Delay-Constrained Energy Efficient Data Transmission over a Wireless Fading Channel

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Abstract—**We study optimal rate control for transmitting deadline-constrained data over a time-varying channel. Specifically, we consider a wireless transmitter where the channel gain varies stochastically over time and the packets in the queue have strict delay constraints. The transmitter can adapt the rate over time by varying the power and the goal is to obtain the rate-control policy that minimizes the expected energy expenditure while meeting the deadline constraints. We first consider the case of B bits of data that must be transmitted by a deadline T and using a novel continuous-time stochastic control formulation obtain the optimal policy. Based on a cumulative curves methodology and a decomposition approach, we then obtain the optimal policy when the queue has packets with variable deadline constraints. Finally, we present a heuristic policy for the case of arbitrary packet arrivals to the queue and compare its performance using simulation results with a non-adaptive scheme.**

Index Terms—**Energy efficiency, Delay constraints, Wireless fading channel, Rate control, Quality of Service.**

I. Introduction

Delay constraints and energy efficiency are important concerns in the design of modern wireless systems. Data services such as video and real-time multimedia streaming, high throughput file transfers and Voice-over-IP require strict delay constraints on data delivery. Similarly, in sensor networks time critical sensing applications impose deadline constraints within which the data must be transmitted back to a central processing entity. Energy consumption is also an important concern and minimizing this cost has numerous advantages in efficient battery utilization of mobile devices, increased lifetime of sensor and ad-hoc networks, and better utilization of limited energy sources in satellites. Furthermore, since transmission energy constitutes the bulk of the total energy expenditure, it is imperative to minimize this cost to achieve significant energy savings. Our focus in this work is to utilize dynamic transmission rate control to address the above concerns.

Modern wireless devices are equipped with channel measurement and rate adaptation capabilities [1]. Channel measurement allows the transmitter-receiver pair to measure the fade state using a pre-determined pilot signal while rate control capability allows the transmitter to adjust the transmission rate over time. Such a control can be achieved in various ways that include adjusting the power level, symbol rate, coding scheme, constellation size or any combination of these approaches; further, in some technologies the receiver can detect these changes directly from the received data without the need for an explicit rate change control information [2]. The transmission rate can also be adapted very rapidly in time over millisecond duration time-slots [1]. These capabilities, thus, provide ample opportunity to utilize rate control algorithms to optimize system performance.

The power-rate function defines the relationship that governs the amount of transmission power required to reliably transmit at a certain rate. Two fundamental aspects of this function, which are exhibited by most encoding/communication schemes and hence are common assumptions in the literature [4]–[6], [9]–[11], [15], are as follows. First, for a fixed bit error probability and channel state, the required transmission power is a convex function of the communication rate as shown in Figure 1(a). This implies, from Jensen's inequality, that transmitting data at low rates over a longer duration is more energy efficient as compared to high rate transmissions. Second, the wireless channel is time-varying which shifts the convex power-rate curves as a function of the channel state as shown in Figure 1(b). As good channel conditions require less transmission power, exploiting this variability over time by adapting the rate in response to the channel conditions leads to reduced energy cost. Thus, by adapting the transmission rate intelligently over time we can minimize the energy cost while also ensuring that the delay constraints are met.

In this paper, we consider the following setup. The system model consists of a wireless transmitter with packets having strict delay constraints. The channel state varies stochastically over time and is modelled as a general Markov process. The transmitter can control the transmission rate over time and the expended power depends on both the chosen rate and the present channel condition. The objective is to obtain a rate-control policy that serves the packets arriving to the queue within the deadline constraints and minimizes the transmission energy expenditure. Towards this end, we first consider a simplified problem where the queue has only B bits of data that must be transmitted by deadline T. Using a novel continuous-time stochastic control formulation we obtain in explicit/closed form the optimal transmission policy. We then consider a cumulative curves methodology and a novel decomposition approach to obtain the optimal policy with variable deadline

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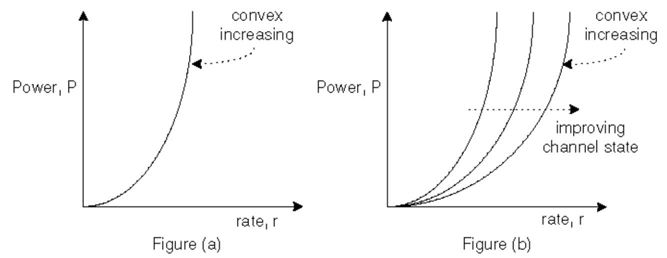

Fig. 1. Transmission power as a function of the rate and the channel state; (a) fixed channel state, (b) variable channel state.

modulation	bits/symbol	SNR/symbol
2 PAM	1	$0.25\,d^2$
4 QAM	2	$0.50\,d^2$
16 QAM	4	$1.25\,d^2$
64 QAM	6	$5.25\,d^2$

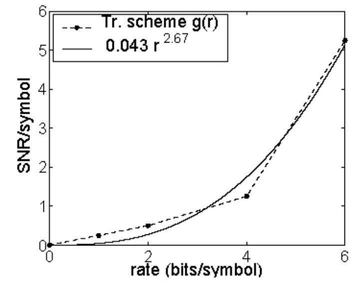

Fig. 2. Modulation scheme considered in [11] as given in the table. The corresponding plot shows the least squares monomial fit, $0.043r^{2.67}$, to the scaled piecewise linear power-rate curve.

constraints. Finally, we present a heuristic policy for the case of arbitrary packet arrivals to the queue and compare its performance using simulation results with a non-adaptive scheme.

Transmission power/rate control is an active area of research in communication networks in various different contexts. Adaptive network control and scheduling has been studied in the context of network stability [11], [13], average throughput [12], [14], average delay [4], [15] and packet drop probability [16]. This literature considers "average metrics" that are measured over an infinite time horizon and hence do not directly apply for delay constrained/real-time data. Furthermore, for strict deadline constraints rate-adaptation simply based on steady state distributions does not suffice and one needs to take into account the system dynamics over time; thus, introducing new challenges and complexity into the problem. Recent work in this direction includes [5]–[10]. The work in [5] studied offline formulations under complete knowledge of the future and devised heuristic policies using the offline solution. The authors in [6] studied several data transmission problems using Dynamic Programming (DP), however, the specific problem that we consider in this work becomes intractable using this methodology. This is due to the large state space in the DP-formulation or the well-known "curse of dimensionality". The works in [9], [10] studied formulations for energy efficient data transmission over a static channel without fading. Similarly, our earlier work in [7] for static non-fading channels used a calculus approach to obtain minimum energy policies with general QoS constraints. The work in this paper is an extension of [8] where we considered a restricted scenario with only a single deadline constraint.

