A Novel Auction Algorithm for Fair Allocation of A Wireless Fading Channel

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***Abstract*—We develop a novel auction-based algorithm to allow users to fairly compete for a wireless fading channel. We use the all-pay auction mechanism whereby user bid for the channel, during each time-slot, based on the fade state of the channel, and the user that makes the higher bid wins use of the channel. Under the assumption that each user has a limited budget for bidding, we show the existence of a unique Nash equilibrium strategy. We show that the strategy achieves a throughput allocation for each user that is proportional to the user's budget and establish that the aggregate throughput received by the users using the Nash equilibrium strategy is at least 3/4 of what can be obtained using an optimal centralized allocation scheme that does not take fairness into account.**

***Index Terms*—Resource allocation, fairness, auction.**

I. Introduction

Network resources such as bandwidth and power are often limited in wireless and satellite networks. When demand exceeds supply, it is desirable to have a systematic procedure in place for fair allocation. However, there is no consensus on the notion of fairness. Any centrally imposed notion of fairness may appear to be unfair from an individual user's perspective. In this paper, we address the problem of fair resource allocation by allowing individual users to compete for resources through bidding for the use of the channel.

A fundamental characteristic of a wireless network is that the channel over which communication takes place is often time-varying. This variation of the channel quality is due to constructive and destructive interference between multipaths and shadowing effects (fading). In a single cell with one transmitter (base station or satellite) and multiple users communicating through time-varying fading channels, the transmitter can send data at higher rates to users with better channels. In time slotted system such as the HDR system, time slots are allocated among users according to their channel qualities.

The problem of resource allocation in wireless networks has received much attention in recent years. In [1] the authors try to maximize the data throughput of an energy and time constrained transmitter sending over a fading channel. A dynamic programming formulation that leads to an optimal transmission schedule is presented. Other works address the similar problem, without consideration to fairness, include [7] and [8]. In [5], the authors consider scheduling policies for *maxmin fairness* allocation of bandwidth, which maximizes the allocation for the most poorly treated sessions while not wasting any network resources, in wireless ad-hoc networks. In [4], the authors designed a scheduling algorithm that achieves *proportional fairness*, a notion of fairness originally proposed by Kelly [6]. In [9], the authors present a slot allocation scheme that maximizes expected system performance subject to the constraint that each user gets a fixed fraction of time slots. The authors did not use a formal notion of fairness, but argue that their system can explicitly set the fraction of time assigned to each user. Hence, while each user may get to use the channel an equal fraction of the time, the resulting throughput obtained by each user may be vastly different.

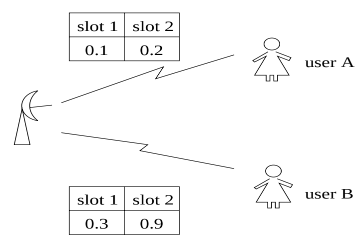

Fig. 1. A communication system with one transmitter and two users with the specified channel condition.

The following simple example illustrates the different allocation that may result from the different notions of fairness. We consider the communication system depicted in Fig. 1 and the allocation schemes that use different notions of fairness discussed in the previous paragraph. Fig. 1 shows a communication system with one transmitter and two users A and B. We assume that the throughput is proportional to the the channel condition. The channel state for user A and user B in the two time slots are (0.1, 0.2) and (0.3, 0.9) respectively (channel coefficient ranges from 0 to 1, and 1 is the best channel condition). The throughput result for each individual user and for total system under different notions of fairness constraint are given in Table I. When there is no fairness constraint, to maximize the total system throughput would require the transmitter to allocate both time slots to user B. To achieve maxmin fair allocation, the transmitter would allocate slot one to user B and slot two to user A, thus resulting in a total throughput of 0.5. If the transmitter wants to maximize the total throughput subject to the constraint that each user gets one time slot (i.e., the approach of [9]), the resulting allocation scheme, denoted as time fraction fair, is to give user A slot one and user B slot two. As a result, the total throughput is 1.0.

	Throughput for user A	Throughput for user B	Total throughput
No fair constraint	0	1.2	1.2
Maxmin fair	0.2	0.3	0.5
Time fraction	0.1	0.9	1.0

TABLE I
THROUGHPUT RESULTS USING DIFFERENT NOTIONS OF FAIRNESS.

In the above example, the transmitter selects an allocation scheme to ensure an artificially chosen notion of fairness. From Table I, we can see that from the user's perspective, no notion is truly fair as both users want slot two. In order to resolve this conflict, we use a new approach which allows users to compete for time slots. In this way, each user is responsible for its own action and resulting throughput. We call the fraction of bandwidth received by each user *competitive fair*. Using this notion of competitive fairness, the resulting throughput obtained for each user can serve as a reference point for comparing various allocation schemes.

