Randomized Algorithms for Throughput-Optimality and Fairness in Wireless Networks

Atilla Eryilmaz, Eytan Modiano and Asuman Ozdaglar
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA, 02139
Emails: {eryilmaz, modiano, asuman}@mit.edu

***Abstract*— In this paper, we study multi-hop wireless networks with general interference models, and describe a practical randomized routing-scheduling-congestion-control mechanism that is guaranteed to *fully* utilize the network capacity, and achieve fair allocation of the resources. Using the framework described in this paper, low complexity distributed algorithms can be developed for a large class of interference models and fairness criteria (such as proportional and max-min fairness). Earlier distributed algorithms proposed in this context have been highly interference model dependent, and can guarantee at most 50% utilization of the achievable throughput. This is the first work that assures 100% utilization and also provides fair allocation in multi-hop wireless networks.**

I. Introduction

The control of data networks to achieve high throughput and fair allocation of the resources among competing users (or flows) is clearly one of the most important problems in data communications. There has been considerable interest and progress in the development of algorithms that address the issues of efficiency and fairness for both wireline and wireless networks.

In the case of fixed arrival rates (i.e. *inelastic traffic*), Tassiulas and Ephremides provided in their seminal work [25] a joint routing-scheduling algorithm that can achieve the highest throughput without violating the stability of the network. Such *throughput-optimal* policies make dynamic routing and scheduling decisions to avoid highly congested nodes. For wireline networks, these policies can be implemented in a distributed fashion by using buffer occupancy information of only the neighboring nodes. In contrast, in wireless multi-hop networks, there is no known scheme for low complexity implementation of throughput-optimal policies. Many other throughput-optimal algorithms developed later share the same weakness [1, 21, 19, 9].

It is well-recognized that reducing the complexity of the centralized computation is an essential requirement for the development and implementation of throughput-optimal policies. For the case of switches, Tassiulas has shown how the use of randomized algorithms can reduce the complexity of the centralized computation [24]. One of the main contributions of our work is to extend the use of randomized algorithms for potential distributed implementation of routing and scheduling in the context of multi-hop wireless networks.

The issue of fair service of *elastic traffic*, where the rate of an elastic flow is assumed to be controllable, is first considered by Kelly et al. [11, 12] in the context of wireline networks. The authors developed de-centralized algorithms that have strong ties to notions in market economics. The main idea behind these algorithms was for each source to measure the congestion level it experiences and to adjust its flow rate accordingly with its utility function in order to fully utilize the resources. These ideas have been extended to different scenarios and algorithms (e.g. [15], see [22] for a review). However, all of these works were developed for wireline networks, assumed fixed routes, and ignored the stochastic nature of the traffic.

More recently, it has been realized that ideas of flow control can be successfully utilized together with the dynamic routing and scheduling algorithm described above [13, 23, 7, 18, 8]. It has been shown that mean rates arbitrarily close to the fair allocation can be achieved without violating stability. However, the routing and scheduling part of the algorithm inherited the centralized optimization problem of [25], which is impractical to solve for multi-hop wireless networks.

The design and analysis of distributed implementations of the above cross-layer approach attracted a lot of attention from the community. In particular, [14, 27] provided algorithms that guarantee 50% utilization of the stability region for node-exclusive-spectrum-sharing (NESS) interference model, where each feasible allocation forms a matching[1] of the graph. It has been shown in [3] that 33% can be guaranteed even when the algorithm operates in a *totally asynchronous manner* (see [2] for a definition). Distributed implementations exist for this particular interference model if a significant portion of the capacity is sacrificed. For other general interference models, it has been shown that the amount of sacrifice is even greater. For example, [28] and [4] consider the two-hop interference model[2] and show that the guaranteed fraction of the capacity region drops with the increasing number of neighbors. In particular, for a grid topology, only 12.5% of the capacity region is achieved, which is very discouraging.

[1] In a matching, no two links that are incident on the same node can be active simultaneously.

[2] In the two-hop interference model, a transmission over a link (n, m) is successful if and only if all the neighbors of n and m are silent at the time.

Our main contributions in this work are listed next.

• By extending the approach in [24] to the multi-hop wireless network scenario, we prove that the capacity region of the network can be *fully* utilized with a practical scheduling-routing algorithm. Thus, no fraction of the throughput need to be sacrificed for a practical implementation.

• In the case of elastic traffic, we propose a cross-layer mechanism that is composed of a decentralized congestion controller operating in parallel with the practical scheduling-routing algorithm discussed in the previous item. We prove that this mechanism asymptotically achieves fair division of the resources across the flows.

• All of our analysis considers a generic interference model that can be applied to a large set of graph theoretic collision models considered in the literature.

It is important to note that the proposed mechanism can be implemented with low complexity distributed algorithms for specific interference models of practical interest (see [6]).