II. System Model

We consider a continuous-time model of the system. Clearly, such a model is an approximation of the actual system but the assumption is justified since in practice, transmission rate can be adapted over a communication-slot duration which is very short, on the order of 1 msec [1]. In contrast the packet delay requirements can be on the order of 100 msec, thus justifying a continuous-time view of the system. A significant advantage of such a model is that it makes the problem mathematically tractable and yields simple solutions. The alternative discrete-time dynamic programming setup is intractable and would only yield numerical solutions without much insights. The results obtained using the continuous-time model can then be applied to the discrete-time system in a very straightforward manner by simply evaluating the solution at discrete times as done for the simulation results in Section V-B.

A. Transmission Model

Let h_t denote the channel gain, $P(t)$ the transmitted signal power and $P^{rcd}(t)$ the received signal power at time t. We make the common assumption [4]–[6], [9]–[11], [15] that the required received signal power for reliable communication, with a certain low bit-error probability, is convex in the rate; i.e. $P^{rcd}(t) = g(r(t))$, where $g(r)$ is a non-negative convex increasing function for $r \geq 0$. Since the received signal power is given as $P^{rcd}(t) = |h_t|^2 P(t)$, the required transmission power to achieve rate $r(t)$ is given by,

$$P(t) = \frac{g(r(t))}{c(t)} \tag{1}$$

where $c(t) \triangleq |h_t|^2$. The quantity $c(t)$ is referred to as the *channel state* at time t. Its value at time t is assumed known either through prediction or direct channel measurement but evolves stochastically in the future. Some specific examples of (1) can be found in [4], [5]. It is worth emphasizing that while we defined $c(t)$ as $|h_t|^2$ to motivate the relationship in (1), more generally, $c(t)$ could include other stochastic variations in the system and (uncontrollable) interference from other transmitter-receiver pairs, as long as the power-rate relationship is given by (1).

In this work, our primary focus will be on $g(r)$ belonging to the class of *Monomial* functions, namely, $g(r) = kr^n$, $n > 1, k > 0$ $(n, k \in \mathbb{R})$. While this assumption restricts the generality of the problem, it serves several purposes. First, mathematically it leads to simple closed-form optimal solutions that can be applied in practice. Second, most importantly, for most practical transmission schemes $g(\cdot)$ is described numerically and its exact analytical form is unknown. In such situations, one can obtain the best approximation of that function to the form kr^n by choosing the appropriate k, n and then applying the results thus obtained. For example, consider the modulation scheme considered in [11] and reproduced here in Figure 2. The table gives the rate and the normalized signal power per symbol, where d is the minimum distance between signal points and the scheme is designed for error

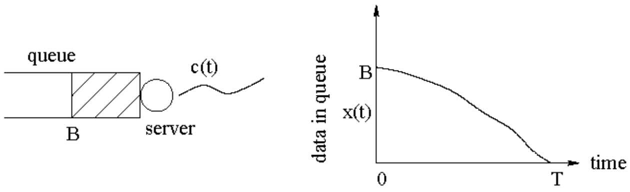

Fig. 3. Schematic description of the system for the BT-problem.

probabilities less than 10^{-6}. The plot gives the least squares monomial fit to the transmission scheme and one can see from the plot that for this example the monomial approximation is fairly close. Third, monomials form the first step towards studying extensions to polynomial functions which would then apply to a general $g(\cdot)$ function using the polynomial expansion. Under a more restrictive setting in Section III-C, we also study the class of *Exponential* functions, namely, $g(r) = k(\alpha^r - 1),\ \alpha > 1, k > 0\ (\alpha, k \in \mathbb{R})$. Finally without loss of generality, throughout the paper we take $k = 1$, since any other value of k simply scales the energy cost without affecting the policy results.

B. Channel Model

We consider a general continuous-time discrete state space Markov model for the channel state process. Markov processes constitute a large class of stochastic processes that exhaustively model a wide set of fading scenarios and there is substantial literature on these models [17], [18] and their applications to communication networks [18], [19]. Denote the channel state process as $C(t)$ and the state space as $\mathcal{C}$. Let $c \in \mathcal{C}$ denote a particular channel state and $\{c(t), t \geq 0\}$ denote a sample path. Starting from state c, let $\mathcal{J}_c$ be the set of all states ($\neq c$) to which the channel can transition when the state changes. Let $\lambda_{c\tilde{c}}$ denote the channel transition rate from state c to $\tilde{c}$, then, the sum transition rate at which the channel jumps out of state c is, $\lambda_c = \sum_{\tilde{c} \in \mathcal{J}_c} \lambda_{c\tilde{c}}$. Clearly, the expected time that $C(t)$ spends in state c is $1/\lambda_c$ and one can view $\frac{1}{\lambda_c}$ as the coherence time of the channel in state c.

Now, define $\lambda \triangleq \sup_c \lambda_c$ and a random variable, $Z(c)$, as,

$$Z(c) \triangleq \begin{cases} \tilde{c}/c, & \text{with prob. } \lambda_{c\tilde{c}}/\lambda,\ \tilde{c} \in \mathcal{J}_c \\ 1, & \text{with prob. } 1 - \lambda_c/\lambda \end{cases} \tag{2}$$

With this definition, we obtain a compact and simple description of the process evolution as follows. *Given a channel state c, there is an Exponentially distributed time duration with rate λ after which the channel state changes. The new state is a random variable which is given as $C = Z(c)c$.* Clearly, from (2) the transition rate to state $\tilde{c} \in \mathcal{J}_c$ is unchanged at $\lambda_{c\tilde{c}}$, whereas with rate $\lambda - \lambda_c$ there are indistinguishable self-transitions. This is a standard Uniformization technique and there is no process generality lost with the new description as it yields a stochastically identical scenario.