In our model, users compete for time-slots. For each time-slot, each user has a different valuation (i.e., its own channel condition). And each user is only interested in getting a higher throughput for itself. Naturally, these characteristics give rise to an auction. In this paper we consider the all-pay auction mechanism. Using the all-pay auction mechanism, users submit a "bid" for the time-slot and the transmitter allocates the slot to the user that made the highest bid. Moreover, in the all-pay auction mechanism, the transmitter gets to keep the bids of all users (regardless of whether or not they win the auction). Each user is assumed to have an initial amount of money. The money possessed by each user can be viewed as fictitious money that serves as a mechanism to differentiate the QoS given to the various users. This fictitious money, in fact, could correspond to a certain QoS for which the user paid in real money. As for the solution of the slot auction game, we use the concept of Nash equilibrium, which is a set of strategies (one for each player) from which there are no profitable unilateral deviation.

In this paper, we consider a communication system with one transmitter and two users. For each time slot, channel states are independent and identically distributed with known probability distribution. Each user wants to maximize its own *expected* total throughput subject to an average money constraint.

We have the following main results:

- We find a unique Nash equilibrium when both channel states are uniformly distributed over $[0, 1]$.
- We show that the Nash equilibrium strategy pair provides an allocation scheme that is fair in the sense that the price per unit of throughput is the same for both users.
- We show that the Nash equilibrium strategy of this auction leads to an allocations at which total throughput is no worse than 3/4 of the throughput obtained by an algorithm that attempts to maximize total system throughput without a fairness constraint.
- We provide an estimation algorithm that enables users to accurately estimate the amount of money possessed by their opponent so that users do not need prior knowledge of each other's money.

Game theoretical approaches to resource allocation problems have been explored by many researchers recently (e.g., [2][11]). In [2], the authors consider a resource allocation problem for a wireless channel, without fading, where users have different utility values for the channel. They show the existence of an equilibrium pricing scheme where the transmitter attempts to maximize its revenue and the users attempt to maximize their individual utilities. In [11], the authors explore the properties of a congestion game where users of a congested resource anticipate the effect of their action on the price of the resource. Again, the work of [11] focuses on a wireline channel without the notion of wireless fading. Our work attempts to apply game theory to the allocation of a wireless fading channel. In particular, we show that auction algorithms are well suited for achieving fair allocation in this environment. Other papers dealing with the application of game theory to resource allocation problems include [3][15][16].

This paper is organized as follows. In Section II, we describe the communication system and the auction mechanism. In Section III, the unique Nash equilibrium strategy pair and the resulting throughput for each user are presented for the case that each user can use only one bidding function. In Section IV, we show the unique Nash equilibrium strategy pair for the case that each user can use multiple bidding functions. In Section V, we compare the throughput results of the Nash equilibrium strategy with two other centralized allocation algorithms. Finally, Section VI concludes the paper.

II. PROBLEM FORMULATION

We consider a communication environment with a single transmitter sending data to two users over two different fading channels. We assume that there is always data to be sent to the users. Time is assumed to be discrete, and the channel state for a given channel changes according to a known probabilistic model independently over time. The two channels are also assumed to be independent of each other. The transmitter can transmit to only one user during a particular slot with a constant power P. The channel fade state thus determines the throughput that can be obtained.

For a given power level, we assume for simplicity that the throughput is a linear function of the channel state. This can be justified by the Shannon capacity at low signal-to-noise ratio, or by using a fixed modulation scheme [1]. For general throughput function, the method used in this paper applies as well. Let X_i be a random variable denoting the channel state for the channel between the transmitter and user i, $i = 1, 2$. When transmitting to user i, the throughput will then be $P \cdot X_i$. Without loss of generality, we assume $P = 1$ throughout this paper.

We now describe the all-pay auction rule used in this paper. Let α and β be the *average* amount of money available to user 1 and user 2 respectively during each time slot. We assume that the values of α and β are known to both users. Both users know the distribution of X_1 and X_2. We also assume that the exact value of the channel state X_i is revealed to user i only at the beginning of each time slot. During each time slot, the following actions take place:

1) Each user submits a bid according to the channel condition revealed to it.
2) The transmitter chooses the one with higher bid to transmit.
3) Once a bid is submitted by the user, it is taken by the transmitter regardless of whether the user gets the slot or not, i.e., no refund for the one who loses the bid.