The rest of the paper is organized as follows. In Section II, we describe the system model. Then, in Section III, we describe the practical algorithm and show its throughput-optimality. Section IV introduces the congestion control mechanism for elastic traffic and establishes its fair characteristics. We provide our concluding remarks in Section V, and some of the proofs in the Appendix.

II. System model

Consider a fixed wireless network that is represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ denotes the set of fixed nodes and $\mathcal{L}$ denotes the set of undirected links. We assume a time slotted system with synchronized nodes, where each slot is just long enough to accommodate a single packet transmission. Suppose there is a set $\mathcal{F}$ of end-to-end flows traversing the network, where each flow $f \in \mathcal{F}$ is given by the node pair $(s(f), d(f))$. Here, $s(f)$ denotes the source node of flow f and $d(f)$ denotes the destination node. We allow for multiple routes between each source-destination pair. We consider both the inelastic and elastic traffic cases.

Associated with each destination a buffer is maintained at each node. We let $q_{n,d}[t]$ denote the length of the queue at node n keeping packets destined for node d at the beginning of slot t.

Definition 1 (Stability). *A given queue is called* stable *if* $\mathbb{E}[q_{n,d}[\infty]] < \infty$, *where* $q_{n,d}[\infty]$ *denotes the random variable with distribution given by the steady-state distribution of* $\{q_{n,d}[t]\}$. *The network is stable if all queues are stable; and unstable otherwise.*

We consider a general interference model formulation that contains all of the graph theoretic collision models considered in the context of scheduling (e.g. NESS [20, 14, 27, 3], or two-hop interference models [28, 4]). We say that two links *interfere* if their concurrent transmissions collide, and assume that if two interfering links are activated in a slot, both of the transmissions fail. We use $\mathcal{I}(l) \subset \mathcal{L}$ to denote the set of links that interfere with link l. Typically, this set will contain links from the local neighborhood of l. As an example, the matching constraint of the NESS model implies that for link $l = (n, m)$, $\mathcal{I}(l) := \{l' \in \mathcal{L} : l' \in i(n) \text{ or } l' \in i(m)\}$, where $i(n)$ denotes the set of links that are incident to node n. We will prove our results for the general model.

We use $\pi = \left\{\pi_{(n,m)}\right\}_{(n,m)\in\mathcal{L}}$ to denote a link activation (or allocation) vector, and Π denote the feasible set of allocations that complies with the interference constraints. An allocation is feasible if and only if no two links in the set interfere with each other. As an example, for the NESS interference model, Π corresponds to the set of matchings of $\mathcal{G}$. We introduce the notation $\pi^d_{(n,m)}$ to distinguish packets destined for different nodes: at any given slot, $\pi^d_{(n,m)}[t] \in \{0, 1\}$ is 1 if link (n, m) serves a packet destined for node d in that slot, and 0 otherwise. Notice that $\pi_{(n,m)}[t] = \sum_{d\in\mathcal{N}} \pi^d_{(n,m)}[t],$ for all t. Also note that it is sufficient to restrict our attention to policies that sets $\pi^d_{(n,m)}[t]$ to zero whenever $q_{n,d}[t] = 0$, for all $(n, m) \in \mathcal{L}$, because any other policy that sets $\pi^d_{(n,m)}[t] = 1$ when $q_{n,d}[t] = 0$ can be replaced by a policy with $\pi^d_{(n,m)}[t] = 0$ without affecting the evolution of the queue-lengths. Then, we can write the evolution of a particular queue, say $q_{n,d}$, as

$$q_{n,d}[t+1] = \left(q_{n,d}[t] - \pi^d_{out(n)}[t] + \sum_{f\in\mathcal{S}_{n,d}} x_f[t] + \pi^d_{into(n)}[t]\right), \quad (1)$$

where $x_f[t]$ is the number of exogenous flow-f arrivals to the network in slot t, and $\mathcal{S}_{n,d} \triangleq \{f \in \mathcal{F} : s(f) = n, d(f) = d\}$ denotes the set of flows that start at node n and are destined to node d. Also, $\pi^d_{into(n)}[t] \triangleq \sum_{k:(k,n)\in\mathcal{L}} \pi^d_{(k,n)}[t]$ is a shorthand for the number of packets entering node n that are destined for node d. Similarly, $\pi^d_{out(n)}[t]$ is the number of packets leaving node n and are destined for node d. We set $q_{d,d}[t] = 0\ \forall t$.

Definition 2 (Capacity (Stability) Region). *Let* $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ *be a given network and* Π *be the set of feasible allocations. The* capacity (or stability) region Λ *of the network is given by the set of vectors* $\mathbf{r} = (r_f)_{f\in\mathcal{F}}$ *for which there exists* $\pi^{d(f)}_{(n,m)} \geq 0$, *for all* $(n, m) \in \mathcal{L}$ *and* $f \in \mathcal{F}$, *such that both the flow conservation constraints at the nodes and the feasibility constraints are satisfied, as given below:*

(C1) For all $n \in \mathcal{N}$ *and* $f \in \mathcal{F}$, *we have*[3]

$$r_f \mathbf{1}_{s(f)=n} + \sum_{k:(k,n)\in\mathcal{L}} \pi^{d(f)}_{(k,n)} = \sum_{m:(n,m)\in\mathcal{L}} \pi^{d(f)}_{(n,m)},$$

(C2) $\left[\pi_{(n,m)}\right]_{(n,m)\in\mathcal{L}} \in Conv(\Pi)$. [4]

It is shown in [25, 19] that Λ is the set of mean arrival rates for which there exists a policy that stabilizes the network.

