H_\nu(x, c, t)$, we then obtain the dual function as simply, $\mathcal{L}(\bar{\nu}) = H_\nu(x_0, c_0, 0) - \frac{(\nu_1 + \ldots + \nu_L)PT}{L}$. In the process of obtaining $H_\nu(x, c, t)$, we will also obtain the optimal rate functions for the lagrange multiplier $\bar{\nu}$.

Now, focus on the $k^{th}$ partition interval so that $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$ and consider a small interval $[t, t+h)$, within this partition. Let some policy $r(\cdot)$ be followed over $[t, t+h)$ and the optimal policy thereafter, then using Bellman's principle [19] we have,

$$\begin{aligned}H_\nu(x, c, t) =& \min_{r(\cdot)} \Big\{E \int_t^{t+h} \frac{(1 + \nu_k)g(r(x(s), c(s), s))}{c(s)} ds \\ &+ E H_\nu(x_{t+h}, c_{t+h}, t + h)\Big\}\end{aligned} \tag{11}$$

where $x_{t+h}$ is short-hand for $x(t + h)$ and the expectation is conditioned on $(x, c, t)$. The left side above is the optimal cost if the optimal policy is followed right from the starting state $(x, c, t)$, whereas on the right side, the expression within the minimization bracket is the total cost with policy $r(\cdot)$

being followed over $[t, t+h]$ and the optimal policy thereafter. Removing the minimization gives the inequality,

$$\begin{aligned} H_\nu(x,c,t) \quad &\leq \quad E\int_t^{t+h} \frac{(1+\nu_k)g(r(x(s),c(s),s))}{c(s)}ds \\ &+ E\left[H_\nu(x_{t+h},c_{t+h},t+h)\right] \end{aligned} \tag{12}$$

Rearranging, dividing by $h$ and taking the limit $h \downarrow 0$ gives,

$$A^r H_\nu(x,c,t) + \frac{(1+\nu_k)g(r)}{c} \geq 0 \tag{13}$$

The above follows since $\frac{E\int_t^{t+h}\left(\frac{(1+\nu_k)g(r(\cdot))}{c_s}\right)ds}{h} \to \frac{(1+\nu_k)g(r)}{c}$ where $r$ is the value of the transmission rate at time $t$, i.e. $r = r(x,c,t)$, and, $A^r H_\nu(x,c,t)$ is defined as $A^r H_\nu(x,c,t) = \lim_{h\downarrow 0} \frac{EH_\nu(x_{t+h},c_{t+h},t+h) - H_\nu(x,c,t)}{h}$. The quantity $A^r H_\nu(x,c,t)$ is called the differential generator of the Markov process $(x(t),c(t))$ for policy $r(\cdot)$ and intuitively, it's a natural generalization of the ordinary time derivative for a function that depends on a stochastic process. An elaborate discussion on this topic can be found in [19]–[21]. For our case, using the time evolution as in (4), the quantity $A^r H_\nu(x,c,t)$ can be evaluated as,

$$\begin{aligned} A^r H_\nu(x,c,t) =& \frac{\partial H_\nu}{\partial t} - r\frac{\partial H_\nu}{\partial x} \\ &+ \lambda(E_z[H_\nu(x,Z(c)c,t)] - H_\nu(x,c,t)) \end{aligned} \tag{14}$$

where $E_z$ is the expectation with respect to the $Z$ variable. Now, in the above steps from (12)-(13) if policy $r(\cdot)$ is replaced with the optimal policy $r^*(\cdot)$, there is equality throughout and we get,

$$A^{r^*} H_\nu(x,c,t) + \frac{(1+\nu_k)g(r^*)}{c} = 0 \tag{15}$$

Hence, for a given system state $(x,c,t)$, the optimal transmission rate, $r^*$, is the value that minimizes (13) and the minimum value of the expression equals zero. Thus, over the $k^{th}$ partition interval with $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$, we get the following optimality condition,

$$\min_{r\in[0,\infty)} \left[\frac{(1+\nu_k)g(r)}{c} + A^r H_\nu(x,c,t)\right] = 0 \tag{16}$$