The formulation of our auction is different from the type of auction used in economic theory in several ways. First, we look at a case where the number of object in the auction goes to infinity. While in the current auction research, the number of object is finite [12][13][14]. Second, in our auction formulation, the money used for bidding does not have a direct connection with the value of the time slot. Money is merely a tool for users to compete for time slots, and it has no value after the auction. Therefore, it is desirable for each user to spend all of its money. However, in auction theory, an object's value is measured in the same unit as the money used in the bidding process, hence their objective is to maximize the difference between the object's value and its cost. Lastly, in our formulation, the valuation of each commodity (time-slot) changes due to the fading channel model; a notion that is not common in economic theory.

Besides the all-pay auction, *first-price* auction and *second-price* auction are two other commonly used auction mechanisms. In the first-price auction, each bidder submits a single bid without seeing the others' bids, and the object is sold to the bidder who makes the highest bid. The winner pays its bid. In the second price auction, each user independently submits a single bid without seeing the others' bids, and the object is sold to the bidder who makes the highest bid. However, the price it pays is the *second*-highest bidder's bid [12]. We choose to use the all-pay auction in this paper to illustrate the auction approach to resource allocation in wireless networks. We believe that other auction mechanisms can be similarly applied.

The objective for each user is to design a bidding strategy, which specifies how a user will act in every possible distinguishable circumstance, to maximize its *own* expected throughput per time slot subject to the expected or average money constraint. Once a user, say user 1, chooses a function, say $f_1^{(i)}$, for its strategy in the ith slot, it bids an amount of money equal to $f_1^{(i)}(x)$ when it sees its channel condition in the ith slot is $X_1 = x$.

Formally, let F_1 and F_2 be the set of continuous and bounded real-valued functions with finite first and second derivative over the support of X_1 and X_2 respectively. Then, the strategy space for user 1, say S_1, and user 2, say S_2, are defined as follows:

$$
\begin{aligned}
S_1 &= \left\{ f_1^{(1)}, \cdots, f_1^{(n)} \in F_1 \;\middle|\; \frac{1}{n}\sum_{i=1}^{n} E[f_1^{(i)}(X_1)] = \alpha \right\} \\
S_2 &= \left\{ f_2^{(1)}, \cdots, f_2^{(n)} \in F_2 \;\middle|\; \frac{1}{n}\sum_{i=1}^{n} E[f_2^{(i)}(X_2)] = \beta \right\}
\end{aligned} \tag{1}
$$

For each user, a strategy is a sequence of bidding functions $f^{(1)}, \cdots, f^{(n)}$. Without loss of generality, we restrict each user to have n different bidding functions, where n can be chosen as an arbitrarily large number. Note that users choose a strategy for a block of n time slots instead of just for a single time slot, one bidding function for each slot. In order to maximize the overall throughput (over infinite horizon), each user chooses bidding functions to maximize the expected total throughput over this block of n slots. The term $E[f_1^{(i)}(X_1)]$ denotes the expected amount of money spent by user 1 if it uses bidding function $f_1^{(i)}$ for the ith slot in the block.

We first consider a special class of strategies in which each user can use only a single bidding function. More specifically, by setting $f_1 = f_1^{(1)} = \cdots = f_1^{(n)}$ and $f_2 = f_2^{(1)} = \cdots = f_2^{(n)}$, we have the following:

$$
\begin{aligned}
\bar{S}_1 &= \left\{ f_1 \in F_1 \;\middle|\; E[f_1(X_1)] = \alpha \right\} \\
\bar{S}_2 &= \left\{ f_2 \in F_2 \;\middle|\; E[f_2(X_2)] = \beta \right\}
\end{aligned} \tag{2}
$$

By considering first the set of strategies in $\bar{S}_1$ and $\bar{S}_2$, we are able to find the Nash equilibrium strategy pair within the set S_1 and S_2.

Given a strategy pair (f_1, f_2), where $f_1 \in \bar{S}_1$ and $f_2 \in \bar{S}_2$, the expected throughput or payoff function for user 1 is defined as the following assuming the constant power $P = 1$:

$$
G_1(\alpha, \beta) = E_{X_1, X_2}[X_1 \cdot 1_{f_1(X_1) \geq f_2(X_2)}] \tag{3}
$$

where

$$
1_{f_1(X_1) \geq f_2(X_2)} = \begin{cases} 1 & \text{if} \quad f_1(X_1) \geq f_2(X_2) \\ 0 & \text{otherwise} \end{cases}
$$

Similarly, the throughput function for user 2 assuming $P = 1$:

$$
G_2(\alpha, \beta) = E_{X_1, X_2}[X_2 \cdot 1_{f_2(X_2) > f_1(X_1)}] \tag{4}
$$

Throughout this paper, for simplicity, we let the channel state X_i be uniformly distributed over $[0, 1]$. However, our approach can be extended to the case where the channel state has a general distribution. Due to space limitations, we omit the more complex analysis for general channel state distribution.