Example: Consider the standard Gilbert-Elliott channel model [18] that has two states b and g denoting the "bad" and the "good" channel conditions respectively. The two states correspond to a two level quantization of the channel gain. If the measured channel gain is below some value, the channel is labelled as "bad" and $c(t)$ is assigned an average value c_b, otherwise $c(t) = c_g$ for the good condition. Let the transition rate from the good to the bad state be λ_{gb} and from the bad to the good state be λ_{bg}. Let $\gamma = c_b/c_g$, and using the earlier notation we get, $\lambda = \max(\lambda_{bg}, \lambda_{gb})$. For state c_g, we obtain $Z(c_g)$ as,

$$Z(c_g) = \begin{cases} \gamma, & \text{with prob. } \lambda_{gb}/\lambda \\ 1, & \text{with prob. } 1 - \lambda_{gb}/\lambda \end{cases} \tag{3}$$

To obtain $Z(c_b)$, replace γ with $1/\gamma$ and λ_{gb} with λ_{bg} in equation (3) above.

III. BT - Problem

We begin first with the following problem. The queue has B units of data that must be transmitted by deadline T, with minimum energy over a time-varying channel. We refer to this as the "BT-problem" where the notation implies that the amount of data under consideration is B, and the deadline is T. The case with variable deadline constraints is treated in the next section. We now describe in detail the control formulation and the optimality conditions for the BT-problem.

A. Optimal Control Formulation

Consider the BT-problem and let $x(t)$ denote the amount of data left in the queue at time t. The system state can be described as (x, c, t), where the notation means that at the present time t, $x(t) = x$ and $c(t) = c$. Let $r(x, c, t)$ denote the chosen transmission rate for the corresponding system state (x, c, t). Since the underlying process is Markov, it is sufficient to restrict attention to transmission policies that depend only on the present system state [23]. Clearly then, (x, c, t) is a Markov process. The system is schematically depicted in Figure 3.

Given a policy $r(x, c, t)$, the system evolves in time as a Piecewise-Deterministic-Process (PDP) as follows. It starts with $x(0) = B$ and $c(0) = c_0$. Until τ_1, where τ_1 is the first time instant after $t = 0$ at which the channel changes, the buffer is reduced at the rate $r(x(t), c_0, t)$. Hence, over the interval $[0, \tau_1)$, $x(t)$ satisfies the ordinary differential equation,

$$\frac{dx(t)}{dt} = -r(x(t), c_0, t) \tag{4}$$

Equivalently, $x(t) = x(0) - \int_0^t r(x(s), c_0, s)ds,\ t \in [0, \tau_1]$. Then, starting from the new state $(x(\tau_1), c_1, \tau_1)$, the above procedure repeats until $t = T$ is reached.

A transmission policy, $r(x, c, t)$, is *admissible*, if it satisfies the following,

(a) $0 \leq r(x, c, t) < \infty$, (non-negativity)
(b) $r(x, c, t) = 0$, if $x = 0$ (no data left to transmit) and,
(c) $x(T) = 0$, a.s. (deadline constraint)[1].

[1] An additional technical requirement is that $r(x, c, t)$ be continuous and locally lipschitz in x (for $x > 0$) which ensures that $x(t)$ is the unique solution of (4).

Consider now an admissible transmission policy $r(\cdot)$ and define a cost-to-go function, $J_r(x, c, t)$, as the expected energy cost starting at time $t < T$ in state (x, c, t). Then,

$$J_r(x, c, t) = E\left[\int_t^T \frac{1}{c(s)} g(r(x(s), c(s), s))ds\right] \quad (5)$$

where the term within the brackets is the total energy expenditure obtained as the integral of the power cost over time and the expectation is conditioned on the starting state $x(t) = x$, $c(t) = c$. Define a *minimum cost function*, $J(x, c, t)$, as the infimum of $J_r(x, c, t)$ over the set of all admissible transmission policies.

$$J(x, c, t) = \inf_{r(\cdot)} J_r(x, c, t), \quad r(x, c, t) \text{ admissible} \quad (6)$$

Now, stated concisely, the optimization problem is to compute the minimum cost function $J(x, c, t)$ and obtain the optimal policy $r^*(x, c, t)$ that achieves this minimum cost.

B. Optimality Conditions

A standard approach towards studying continuous time problems is to investigate their behavior over a small time interval. In the context of the BT-problem, this methodology applies as follows. Suppose that the system is in state (x, c, t). We first apply a transmission policy, $r(\cdot)$, in the small interval $[t, t+h]$ and thereafter, starting from $(x(t+h), c(t+h), t+h)$ we assume that the optimal policy is followed. By assumption, the energy cost is optimal over $[t+h, T]$, hence, investigating the system over $[t, t+h]$ would give conditions for the optimality of the chosen rate at time t. Since t is arbitrary, we obtain formal conditions for an optimal policy.

Following the above approach, we now present the details of the analysis. Consider $t \in [0, T)$ and a small interval $[t, t+h]$, where $t + h < T$. From Bellman's principle [21] we get,

$$\begin{aligned} J(x, c, t) \quad = \quad & \min_{r(\cdot)} \Big\{ E \int_t^{t+h} \frac{1}{c(s)} g(r(x(s), c(s), s))ds \\ & + EJ(x_{t+h}, c_{t+h}, t+h) \Big\} \end{aligned} \quad (7)$$

where x_{t+h}, c_{t+h} is a short-hand notation for $x(t+h)$ and $c(t+h)$ respectively. The expression within the minimization bracket in (7) denotes the total cost with policy $r(\cdot)$ being followed over $[t, t+h]$ and the optimal policy thereafter. This cost must be greater than the cost of applying the optimal policy directly from the starting state (x, c, t). Thus, we get,

$$\begin{aligned} J(x, c, t) \quad \le \quad & E \int_t^{t+h} \frac{1}{c(s)} g(r(x(s), c(s), s))ds \\ & + E\left[J(x_{t+h}, c_{t+h}, t+h)\right] \end{aligned} \quad (8)$$

$$\begin{aligned} & E[J(x_{t+h}, c_{t+h}, t+h)] - J(x, c, t) \\ & \quad + E \int_t^{t+h} \frac{1}{c(s)} g(r(x(s), c(s), s))ds \ge 0 \end{aligned} \quad (9)$$