Substituting $A^r H_\nu(\cdot)$ from (14), we see that (16) is a partial differential equation in $H_\nu(x,c,t)$, also referred to as the Hamilton-Jacobi-Bellman (HJB) equation.

$$\begin{aligned} \min_{r\in[0,\infty)} \; &\Big\{\frac{(1+\nu_k)g(r)}{c} + \frac{\partial H_\nu}{\partial t} - r\frac{\partial H_\nu}{\partial x} \\ &+ \lambda(E_z[H_\nu(x,Z(c)c,t)] - H_\nu(x,c,t))\Big\} = 0 \end{aligned} \tag{17}$$

In summary, the above arguments state that if $H_\nu(\cdot)$ is sufficiently smooth, it satisfies the optimality condition in (17) over all the partition intervals $k = 1,\ldots,L$ and the optimal rate functions are the corresponding $r$ values that minimize (17). The boundary conditions for $H_\nu(\cdot)$ are as follows. At $t = T$, $H_\nu(x,c,T) = \frac{\tau g(\frac{x}{\tau})}{c}$, since starting in state $(x,c)$ at time $T$, the optimal cost simply equals the penalty cost. Over each partition interval, we require that $H_\nu(\cdot)$ is continuous at the edges, so that the functions evaluated for the various intervals are consistent. An important caveat to note is that the optimality condition alone doesn't suffice and we need sufficiency arguments to verify that a solution of the PDE in (17) is the optimal solution. This is indeed the case, but, the verification theorems are very detailed and omitted for brevity.

We now present the results for the function, $H_\nu(x,c,t)$, and the corresponding optimal rate function, denoted as $r^*_\nu(x,c,t)$, where the subscript $\nu$ is used to indicate explicit dependence on the lagrange multipliers, $\bar{\nu}$. Theorem I gives the solution which is further explained later, but first, we need some additional notation regarding the channel process. Let there be $m$ channel states in the Markov model and denote the various states $c \in \mathcal{C}$ as $c^1, c^2, \ldots, c^m$. Given a channel state $c^i$, the values taken by the random variable $Z(c^i)$ (defined in (2)) are denoted as $\{z_{ij}\}$, where $z_{ij} = c^j/c^i$. The probability that $Z(c^i) = z_{ij}$ is denoted as $p_{ij}$. Clearly, if there is no transition from state $c^i$ to $c^j$, $p_{ij} = 0$.

**Theorem** *I:* Consider the minimization in (10) with $g(r) = r^n$, $(n > 1, n \in \mathbb{R})$. For $k = 1,\ldots,L$ and $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$ ($k^{th}$ partition interval),

$$H_\nu(x,c^i,t) \quad = \quad \frac{(1+\nu_k)x^n}{c^i(f^k_{c^i}(t))^{n-1}}, \quad i = 1,\ldots,m \tag{18}$$

$$r^*_\nu(x,c^i,t) \quad = \quad \frac{x}{f^k_{c^i}(t)}, \qquad i = 1,\ldots,m \tag{19}$$

where over the $k^{th}$ interval, $\{f^k_{c^i}(t)\}_{i=1}^m$ is the solution of the following ODE system,

$$(f^k_{c^1}(t))' = -1 - \frac{\lambda f^k_{c^1}(t)}{n-1} + \frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{1j}}{z_{1j}}\frac{(f^k_{c^1}(t))^n}{(f^k_{c^j}(t))^{n-1}} \tag{20}$$

$$\vdots$$

$$(f^k_{c^m}(t))' = -1 - \frac{\lambda f^k_{c^m}(t)}{n-1} + \frac{\lambda}{n-1}\sum_{j=1}^m \frac{p_{mj}}{z_{mj}}\frac{(f^k_{c^m}(t))^n}{(f^k_{c^j}(t))^{n-1}} \tag{21}$$

The following boundary conditions apply; if $k = L$, $f^L_{c^i}(T) = \tau(1+\nu_L)^{\frac{1}{n-1}}, \forall i$ (at the deadline) and if $k = 1,..,L-1$, $f^k_{c^i}\left(\frac{kT}{L}\right) = \left(\frac{1+\nu_k}{1+\nu_{k+1}}\right)^{\frac{1}{n-1}} f^{k+1}_{c^i}\left(\frac{kT}{L}\right), \forall i$ (at the partition boundaries). The dual function in (8) is then given as,

$$\mathcal{L}(\bar{\nu}) = \frac{(1+\nu_1)x_0^n}{c_0(f^1_{c_0}(0))^{n-1}} - \frac{(\nu_1+\ldots+\nu_L)PT}{L} \tag{22}$$