III. Unique Nash equilibrium strategy with a single bidding function

We present in this section a unique Nash equilibrium strategy pair (f_1^*, f_2^*). A strategy pair (f_1^*, f_2^*) is said to be in Nash equilibrium if f_1^* is the best response for user 1 to user 2's strategy f_2^*, and f_2^* is the best response for user 2 to user 1's strategy f_1^*. We consider here the case where both users choose their strategies from the strategy space $\bar{S}_1$ and $\bar{S}_2$ (i.e., the single bidding function strategy) and the value of α and β are known to both users.

To get the Nash equilibrium strategy pair, we first argue that an equilibrium bidding function must be nondecreasing. To see this, consider an arbitrary bidding function f such that $f(a) > f(b)$ for some $a < b$. If user 1 chooses f as its bidding function, user 1 will be better off if it bids $f(b)$ when the channel state is a and $f(a)$ when the channel state is b. This way, its odds of winning the slot when the channel state is b, which is more valuable to it, will be higher than before, and it has an incentive to change its strategy (i.e., f is not an equilibrium strategy). Hence, we conclude that, for each user, an equilibrium bidding function must be nondecreasing.

We further restrict users' bidding functions to be *strictly* increasing for technical reason which will be explained later. There is no loss of generality in this assumption because any continuous, bounded, nondecreasing function can be approximated by a *strictly* increasing function arbitrarily closely.

Next, we show some useful properties associated with the equilibrium strategy pair (f_1^*, f_2^*). All of the proofs in this paper are omitted for brevity.

***Lemma** 1:* If (f_1^*, f_2^*) is a Nash equilibrium strategy pair, $f_1^*(1) = f_2^*(1)$.

We have just established that $f_1^*(1) = f_2^*(1)$ is a necessary condition for (f_1^*, f_2^*) to be an equilibrium strategy pair. We also find that $f_1^*(0) = f_2^*(0) = 0$ since it does not make sense to bid for a slot with zero channel state. Thus, from now on, to find the Nash equilibrium strategy pair (f_1^*, f_2^*), we will consider only the function pair $f_1 \in \bar{S}_1$ and $f_2 \in \bar{S}_2$ that are strictly increasing and satisfying the above two boundary conditions (i.e., $f_1(1) = f_2(1)$ and $f_1(0) = f_2(0) = 0$).

These two boundary conditions, together with strictly increasing property of $f_1 \in \bar{S}_1$ and $f_2 \in \bar{S}_2$, make the inverse of f_1 and f_2 well defined. Thus, we are able to define the following terms. With user 2's strategy f_2 fixed, let $g_{f_2}^{(1)} : (x_1, b) \rightarrow \mathcal{R}$ denote user 1's expected throughput of a slot conditioning on the following events:

- User 1's channel state is $X_1 = x_1$.
- User 1's bid is b.

Specifically, we can the write the equation:

$$g_{f_2}^{(1)}(x_1, b) = x_1 \cdot P(f_2(X_2) \leq b) \tag{5}$$

where $P(f_2(X_2) \leq b)$ is the probability that user 1 wins the time slot. Consequently, using a strategy f_1, user 1's throughput is given by:

$$\begin{aligned} G_1(\alpha, \beta) &= \int_0^1 g_{f_2}^{(1)}(x_1, f_1(x_1)) \cdot p_{X_1}(x_1)\, dx_1 \\ &= \int_0^1 g_{f_2}^{(1)}(x_1, f_1(x_1))\, dx_1. \end{aligned} \tag{6}$$

where the last equality results from the uniform distribution assumption.