Divide (9) by h and take the limit $h \downarrow 0$. In the limit we have,

$$\frac{E \int_t^{t+h} \left(\frac{g(r(x_s, c_s, s))}{c_s}\right) ds}{h} \to \frac{g(r)}{c} \quad (10)$$

where r is the transmission rate at time t, i.e. $r = r(x, c, t)$. Define $\lim_{h \downarrow 0} \frac{EJ(x_{t+h}, c_{t+h}, t+h) - J(x, c, t)}{h} = A^r J(x, c, t)$, then in the limit, (9) simplifies to,

$$A^r J(x, c, t) + \frac{1}{c} g(r) \ge 0 \quad (11)$$

The quantity $A^r J(x, c, t)$ is called the differential generator of the Markov process $(x(t), c(t))$ for policy $r(\cdot)$. Intuitively, it can be viewed as a natural generalization of the ordinary time derivative for a function that depends on a stochastic process. An elaborate discussion on this topic can be found in [21]–[23]. For the process $(x(t), c(t))$, using the time evolution in (4), the quantity $A^r J(x, c, t)$ can be evaluated as [21],

$$\begin{aligned} A^r J(x, c, t) = & \frac{\partial J}{\partial t} - r(x, c, t)\frac{\partial J}{\partial x} \\ & + \lambda(E_z[J(x, Z(c)c, t)] - J(x, c, t)) \end{aligned} \quad (12)$$

where E_z is the expectation with respect to the Z variable as defined in (2).

Now, in the above steps from (8)-(11) if policy $r(\cdot)$ is replaced with the optimal policy $r^*(\cdot)$ there is equality throughout and we get,

$$A^{r^*} J(x, c, t) + \frac{1}{c} g(r^*) = 0 \quad (13)$$

Thus, for a given system state (x, c, t), the optimal transmission rate r^* is that value of r that minimizes (11) and the minimum value of this expression equals zero. This gives,

$$\min_{r \in [0, \infty)} \left[\frac{g(r)}{c} + A^r J(x, c, t)\right] = 0 \quad (14)$$

Substituting $A^r J$ from (12), we get a partial differential equation (PDE) in $J(x, c, t)$ which is also referred as the Hamilton-Jacobi-Bellman (HJB) equation. This is the *Optimality Equation* for the BT-problem.

$$\begin{aligned} \min_{r \in [0, \infty)} \quad & \Big\{ \frac{g(r)}{c} + \frac{\partial J(x, c, t)}{\partial t} - r\frac{\partial J(x, c, t)}{\partial x} \\ & + \lambda(E_z[J(x, Z(c)c, t)] - J(x, c, t)) \Big\} = 0 \end{aligned} \quad (15)$$

The boundary conditions for the above PDE are, $J(0, c, t) = 0$, and $J(x, c, T) = \infty$, if $x > 0$. The last condition follows due to the deadline constraint of T on the data.

C. Optimal Transmission Policy

We have, so far, presented general results on the optimality condition for the BT-problem. We, now, give specific analytical results for the optimal policy and discuss some of the insights that can be drawn from it. However, before proceeding further, a few additional notations regarding the channel process are required. Let there be total m channel states in the Markov model and denote the various states $c \in \mathcal{C}$ as $c^1, c^2, \ldots, c^m$. Given a channel state c^i, the values taken by the random variable $Z(c^i)$ (defined in (2)) are denoted as $\{z_{ij}\}$, where $z_{ij} = c^j/c^i$. The probability that $Z(c^i) = z_{ij}$ is denoted as p_{ij}. Clearly, if there is no transition from state c^i to c^j, $p_{ij} = 0$. Also, as pointed out earlier, without loss of generality we take the multiplicative constant $k = 1$ in the power-rate function.

Theorem *I: Consider the BT-problem with $g(r) = r^n$, $n > 1, n \in \mathbb{R}$ and a Markov channel model. The optimal policy, $r^*(x, c, t)$, and the function, $J(x, c, t)$, are,*

$$r^*(x, c^i, t) = \frac{x}{f_i(T-t)}, \; i = 1, \ldots, m \quad (16)$$

$$J(x, c^i, t) = \frac{x^n}{c^i (f_i(T-t))^{n-1}}, \; i = 1, \ldots, m \quad (17)$$

The functions $\{f_i(s)\}_{i=1}^m$ are the solution of the following ODE with the boundary conditions $f_i(0) = 0, f_i'(0) = 1, \forall i$[2]*,*

$$f_1'(s) = 1 + \frac{\lambda f_1(s)}{n-1} - \frac{\lambda}{n-1} \sum_{k=1}^{m} \frac{p_{1k}}{z_{1k}} \frac{(f_1(s))^n}{(f_k(s))^{n-1}} \quad (18)$$

$$\vdots$$

$$f_m'(s) = 1 + \frac{\lambda f_m(s)}{n-1} - \frac{\lambda}{n-1} \sum_{k=1}^{m} \frac{p_{mk}}{z_{mk}} \frac{(f_m(s))^n}{(f_k(s))^{n-1}} \quad (19)$$

Proof: It can be shown that the functions in (16)-(17) satisfy the optimality equation in (15). The proof is omitted for brevity and can be found in [3]. ■

The results in the above theorem can be interpreted as follows. From (16), the optimal rate given x amounts of data left, channel state c^i and time t, is $\frac{x}{f_i(T-t)}$, where the function $f_i(s)$ is associated with the channel state c^i. The corresponding minimum expected cost starting from state (x, c^i, t) is $\frac{x^n}{c^i f_i(T-t)^{n-1}}$. The boundary condition $f_i(0) = 0$ is due to the deadline constraint, since at the deadline, $(T - t) = 0$ and we require $J(x, c^i, T) = \infty$, if $x \neq 0$. The system of ODE above can be easily solved offline numerically using standard techniques (eg. ODE solvers in MATLAB). Furthermore, this computation needs to be done only once before the system starts operating. In fact, $\{f_i(s)\}$ can be pre-determined and stored in a table in the transmitter memory. Once $\{f_i(s)\}$ are known, the closed form structure of the optimal policy in (16) warrants no further computation. At time t, the transmitter simply looks at the amount of data left in the queue, x, the channel state, c^i, and then using the appropriate $f_i(\cdot)$ function it computes the transmission rate $\frac{x}{f_i(T-t)}$.

The solution in (16) provides several interesting observations and insights as follows. At time t, the optimal rate depends on the channel state c^i through the function $f_i(T-t)$ and this rate is linear in x with slope $\frac{1}{f_i(T-t)}$. We can view the quantity $\frac{1}{f_i(T-t)}$ as the "urgency" of transmission at time t under the channel state c^i and with $(T - t)$ time left until the deadline. This view gives a nice separation form for the optimal rate:

*rate = amount of data left * urgency of transmission*

[2] For numerical evaluation of the ODE solution, the two boundary conditions can be combined by taking a small $\epsilon > 0$, letting $f_i(\epsilon) = \epsilon, \forall i$ and then using an initial-value ODE solver to obtain $\{f_i(s)\}, \; s \geq \epsilon$.