*Proof:* The proof is omitted for length considerations but it can be checked that the solution satisfies the optimality condition. ■

The above solution can be understood as follows. There is a set of functions $\{f^k_{c^i}(t)\}$ for the $L$ partitions and the $m$ channel states; i.e. for each partition interval, $k$, there are $m$ functions $\{f^k_{c^i}(t)\}_{i=1}^m$ for the corresponding channel states. The subscript refers to the channel state while the superscript refers to the partition interval. Now, given that the present time $t$ lies in the $k^{th}$ interval, the rate function has the simple closed form expression $\frac{x}{f^k_{c^i}(t)}$ as given in (19) while $H_\nu(\cdot)$ is as given in (18). The functions $\{f^k_{c^i}(t)\}_{i=1}^m$ for the $k^{th}$ interval are the solution of the system of ODE in (20)-(21) with the boundary condition at the right edge of the interval given as, $f^k_{c^i}\left(\frac{kT}{L}\right) = \left(\frac{1+\nu_k}{1+\nu_{k+1}}\right)^{\frac{1}{n-1}} f^{k+1}_{c^i}\left(\frac{kT}{L}\right)$. This ensures that $H_\nu(x,c,t)$ is continuous at the partition edge, $t = \frac{kT}{L}$. For the $L^{th}$ interval the boundary condition is, $f^L_{c^i}(T) = \tau(1+\nu_L)^{\frac{1}{n-1}}$; this ensures that at $t = T$, $H_\nu(x,c^i,T) = \frac{\tau g(\frac{x}{\tau})}{c^i} = \frac{x^n}{c^i \tau^{n-1}}$. Now, the functions $\{f^k_{c^i}(t)\}$ can be evaluated starting at the $L^{th}$ interval to obtain $\{f^L_{c^i}(t)\}_{i=1}^m$ and then $\{f^{L-1}_{c^i}(t)\}_{i=1}^m$ using the boundary condition above and proceeding backwards to the first interval.

In complete generality, a closed form solution for the system of ODE as given above is difficult to obtain, however, there is a special case which can be solved in closed form as discussed next. Nevertheless, in the general case, the system of ODE can be easily solved numerically using standard techniques with minimal computational requirement. An important point to note is that this computation needs to be done offline before the system operation. Once the $\{f^k_{c^i}(t)\}$ are known, the closed form structure of the policy in (19) warrants no further computation.

Constant Drift Channel Model: Under a special structure in the Markov channel model which we refer to as the *constant drift* channel, the functions $f^k_{c^i}(t)$ are independent of the channel state (i.e. $f^k_{c^i}(t) = f^k(t), \forall i$) and the common functions $\{f^k(t)\}_{k=1}^L$ can be obtained in closed form. The particular assumption on the channel model is that the expected value of $1/Z(c)$ is independent of the channel state, i.e. $E[1/Z(c)] = \beta$ (a constant). Since $\tilde{c} = Z(c)c$, starting in state $c$, the next transition state satisfies $E\left[\frac{1}{\tilde{c}}\right] = E\left[\frac{1}{Z(c)}\right]\frac{1}{c} = \beta/c$. Thus, if we look at the process $1/c(t)$, the above assumption means that over the interval of interest, the expected value of the next state (given the present state $1/c$) is a constant multiple of the present state. We refer to $\beta$ as the "drift" parameter of the channel process. If $\beta > 1$, the process $1/c(t)$ drifts upwards in an expected sense, if $\beta = 1$, there is no expected drift and if $\beta < 1$, the drift is downwards. In practice, this could be a good model for slow fading channels which over the deadline interval are drifting towards improving or worsening conditions.