With user 1's strategy f_1 fixed, similar terms for user 2 can be defined.

$$g_{f_1}^{(2)}(x_2, b) = x_2 \cdot P(f_1(X_1) \leq b)$$

Then, user 2's throughput is given by:

$$\begin{aligned} G_2(\alpha, \beta) &= \int_0^1 g_{f_1}^{(2)}(x_2, f_2(x_2)) \cdot p_{X_2}(x_2)\, dx_2 \\ &= \int_0^1 g_{f_1}^{(2)}(x_2, f_2(x_2))\, dx_2. \end{aligned} \tag{7}$$

Due to the uniformly distributed channel state, $P(f_2(X_2) \leq b)$ is given by

$$P(f_2(X_2) \leq b) = P(X_2 \leq f_2^{-1}(b)) = f_2^{-1}(b)$$

where f_2^{-1} is well defined. Thus, we can rewrite Eq. (5) as

$$g_{f_2}^{(1)}(x_1, b) = x_1 \cdot f_2^{-1}(b).$$

Hence we have,

$$G_1(\alpha, \beta) = \int_0^1 x_1 \cdot f_2^{-1}(f_1(x_1))\, dx_1 \tag{8}$$

$$G_2(\alpha, \beta) = \int_0^1 x_2 \cdot f_1^{-1}(f_2(x_2))\, dx_2 \tag{9}$$

The following lemma gives a necessary and sufficient condition of a Nash equilibrium strategy pair. For convenience, we denote $\frac{\partial g_{f_2}^{(1)}(x_1,b)}{\partial b}|_{b=b^*}$ (i.e., the marginal gain at $b = b^*$) as $Dg_{f_2}^{(1)}(x_1, b^*)$.

***Lemma** 2:* A strategy pair (f_1^*, f_2^*) is a Nash equilibrium strategy pair if and only if $Dg_{f_2^*}^{(1)}(x_1, f_1^*(x_1)) = c_1$ and $Dg_{f_1^*}^{(2)}(x_2, f_2^*(x_2)) = c_2$, for some constants c_1 and c_2, for all $x_1 \in [0, 1]$ and all $x_2 \in [0, 1]$.

To understand the lemma intuitively, suppose there exists $x \neq \tilde{x}$ such that $Dg_{f_2^*}^{(1)}(x, f_1^*(x)) > Dg_{f_2^*}^{(1)}(\tilde{x}, f_1^*(\tilde{x}))$. Reducing the bid at $\tilde{x}$ to $f_1^*(\tilde{x}) - \delta$ and increasing the bid at x to $f_1^*(x) + \delta$ will result in an increase in the throughput by $(Dg_{f_2^*}^{(1)}(x, f_1^*(x)) - Dg_{f_2^*}^{(1)}(\tilde{x}, f_1^*(\tilde{x}))) \cdot \delta$. Thus, user 1 has an incentive to change its bidding function, and (f_1^*, f_2^*) cannot be a Nash equilibrium strategy pair in this case.

With Lemma 2, we are able to find the unique Nash equilibrium strategy pair. The exact form of the equilibrium bidding strategies are presented in the following Theorem.

***Theorem** 1:* Under the assumption of a single bidding function, the following is a unique Nash equilibrium strategy pair for the auction:

$$f_1^*(x) = c \cdot x^{\gamma+1} \tag{10}$$

$$f_2^*(x) = c \cdot x^{\frac{1}{\gamma}+1} \tag{11}$$

where the constant γ and c are chosen such that

$$\int_0^1 c \cdot x^{\gamma+1}\, dx = \alpha \tag{12}$$

$$\int_0^1 c \cdot x^{\frac{1}{\gamma}+1}\, dx = \beta \tag{13}$$

Equations (12) and (13) impose the average money constraints. Fig. 2 shows an example of the Nash equilibrium bidding strategy pair when $\alpha = 1$ and $\beta = 2$. Since user 1 has less money than user 2, user 1 concentrates its bidding on time slots with very good channel state.

Fig. 3 shows the resulting allocation scheme when both users employ the Nash equilibrium strategy shown in Fig. 2. Above the curve, time slots will be allocated to user 2 since user 2's

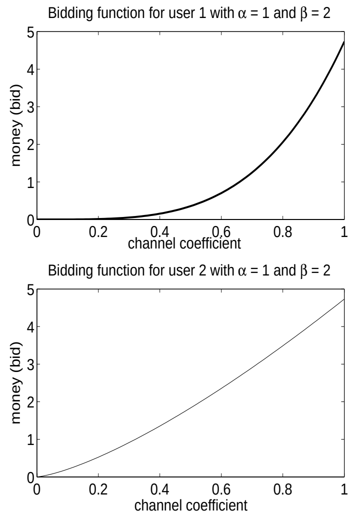

Fig. 2. An example of Nash equilibrium strategy pair for $\alpha = 1$ and $\beta = 2$.