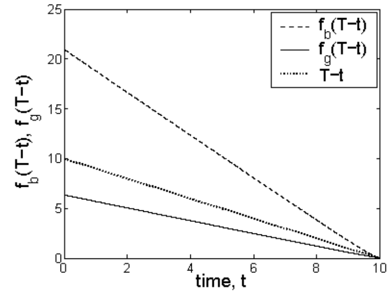

Fig. 4. $f_b(T-t)$ and $f_g(T-t)$ plot for the bad and the good channel respectively. Other parameters include, $g(r) = r^2$, $T = 10, \lambda = 5, \gamma = 0.3$.

Due to the boundary condition, as t approaches T, $f_i(T - t)$ goes to zero; thus, as expected the urgency of transmission, $\frac{1}{f_i(T-t)}$, increases as t approaches the deadline. Finally, one last observation is that if we set $\lambda = 0$ (no channel variations), $f_i(T - t) = T - t, \forall i$ and $r^*(x, c, t) = \frac{x}{T-t}$. Thus, with no channel variations the optimal policy is to transmit at a rate that just empties the buffer by the deadline. This observation is consistent with the earlier results in the literature for non-fading/time-invariant channels [5], [7], [9]. We refer to this policy as the "Direct Drain" (DD) policy.

Numerical Example: Consider the Gilbert-Elliott (GE) channel model with two states "bad" and "good" as described in Section II-B. Let $g(r) = r^2$ and for simplicity take $\lambda_{bg} = \lambda_{gb} = \lambda$. Denoting $\gamma = c_b/c_g$, we have, $Z(c_g) = \gamma$, $w.p.$ 1, and $Z(c_b) = 1/\gamma$, $w.p.$ 1. Denoting $f_b(s), f_g(s)$ as the respective functions in the bad and the good states, we have,

$$f_b'(s) = 1 + \lambda f_b(s) - \frac{\gamma \lambda (f_b(s))^2}{f_g(s)} \quad (20)$$

$$f_g'(s) = 1 + \lambda f_g(s) - \frac{\lambda (f_g(s))^2}{\gamma f_b(s)} \quad (21)$$

Figure 4 plots these functions, evaluated using MATLAB, for $T = 10, \lambda = 5, \gamma = 0.3$. First, as expected $f_g(T - t) \leq f_b(T - t), \forall t$, which implies that given x units of data in the buffer and time t, the rate $\frac{x}{f_g(T-t)}$ is higher under the good state than the bad state. Second, $f_g(T-t) \leq T-t \leq f_b(T-t)$, where the function, $T - t$, gives the rate, $\frac{x}{T-t}$, corresponding to the direct drain (DD) policy. Thus, the optimal policy both spreads the data over time and adapts the rate in response to the time-varying channel condition and this adaptation is governed by the respective functions $\{f_i(\cdot)\}$.

The following simulation results compare the performance of the optimal policy with the direct drain (DD) policy. For the simulations, we consider the GE channel model with $c_g = 1$, $c_b = \gamma$ and take $g(r) = r^2$. We let, $T = 10$ and partition the interval $[0, 10]$ into slots of length $dt = 10^{-3}$, thus, having $10,000$ time slots. The transmission rate chosen in each slot is obtained by evaluating the respective policies at the time corresponding to the start of that slot. A channel sample path is simulated using a Bernoulli process, where in a slot the channel transitions with probability λdt and with probability $1-\lambda dt$ there is no transition. At each transition, the new state is

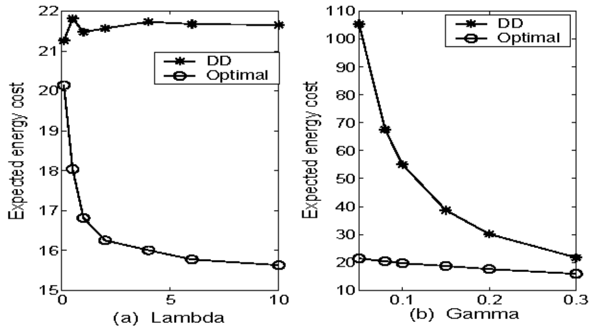

Fig. 5. Expected energy cost for the optimal and the DD policy.

$\tilde{c} = Z(c)c$ which for the GE model amounts to jumps between the two states. Expected energy cost is computed by taking an average over 10^4 sample paths. Figure 5(a) plots the energy costs of the two policies as λ is varied with $\gamma = 0.3$, $B = 10$. When λ is small the channel is essentially time-invariant over the deadline interval and the two policies are comparable. As λ increases, the optimal cost substantially decreases due to the channel adaptation. In Figure 5(b), γ is varied with $\lambda = 5$, $B = 10$. As γ decreases the good and bad channel quality differ significantly and the optimal rate adaptation leads to a much lower energy cost in terms of an order of magnitude as compared to the DD policy.

Constant Drift Channel: Theorem I gives the optimal policy for a general Markov channel model. By considering a special structure on the channel model which we refer to as the *Constant Drift* channel, two specialized results can be obtained. First, we obtain the $f(\cdot)$ function in closed form for the Monomial class ($g(r) = r^n$), and second, we obtain the optimal policy for the Exponential class ($g(r) = \alpha^r - 1$).

In the constant drift channel model, we assume that the expected value of the random variable $1/Z(c)$ is independent of the channel state, i.e. $E[1/Z(c)] = \beta$, a constant. Thus, starting in state c, if $\tilde{c}$ denotes the next transition state we have $E\left[\frac{1}{\tilde{c}}\right] = E\left[\frac{1}{Z(c)}\right]\frac{1}{c} = \frac{\beta}{c}$. This means that if we look at the process $1/c(t)$, the expected value of the next state is a constant multiple of the present state. We refer to β as the "*drift*" parameter of the channel process. If $\beta > 1$, the process $1/c(t)$ has an upward drift; if $\beta = 1$, there is no drift and if $\beta < 1$, the drift is downwards. There are various situations where the above model is applicable over the time scale of the deadline interval. For example, when a mobile device is moving in the direction of the base station, the channel has an expected drift towards improving conditions and vice-versa. Similarly, in case of satellite channels, changing weather conditions such as cloud cover makes the channel drift towards worsening conditions and vice-versa. In cases when the time scale of these drift changes is longer than the packet deadlines, the constant drift channel serves as an appropriate model.