**Theorem** *II:* Consider the minimization in (10) with $g(r) = r^n$ and the constant drift channel model with parameter $\beta$. For $k = 1,\ldots,L$, $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$,

$$H_\nu(x,c,t) = \frac{(1+\nu_k)x^n}{c(f^k(t))^{n-1}} \tag{23}$$

$$r^*_\nu(x,c,t) = \frac{x}{f^k(t)} \tag{24}$$

Let $\eta = \frac{\lambda(\beta-1)}{n-1}$, then,

$$f^k(t) = \tau(1+\nu_k)^{\frac{1}{n-1}} e^{-\eta(T-t)} + \frac{1}{\eta}\Big\{\sum_{j=0}^{L-k-1}\left(\frac{1+\nu_k}{1+\nu_{L-j}}\right) \left(e^{-\eta\left(\frac{T(L-j-1)}{L}-t\right)} - e^{-\eta\left(\frac{T(L-j)}{L}-t\right)}\right)\Big\} + \frac{1}{\eta}\left(1 - e^{-\eta\left(\frac{kT}{L}-t\right)}\right)$$

### B. Lagrange Duality

From Lemma 1, we see that given a lagrange vector $\bar{\nu} \geq 0$, the dual function is a lower bound to the optimal cost of the constrained problem, $\mathcal{P}$. Thus, it makes sense to maximize $\mathcal{L}(\bar{\nu})$ over $\bar{\nu} \geq 0$. Theorem III below, states that strong duality holds or that maximizing $\mathcal{L}(\bar{\nu})$ over $\bar{\nu} \geq 0$ gives the optimal cost of $\mathcal{P}$, and, that if $\mathcal{P}$ has an optimal policy, then the optimal rate function is the same as that obtained in Theorem I with $\bar{\nu} = \bar{\nu}^*$, the maximizing lagrange vector.

As in Lemma 1, let $J(x_0, c_0)$ be the optimal cost of $(\mathcal{P})$ starting at $t = 0$ in state $(x_0, c_0)$, where $x_0 \in [0,\infty), c_0 \in \mathcal{C}$. Note that for $(\mathcal{P})$, the starting state is known and hence is fixed for the optimization. Problem $(\mathcal{P})$ is feasible since a policy that does not transmit any data and simply incurs the penalty cost is an admissible policy. Its cost is finite and hence $J(x_0, c_0)$ is finite.

**Theorem** *III: **(Strong Duality)*** Consider the dual function defined in (8) for $\bar{\nu} \geq 0$, then, we have,

$$J(x_0, c_0) = \max_{\bar{\nu}\geq 0} \mathcal{L}(\bar{\nu}) \tag{25}$$

and the maximum on the right is achieved by some $\bar{\nu}^* \geq 0$. If $(\mathcal{P})$ has an optimal solution, which we denote as $r^*(x,c,t)$, then, $r^*(x,c,t)$ is the minimizing $r(\cdot)$ in (8) for $\bar{\nu} = \bar{\nu}^*$.

*Proof:* The proof follows from the lagrange duality result in [22] and is omitted here for length considerations. ■

Interestingly, the dual function is concave [22] which makes the maximization in (25) much simpler as there are no issues of local maxima and a direct gradient search algorithm would numerically yield $\bar{\nu}^*$. For our case, the dual functions for a general $\bar{\nu} \geq 0$, are given in Theorems I (general markov channel) and II (constant drift channel). While a closed form solution of $\bar{\nu}^*$ is difficult to obtain, one can easily obtain $\bar{\nu}^*$ numerically using standard techniques.

### C. Optimal Policy for $(\mathcal{P})$

The optimal policy for problem $(\mathcal{P})$ can now be obtained by combining Theorems I and III and is given as follows. For $k = 1,\ldots,L$ and $t \in [\frac{(k-1)T}{L}, \frac{kT}{L})$ ($k^{th}$ partition interval),

$$r^*(x,c^i,t) = r^*_{\nu^*}(x,c^i,t) = \frac{x}{f^k_{c^i}(t)}, \qquad i = 1,\ldots,m \tag{26}$$

where the functions $\{f^k_{c^i}(t)\}$ are evaluated with $\bar{\nu} = \bar{\nu}^*$. As mentioned earlier in Section III-A, the computation for $\bar{\nu}^*$ and $\{f^k_{c^i}(t)\}$ needs to be done offline before the data transmission. In practice, if the transmitter has computational capabilities, these computations can be carried out at $t = 0$ for the given problem parameters, otherwise, the $\bar{\nu}^*$ and $\{f^k_{c^i}(t)\}$ can be pre-determined and stored in a table in the transmitter memory. Having known $\{f^k_{c^i}(t)\}$, the closed form structure of the optimal policy as given in (26) warrants no further computation and is simple to implement. At time $t$, the transmitter looks at the amount of data in the buffer, $x$, the channel state, $c$, the partition interval $k$ in which $t$ lies and computes the rate for the communication slot as simply $\frac{x}{f^k_c(t)}$.