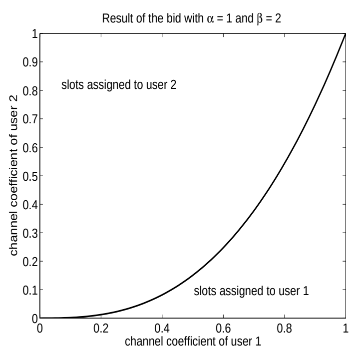

Fig. 3. Allocation scheme from Nash equilibrium strategy pair for $\alpha = 1$ and $\beta = 2$.

bid is higher than user 1's in this region. Similarly, user 1 gets the slots below the curve. Here, user 2 is allocated more slots than user 1 since it has more money.

If both players use the Nash equilibrium strategy, the expected throughput obtained are given by:

$$G_1(\alpha, \beta) = \frac{\alpha}{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \alpha\beta}} \tag{14}$$

$$G_2(\alpha, \beta) = \frac{\beta}{\alpha + \beta + \sqrt{(\alpha - \beta)^2 + \alpha\beta}} \tag{15}$$

As can be seen, the ratio of the throughput obtained $\frac{G_1(\alpha,\beta)}{G_2(\alpha,\beta)}$ is equal to $\frac{\alpha}{\beta}$ which is the ratio of the money each user had initially. Thus, the Nash equilibrium strategy pair provides an allocation scheme that is fair in the sense that the price per unit of throughput is the same for both users.

IV. Unique Nash Equilibrium Strategy with Multiple Bidding Functions

In the previous section, we restricted the strategy space of each user to be a single bidding function (i.e., $\bar{S}_1$ and $\bar{S}_2$) instead of a sequence of bidding functions (i.e., S_1 and S_2). However, the money constraint imposed upon each user is a long term average money constraint. A natural question to ask is the following: Is it profitable for an individual user to change its bidding functions over time while satisfying the long term average money constraint? Therefore, in this section, we allow the users to use a strategy within a broader class of strategy space, S_1 and S_2, and explore whether there is an incentive for a user to do so (i.e., whether there exists a Nash equilibrium strategy so that it can increase its throughput).

To choose a strategy (i.e., a sequence of bidding functions) from the strategy space S_1 or S_2, a user encounters two problems. First, it must decide how to allocate its money among these n bidding functions so that the average money constraint is still satisfied. Second, once the money allocated to the ith bidding function is specified, a user has to choose a bidding function for the ith slot. The second problem is already solved in the previous section (see Theorem 1). In this section, we will focus on the first problem that a user encounters, specifically, the problem of how to allocate money between the bidding functions while satisfying the following condition: The total expected amount of money for the *sequence* of n bidding functions is $n \cdot \alpha$ for user 1 and $n \cdot \beta$ for user 2.

More precisely, the strategy space or possible actions that can be taken by users are the following:

$$\hat{S}_1 = \{\alpha_1, \cdots, \alpha_n \mid \alpha_1 + \cdots + \alpha_n = n \cdot \alpha\}$$
$$\hat{S}_2 = \{\beta_1, \cdots, \beta_n \mid \beta_1 + \cdots + \beta_n = n \cdot \beta\}$$

The objective of each user is still to maximize its own throughput. When user 1 and user 2 allocate α_i and β_i for their ith bidding function which is given in Theorem 1, the payoff functions are $G_1(\alpha_i, \beta_i)$ for user 1 and $G_2(\alpha_i, \beta_i)$ for user 2.

The following lemma gives us a Nash equilibrium strategy pair for the auction game described in this section.

Lemma *3:* Given that user 2's strategy is to allocate its money evenly among its bidding functions (i.e., $\beta_i = \beta, i = 1 \cdots n$), user 1's best response is to allocate its money evenly as well (i.e., $\alpha_i = \alpha, i = 1 \cdots n$); and vice versa. Therefore, a Nash equilibrium strategy pair for this auction is for both users to allocate their money evenly.

We have already obtained a Nash equilibrium strategy pair from the above Lemma. The following theorem states that this Nash equilibrium strategy pair is in fact unique within the strategy space considered.

Theorem *2:* For the auction in this section, a unique Nash equilibrium strategy for both users is to allocate their money evenly among the bidding functions.

In this section, users are given more freedom in choosing their strategies (i.e., they can choose n different bidding functions). However, as Theorem 2 shows, the unique Nash equilibrium strategy pair is for each user to use a single bidding function from its strategy space. Thus, the throughput result obtained in this broader strategy space-S_1 and S_2–is the same as the throughput result from previous section. Therefore, there is no incentive for a user to use different bidding functions.