The next theorem, Theorem II, gives the optimal policy result for the monomial class of functions while Theorem III gives the result for the exponential class.

Theorem *II: Consider the BT-problem with $g(r) = r^n$, $n > 1, n \in \mathbb{R}$ and a constant drift channel with drift β. The optimal policy, $r^*(x,c,t)$, and the function, $J(x,c,t)$, are,*

$$r^*(x,c,t) = \frac{x}{f(T-t)} \tag{22}$$

$$J(x,c,t) = \frac{x^n}{c(f(T-t))^{n-1}} \tag{23}$$

where $f(T-t) = \frac{(n-1)}{\lambda(\beta-1)}(1 - \exp(-\frac{\lambda(\beta-1)}{n-1}(T-t)))$.

Proof: For the constant drift channel model, the functions $f_i(s)$ are the same for all the channel states. Denoting this common function as $f(s)$ the ODE equation from Theorem I becomes $f'(s) = 1 - \frac{\lambda(\beta-1)}{n-1}f(s)$. Evaluating this for the boundary conditions in Theorem I gives the above result. The details of the proof are omitted for brevity and can be found in [3]. ■

The closed-form expression of $f(\cdot)$ above provides an interesting intuitive observation related to the parameter β. Suppose that the present channel state is c, then for a fixed r the expected power cost for the next channel state is $\left(E\left[\frac{g(r)}{Z(c)c}\right] = \frac{g(r)}{c}\beta\right)$ which is β times the present cost, $\frac{g(r)}{c}$. This means that for higher values of parameter β, the channel on every transition drifts in an expected sense towards higher expected power cost or worsening conditions and vice-versa as β decreases. Hence, as expected, the urgency of transmission $1/f(t)$ is an increasing function with respect to β since for large β values it becomes more energy efficient to utilize the present channel conditions. Interestingly, when $\beta = 1$, the expected future power cost does not change and in this case the optimal policy reduces to the direct drain (DD) policy $r^*(x,c,t) = \frac{x}{T-t}$. Thus, we see that the direct drain policy is optimal both under no channel variations and under a constant drift channel model with $\beta = 1$.

Theorem *III: Consider the BT-problem with $g(r) = \alpha^r - 1$, $\alpha > 1$ and a constant drift channel with drift β. The optimal policy, $r^*(x,c,t)$, is the following,*

Case 1: $\beta \geq 1$,

$$r^*(x,c,t) = \begin{cases} \sqrt{\frac{2x\lambda(\beta-1)}{\ln \alpha}}, & 0 \leq x < \frac{\lambda(\beta-1)(T-t)^2}{2\ln\alpha} \\ \frac{x}{T-t} + \frac{\lambda(\beta-1)(T-t)}{2\ln\alpha}, & x \geq \frac{\lambda(\beta-1)(T-t)^2}{2\ln\alpha} \end{cases} \tag{24}$$

Case 2: $0 < \beta < 1$,

$$r^*(x,c,t) = \begin{cases} 0, & 0 \leq x < \frac{\lambda(1-\beta)(T-t)^2}{2\ln\alpha} \\ \frac{x}{T-t} - \frac{\lambda(1-\beta)(T-t)}{2\ln\alpha}, & x \geq \frac{\lambda(1-\beta)(T-t)^2}{2\ln\alpha} \end{cases} \tag{25}$$

Proof: To obtain the above functions we consider a discrete approximation of the time interval $[0,T]$ with step size dt. Using discrete dynamic programming (DP), we obtain the optimal functions for the discrete system and then take the limit $dt \to 0$; the limiting functions given above can be shown to satisfy the optimality equation in (15). The details are omitted for brevity and can be found in [3]. ■

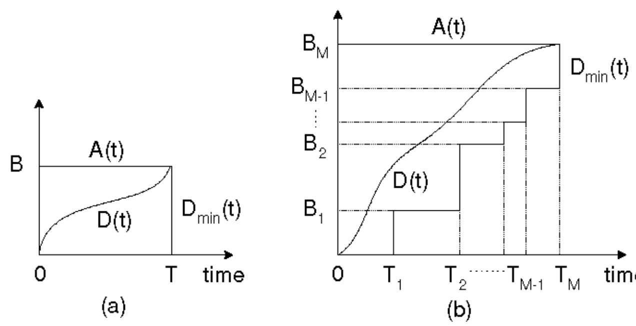

Fig. 6. Cumulative curves for (a) BT-problem, (b) Variable deadlines case.

From above, we see that the optimal rate function has all the properties discussed earlier - it is monotonically increasing in x, increasing as t approaches the deadline and also increasing in the drift parameter β.

IV. Variable Deadlines Setup

In the last section, we dealt with a specific case of the energy minimization problem involving B bits of data and a single deadline T. We now extend the results to a more general setup where the data in the queue has variable deadlines. We adopt a cumulative curves methodology [20] and using a decomposition of the problem in terms of the BT-problem, obtain the optimal policy for this setup. As will be evident, the novel cumulative curves formulation provides a very appealing and simple visualization of the problem.

A. Problem Setup

Let us first define the following cumulative curves. Define the *Arrival Curve*, $A(t)$, as the total number of bits that have arrived to the queue in time $[0, t]$; the *Departure Curve*, $D(t)$, as the total number of bits that have departed (served) in time interval $[0, t]$ and the *Minimum Departure Curve*, $D_{min}(t)$, as the minimum number of bits that must depart by time t to satisfy the deadline constraints. For example, in the BT-problem case, we have $A(t) = B,\ t \in [0, T]$ since the queue has B bits to begin with at time 0 and no more data is added. We have $D_{min}(t) = 0,\ t \in [0, T), D_{min}(T) = B$ since until the deadline $t < T$ there is no minimum data transmission requirement while at T the entire B bits must have been transmitted. Finally, the curve $D(t)$ represents the data departure over time which depends on the chosen transmission policy. A schematic diagram of this is given in Figure 6(a).