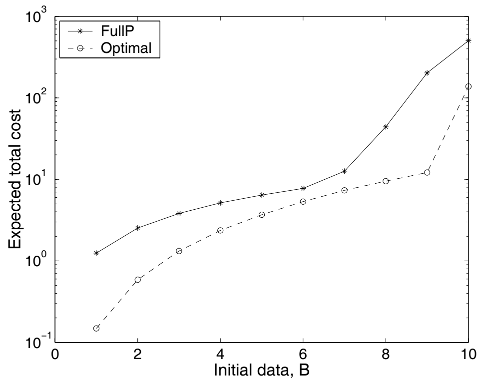


Fig. 3. Total cost comparison of the optimal and the full power policy.

## IV. Simulation Results

In this section, we consider an illustrative example and present energy cost comparisons for the optimal and the Full Power (FullP) policy. In FullP policy, the transmitter always transmits at full power, $P$, and so given the system state $(x, c, t)$ the rate is chosen as, $r(x, c, t) = g^{-1}(cP) = (cP)^{1/n}$, for $g(r) = r^n$. The simulation setup is as follows. The channel model is the GE model as described earlier in Section II-B, with parameters $\lambda_{bg} = 1$, $\lambda_{gb} = 3/7$, $c_g = 1$ and $c_b = 0.2$; thus, $\lambda = \max(\lambda_{bg}, \lambda_{gb}) = 1$ and $\gamma = c_b/c_g = 0.2$. It can be easily checked that with the above parameters, in steady state the fraction of time spent in the good state is $0.7$ and $0.3$ in the bad state. The deadline is taken as $T = 10$ and the number of partition intervals as $L = 20$. The power-rate function is, $g(r) = r^2$ and the value of $\tau$ in the penalty cost function is taken as $0.01$ which is $0.1\%$ of the deadline; thus, a time window of $0.1\%$ is provided at $T$. To simulate the process, communication slot duration is taken as $dt = 10^{-3}$ implying that there are $T/10^{-3} = 10000$ slots over the deadline interval. For each slot, the transmission rate is computed as given by the corresponding policy and the total cost is obtained as the sum of the energy costs in the slots plus the penalty cost. Expectation is then taken as an average over the sample paths.

Figure 3 is a plot of the expected total cost of the two policies with the initial data amount $B$ varied from $1$ to $10$. The value of $P$ is chosen such that at $B = 5$, even with bad channel condition over the entire deadline interval, the entire data can be served at full power. This implies, $P = \frac{1}{\gamma}(5/T)^2 = 1.25$ ($5/T$ is the rate required to serve 5 units in time $T$). Thus, $B \leq 5$ gives the regime in which full power always meets the deadline and $B > 5$ is the regime in which data is left out which then incurs the penalty cost. It's evident from the plot that the optimal policy gives a significant gain in the total cost (note that the y-axis is on a log scale) and at around $B = 1$ FullP policy incurs almost 10 times the optimal cost. Thus, dynamic rate adaptation can yield significant energy savings.

## V. Conclusion

We considered energy efficient transmission of data over a fading channel with deadline and power constraints. Specifically, we addressed the scenario of a wireless transmitter with short-term power limit constraints, having $B$ units of data that must be transmitted by deadline $T$ over a fading channel. Using a novel continuous-time formulation and lagrangian duality, we obtain in closed form the optimal transmission policy that dynamically adapts the rate over time and in response to the time-varying channel variations to minimize the transmission energy cost. This work opens various interesting research directions which include data transmission with multiple deadlines and extensions to scenarios involving control of multiple transmitters having deadline constraints.

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