V. Comparison with Other Allocation Schemes

To this end, we have a unique Nash equilibrium strategy pair and the resulting throughput when both players choose to use the Nash equilibrium strategy. Inevitably, due to the fairness constraint, total system throughput will decrease as compared to the maximum throughput attainable without any fairness constraint. Hence we would like to compare the total throughput of the Nash equilibrium strategy to that of an unconstrained strategy. We address this question by first considering an allocation scheme that maximizes total throughput subject to no constraint. Then, we investigate the throughput of another centralized allocation scheme that maximize the total throughput subject to the constraint that the resulting throughput of individual user is kept at certain ratio.

A. Maximizing Throughput with No Constraint

To maximize throughput without any constraints, the transmitter sends data to the user with a better channel state during each time slot. Then the expected throughput is $E[\max\{X_1, X_2\}]$. Since X_1 and X_2 are independent uniformly distributed in $[0, 1]$, we have $E[\max\{X_1, X_2\}] = \frac{2}{3}$. Using the Nash equilibrium playing strategy, the total expected system throughput, $G_1(\alpha, \beta) + G_2(\alpha, \beta)$, is $\frac{1}{2}$ in the worst case (i.e., one users gets all of the time slots while the other user is starving). *Thus, the channel allocation scheme proposed in this paper can achieve at least* 75 *percent of the maximum attainable throughput*. This gives us a lower bound of the throughput performance of the allocation scheme derived from the Nash equilibrium pair.

B. Maximizing Throughput with Constant Throughput Ratio Constraint

Now, we investigate an allocation scheme with a fairness constraint that requires the resulting throughput of the users to be kept at a constant ratio. Specifically, let G_1 and G_2 denote the expected throughput for user 1 and user 2 respectively. We have the following optimization problem:

$$\begin{aligned} &\max \ G_1 + G_2 \\ &\text{subj. } \frac{G_1}{G_2} = a \end{aligned} \tag{16}$$

where a is a positive real number.

The resulting optimal allocation scheme for the above problem is of the form shown in Fig. 4. The space spanned by X_1 and X_2 is divided into two regions by the separation line $X_2 = c \cdot X_1$, where c is some positive real number. Above the line (i.e., $X_2 > c \cdot X_1$), the transmitter will assign the slot to user 2. Below the line (i.e., $X_2 < c \cdot X_1$), the transmitter will assign the slot to user 1.

To prove the above, we use a method that is similar to the one in [9]. Specifically, let $A : (X_1, X_2) \to \{1, 2\}$ be an allocation scheme that maps a slot, in which channel states are X_1 and X_2 to either user 1 or user 2. By using an allocation scheme A, the resulting throughput for user 1 and user 2 are $G_1^A = E[X_1 \cdot 1_{A(X_1,X_2)=1}]$ and $G_2^A = E[X_2 \cdot 1_{A(X_1,X_2)=2}]$ respectively. Now, we define an allocation scheme as follows:

$$A^*(X_1, X_2) = \begin{cases} 1 & \text{if} \quad X_1(1 + \lambda^*) \geq X_2(1 - a \cdot \lambda^*) \\ 2 & \text{otherwise} \end{cases}$$

where λ^* is chosen such that $G_1^{A^*}/G_2^{A^*} = a$ is satisfied. It is straightforward to verify that such λ^* exists.

Consider an arbitrary allocation scheme A that satisfies $G_1^A/G_2^A = a$. We have

$$\begin{aligned} &E[X_1 \cdot 1_{A(X_1,X_2)=1}] + E[X_2 \cdot 1_{A(X_1,X_2)=2}] \\ &= E[X_1 \cdot 1_{A(X_1,X_2)=1}] + E[X_2 \cdot 1_{A(X_1,X_2)=2}] \\ &\quad + \lambda^*(E[X_1 \cdot 1_{A(X_1,X_2)=1}] - aE[X_2 \cdot 1_{A(X_1,X_2)=2}]) \\ &= E[(X_1 + \lambda^* X_1) \cdot 1_{A(X_1,X_2)=1}] \\ &\quad + E[(X_2 - a\lambda^* X_2) \cdot 1_{A(X_1,X_2)=2}] \\ &\leq E[(X_1 + \lambda^* X_1) \cdot 1_{A^*(X_1,X_2)=1}] \\ &\quad + E[(X_2 - a\lambda^* X_2) \cdot 1_{A^*(X_1,X_2)=2}] \\ &= E[X_1 \cdot 1_{A^*(X_1,X_2)=1}] + E[X_2 \cdot 1_{A^*(X_1,X_2)=2}] \\ &\quad + \lambda^*(E[X_1 \cdot 1_{A^*(X_1,X_2)=1}] - aE[X_2 \cdot 1_{A^*(X_1,X_2)=2}]) \\ &= E[X_1 \cdot 1_{A^*(X_1,X_2)=1}] + E[X_2 \cdot 1_{A^*(X_1,X_2)=2}] \end{aligned} \tag{17}$$