Consider now the variable deadlines problem. Here, the queue has M packets that are arranged and served in the earliest-deadline-first order. Let b_j be the number of bits in the j^{th} packet and T_j be the deadline for this packet; assume $0 < T_1 < T_2 < \ldots < T_M$. There are no new arrivals and the objective is to obtain a transmission policy that serves this data over the time-varying channel with minimum expected energy cost while meeting the deadline constraints. In terms of the cumulative curves, the setup can be visualized as depicted in Figure 6(b). Let $B_j = \sum_{l=1}^{j} b_l$; where B_j is the cumulative amount of data of the first j packets. Then, $A(t) = B_M, \forall t$, since a total B_M bits are in the queue at time 0 and no more data is added. And, $D_{min}(t)$ is a piecewise-constant curve with jumps at times T_j, i.e. at time T_j, $D_{min}(T_j) = B_j$ since the first B_j bits must be transmitted by T_j. Finally, we require that for admissibility a transmission policy must be such that the departure curve, $D(t)$, satisfy $D_{min}(t) \le D(t) \le A(t)$; in other words, data must be served such that the cumulative amount lies above the minimum departure curve (to satisfy the deadline constraints) and below the arrival curve (to satisfy the causality constraints).

B. Optimal Policy

It is evident that a direct solution of the variable deadlines problem stated in the previous section is fairly difficult due to the complexity of the multiple deadline constraints involved. Interestingly, however, the cumulative curves formulation with its graphical visualization provides an intuitive and natural decomposition of the problem in terms of multiple BT-problems which then yields the optimal solution. A visual comparison of the two diagrams in Figure 6 suggests the following approach. First, we can visualize the deadline constraints in terms of the cumulative amounts as $\{B_j T_j\}_{j=1}^{M}$ constraints, that is, a total of B_j bits must be transmitted by deadline T_j ($j = 1, \ldots, M$). Clearly, each $B_j T_j$ constraint is like a BT-problem except that now there are multiple such constraints that all need to be satisfied. For every time t and channel state c, we know the optimal transmission rate to meet each of the $B_j T_j$ constraint individually (assuming only this constraint existed), thus, to meet all the constraints a natural solution is to simply choose the maximum value among the transmission rates evaluated for all of the $B_j T_j$ constraints.

More precisely, the transmission policy is described as follows. Let the system state be denoted as (D, c, t), where D is the cumulative data that has been transmitted by time t and c is the corresponding channel state. Using the optimal rate function in (16), the rate for an individual $B_j T_j$ constraint for channel state c^i, is $\frac{B_j - D}{f_i(T_j - t)}$, since $(B_j - D)$ is the amount of data left and $(T_j - t)$ is the time left until the deadline T_j. Let $r^*(D, c, t)$ denote the transmission rate, then $r^*(\cdot)$ is the maximum value among the rates for all $B_j T_j$ constraints for which $(B_j \ge D, T_j \ge t)$.

$$r^*(D, c^i, t) = \max_{j:(B_j \ge D, T_j \ge t)} r_j^*(B_j - D, c^i, t) \quad (26)$$

$$= \max_{j:(B_j \ge D, T_j \ge t)} \frac{B_j - D}{f_i(T_j - t)} \quad (27)$$

Clearly, by construction all the $B_j T_j$ constraints are satisfied since at all times we choose the maximum rate among those needed to meet each of the remaining constraints. Hence, the policy in (26) is admissible and starting with $D(0) = 0$ the departure curve obtained using (26) satisfies $D_{min}(t) \le D(t) \le A(t),\ t \in [0, T_M]$. Furthermore, the following theorem also shows that the above policy is in fact optimal.

Theorem *IV: **(Variable Deadlines Case)** Consider the variable deadlines problem with $g(r) = r^n$, $n > 1, n \in \mathbb{R}$ and the Markov channel model. The optimal policy, $r^*(D, c, t)$ for $D_{min}(t) \leq D \leq A(t)$, $t \in [0, T_M)$ is given as,*

$$r^*(D, c^i, t) = \max_{j:(B_j \geq D, T_j \geq t)} \frac{B_j - D}{f_i(T_j - t)}, \quad i = 1, \ldots, m \quad (28)$$

The functions $\{f_i(s)\}_{i=1}^m$ are the solution of the ODE system in (18)-(19) with the boundary conditions $f_i(0) = 0, f_i'(0) = 1, \forall i$

Proof: The proof is based on a direct verification that the above functional form satisfies the HJB equation and the boundary conditions. It is omitted here for brevity and can be found in [3]. ■

The optimal solution in (28) is based on the BT-solution, therefore clearly it inherits all the properties of that solution. As before the functions $\{f_i(s)\}_{i=1}^m$ can be obtained numerically using a standard ODE solver. Furthermore, this computation needs to be done only once before the system starts operating and $\{f_i(s)\}$ can be pre-determined and stored in a table in the transmitter memory. Once $\{f_i(s)\}$ are known, the online computation is minimal. At time t, the transmitter simply looks at the cumulative amount of data transmitted D, the channel state, c^i, and then using the appropriate $f_i(\cdot)$ function it computes the maximum among a set of values as given in (28).

V. Packet Arrivals with Deadlines

The optimal solution to the variable deadlines problem provides a simple heuristic way to extend the results to a more general setup involving packet arrivals to the queue. Consider now an arbitrary stream of packet arrivals to the queue with each packet having a deadline by which it must depart. Regardless of the underlying stochastic process generating the packets, we next present a heuristic energy-efficient policy based on the variable deadlines solution. We call it the "***BT-Adaptive***" (BTA) policy. Later, we present simulation results comparing the BTA policy with a non-adaptive scheme.

A. BT-Adaptive (BTA) Policy

Consider packet arrivals to the queue and assume that the arrivals occur at discrete times with each packet having a deadline associated with it. Clearly, at the instant immediately following a packet arrival the transmitter queue consists of, (a) earlier remaining packets with their deadlines and (b) the new packet with its own deadline. Re-arranging the data in the queue in the earliest-deadline-first order we can view the queue as consisting of some amount B_M of data with variable deadlines, identical to the case considered in the last section. Neglecting the future arrivals and using (28), we have the optimal policy to empty the transmitter buffer. As this policy is followed, at the next packet arrival instance the above procedure is repeated by updating the data amount taking into account the new packet. Summarizing, the BTA policy is as follows:

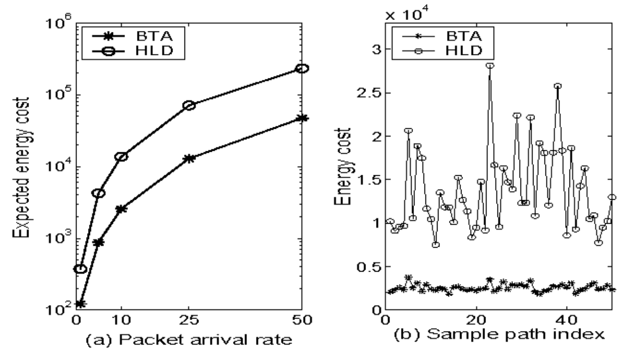

Fig. 7. Energy cost comparison for Poisson arrival process for (a) different arrival rate, (b) different sample paths.