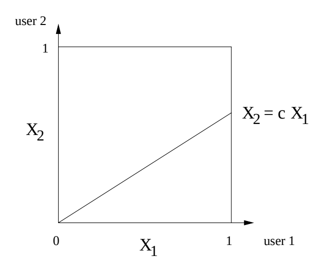

Fig. 4. The optimal allocation scheme to achieve constant throughput ratio fairness.

The inequality in the middle is from the definition of A^*. Specifically, if we were asked to choose an allocation scheme A to maximize $E[(X_1 + \lambda^* X_1) \cdot 1_{A(X_1,X_2)=1}] + E[(X_2 - a\lambda^* X_2) \cdot 1_{A(X_1,X_2)=2}]$. Then, A^* will be an optimal scheme from its definition. Thus, we are able to show that $A^*(X_1, X_2)$ is an optimal solution to the optimization problem in (16).

To find the slope c in Fig.4, we first write the throughput for each user:

$$G_1^{A^*} = \int_0^1 \int_0^{cx_1} x_1 \, dx_1 \, dx_2 = \frac{1}{3} c \tag{18}$$

and

$$\begin{aligned} G_2^{A^*} &= \int_0^c \int_0^{\frac{1}{c} x_2} x_2 \, dx_1 dx_2 + \int_c^1 x_2 \, dx_2 \\ &= \frac{1}{2} - \frac{1}{6} c^2 \end{aligned} \tag{19}$$

Since $G_1^A/G_2^A = a$, we get $c = \frac{-1+\sqrt{1+3a^2}}{a}$.

Using the Nash equilibrium strategy pair, the ratio of the resulting throughput pair $\frac{G_1(\alpha,\beta)}{G_2(\alpha,\beta)}$ is the same as the ratio of money individual user possess ($\frac{\alpha}{\beta}$). For the optimization problem described in (16), by setting $a = \alpha/\beta$, we compare the resulting throughput with the throughput obtained when both users employ the Nash equilibrium strategy. Fig. 5 and Fig. 6 show the comparison. For both users, the Nash equilibrium throughput result is very close to the throughput obtained by solving the constrained optimization problem (within 97 percent to be precise).

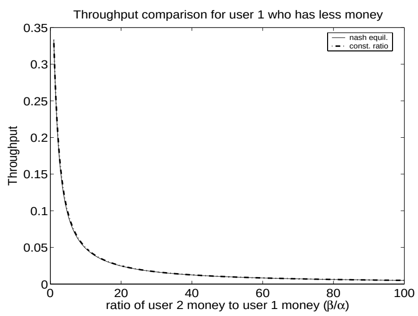

Fig. 5. Throughput result comparison for user 1.

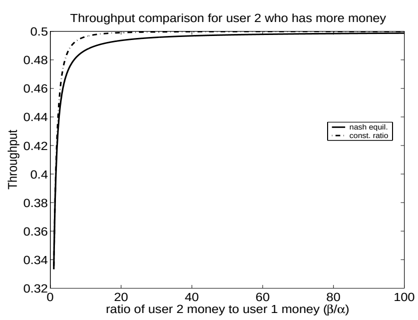

Fig. 6. Throughput result comparison for user 2.

VI. CONCLUSION

We apply an auction algorithm to the problem of fair allocation of a wireless fading channel. Using the all-pay auction mechanism, we are able to obtain a unique Nash equilibrium strategy. Our strategy allocated bandwidth to the users in accordance with the amount of money that they possess. Hence, this scheme can be viewed as a mechanism for providing quality of service (QoS) differentiation; whereby users are given fictitious money that they can use to bid for the channel. By allocating users different amounts of money, the resulting QoS differentiation can be achieved.

We also show that the Nash equilibrium strategy of this auction leads to an allocation at which total throughput is no worse than 3/4 the maximum possible throughput when fairness constraints are not imposed (i.e., slots are allocated to the user with the better channel). In this paper, we focused on finding a Nash equilibrium strategy when both channels are uniformly distributed. However, as we mentioned earlier, our analysis can be extended to channel state with general distribution. An interesting extension could be to find the exact form of a Nash equilibrium with general channel state distribution. Another direction for future work is the application of different auction mechanisms (e.g., first-price and second-price).

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