Transmit the data in the queue with the rate as given in (28); at every packet arrival instant re-arrange the data in the earliest-deadline-first order to obtain a new set of B_iT_i values including the new packet and its deadline; re-initialize D to zero and follow (28) thereafter.

Note that since this policy is not based on a specific arrival process, an interesting feature of it is that it is robust to changes in the arrival statistics and can accommodate multiple deadline classes of packet arrivals to the queue.

B. Simulation Results

In this section, we present simulation results to evaluate the performance of the BT-Adapative policy. For comparison purposes we consider a policy that can be easily implemented in practice and refer to it as the "Head-of-Line Drain" (HLD) policy. In HLD policy, the data in the queue is arranged in the earliest-deadline-first order and the packets are served in that order. At time t, let H_t be the amount of data left in the head-of-the-line packet and T_H be its deadline, then the rate chosen is $r_t = \frac{H_t}{T_H}$. Thus, the transmitter serves the first packet in queue at a rate to transmit it out by its deadline, then moves to the next packet in line and so on. At every packet arrival instant, the data in the queue is re-arranged in the earliest-deadline-first order and the above policy is repeated with the new packet taken into account.

The setup is as follows. The queue has packet arrivals and each packet has a deadline associated with it. On each simulation run, the total time over which the packets arrive and the system is operated is taken as $L = 10$ seconds. This interval $[0, 10]$ is partitioned into $10,000$ slots, thus each slot is of duration $dt = 1$ msec. The channel model is the two state model, described in Section II-B, with the parameters, $c_g = 1$, $c_b = 0.2$, $\lambda_{bg} = \lambda_{gb} = \lambda = 50$. Thus, the average time spent in a state before the channel transitions is $1/50$ seconds, or 20 msec. A channel sample path is simulated using a Bernoulli process where in a slot the channel transitions with probability λdt; otherwise there is no transition. For simplicity, the packet arrival and the channel state transitions occur only at the slot boundaries. For both the BTA and the HLD policies, the rate chosen in a slot is obtained by evaluating the respective

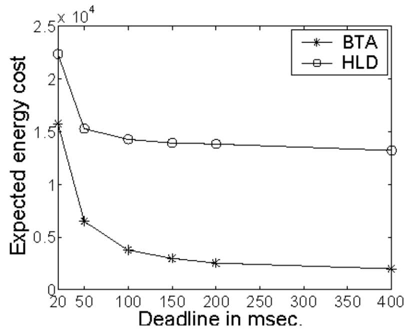

Fig. 8. Energy cost versus packet deadline for Poisson arrival process.

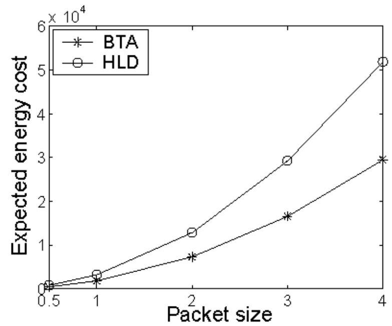

Fig. 9. Energy cost versus packet size for Uniform arrival process.

policies at the time corresponding to the start of that slot. We take $g(r) = r^2$, with appropriate units that have been omitted. Energy cost per slot is $\frac{r^2 dt}{c}$ and the total expected energy cost is obtained as an average of the total cost over sample runs.

We first consider a Poisson packet arrival process with each packet having 1 unit of data and a deadline of 200 msec. Figure 7(a) is a plot of the expected energy cost, plotted on a log scale, versus the packet arrival rate. Note that a packet arrival rate of 10 implies that the average inter-arrival time of a packet is $1/10$ sec. or 100 msec. As is evident from the plot, the BTA policy has a much lower energy cost compared to the HLD policy and as the arrival rate increases the two costs are roughly an order of magnitude apart. This can be intuitively explained as follows. When the arrival rate is low, most of the time the queue has at most a single packet. Hence, both policies choose a rate based on the head-of-line packet with the BTA policy also adapting the rate with the channel state. As the arrival rate increases and due to the bursty nature of the Poisson process, the queue tends to have more packets. The BTA policy then adapts based on the channel and the deadlines of all the packets in the queue, whereas, the HLD policy chooses a rate based solely on the head-of-line packet. The energy efficiency of the BTA policy is not just in an average sense but even on individual sample paths. This is shown in Figure 7(b) for 50 sample paths for arrival rate 10.

In Figure 8, the packet arrival process is Poisson with rate 10 but now the packet deadline is varied. Clearly, as seen in the figure, the energy cost decreases as the packet deadline increases since lower transmission rates are required to meet the deadlines. Also, as the deadline increases the energy cost difference between the BTA and the HLD policy increases. This is because with a larger delay constraint there is more room for the adaptive techniques employed in the BTA policy to have a larger effect.

In Figure 9, we consider a Uniform packet arrival process where now the inter-arrival time between packets is uniformly distributed between 50 and 150 msec. The deadline for each packet is taken as 200 msec while the packet size is varied. First, as expected the energy cost for both the policies increases with packet size and second, as before the BTA policy has a much less energy cost compared with the HLD policy even when the arrival process is less bursty as compared to the Poisson process.

VI. Conclusion

We considered transmission of delay-constrained data over time-varying channels with the objective of minimizing the total transmission energy expenditure. We adopted a novel approach based on a continuous-time formulation and stochastic control theory to obtain optimal solutions for an otherwise difficult set of problems. We first considered the problem of transmitting B bits of data by deadline T and obtained the optimal rate adaptation policy. Using a cumulative curves methodology and a decomposition approach, we then obtained the optimal solution when the data in the queue has variable deadline constraints. Finally, we presented an energy-efficient transmission policy for arbitrary packet arrival process and compared its performance through simulations. We believe that the framework of this paper holds promise for various extensions addressing QoS-constrained data transmission in wireless systems. Some of the natural extensions include a network model with multiple transmitter-receiver pairs and multi-hop transmissions with end-to-end delay constraints.

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