III. Stability for Inelastic Traffic

The focus of this section is throughput-optimality under stability for the inelastic traffic scenario. We provide simple

[3] We use $\mathbf{1}_A$ as the indicator function of event A.

[4] $Conv(A)$ denotes the convex hull of set A, which is the smallest convex set that includes A. The convex hull is included due to the possibility of timesharing between feasible allocations.

routing-scheduling mechanisms that can support any mean arrival rate in the capacity region without violating stability (such mechanisms are said to be *throughput-optimal* [25, 21, 19, 7]). In earlier work [24], Tassiulas used randomized schemes to provide a low complexity stabilizing algorithm for switches using a centralized controller. In this section, we will extend the use of randomized throughput-optimal schemes for multi-hop networks with general interference models. In particular, we show that practical algorithms satisfying several simple conditions can be designed to achieve full utilization of the capacity of multi-hop wireless networks with general interference models.

To this end, we introduce the following notation for link weights: $w_{(n,m)}[t] = w_{(m,n)}[t] \triangleq \max_d |q_{n,d}[t] - q_{m,d}[t]|$. This is also referred to as the *differential backlog*[5] over link (n, m) and can be interpreted as a measure of the importance of the link. Consider the following allocation vector

$$\overset{\star}{\pi}_{\mathbf{w}}\ [t] \in \arg\max_{\pi\in\Pi} \sum_{l\in\mathcal{L}} w_l[t]\pi_l \equiv \arg\max_{\pi\in\Pi}(\mathbf{w}[t]\cdot\pi). \qquad (2)$$

This allocation rule is called the *back-pressure policy*. Once the $\overset{\star}{\pi}_{\mathbf{w}}$ is determined according to (2), only the commodity that maximizes the differential backlog over link (n, m) is served over that link at the chosen rate. If the back-pressure policy is applied at every time slot, it is known to be throughput-optimal. However, it requires a centralized controller that knows $\mathbf{w}[t]$ at every time-slot, and also communicates the allocation vector $\overset{\star}{\pi}_{\mathbf{w}}\ [t]$ instantly to all the nodes of the network. These requirements make the implementation of this algorithm impractical for the multi-hop wireless network scenario.

The idea is to use a random algorithm, instead of the optimal one described in (2), which yields a feasible allocation $\tilde{\pi}[t] \in \Pi$ at every time slot, which is not necessarily equal to $\overset{\star}{\pi}_{\mathbf{w}}\ [t]$, but has a positive probability $\delta > 0$ of being equal to $\overset{\star}{\pi}_{\mathbf{w}}\ [t]$. Thus, we have

$$P(\tilde{\pi}[t] = \overset{\star}{\pi}_{\mathbf{w}}\ [t]) \geq \delta, \qquad \text{for all } \mathbf{w}[t] \text{ and } t. \qquad (3)$$

Once the allocation $\tilde{\pi}[t]$ is chosen, the actual allocation $\pi[t]$ is determined according to the following evolution.

$$\pi[t+1] = \begin{cases} \pi[t] & \text{if } \mathbf{w}[t]\cdot\pi[t] \geq \mathbf{w}[t]\cdot\tilde{\pi}[t] \\ \tilde{\pi}[t] & \text{otherwise} \end{cases} \qquad (4)$$

The above randomized algorithm was introduced in [24] in the context of switches, where there exists a centralized scheduler. A similar approach has been used in developing a distributed implementation for networks restricted to single-hop communication with matching constraints and Bernoulli arrival processes [17]. As we will prove shortly, it turns out that the two conditions (3) and (4) are sufficient to achieve throughput-optimality in a more general setting.

The algorithm updates its allocation vector only if the proposed allocation of the randomized algorithm yields a better objective function. Such an algorithm can be divided into two parts: PICK and COMPARE, where PICK chooses a feasible schedule $\tilde{\pi}[t]$ satisfying (3), and COMPARE communicates the relevant weight information to other nodes in the network to perform (4). Distributed implementations of both PICK and COMPARE can be developed for a given interference model (see [6] for an example).

Theorem 1. *Assume that $x_f[t]$ is independent and identically distributed (i.i.d.)* [6] *for all t and f with $\mathbb{E}[x_f^2[1]] \leq A < \infty$. Then, for any mean arrival vector $\lambda := (\mathbb{E}[x_f[1]])_f \in interior(\Lambda)$, the above randomized policy is stabilizing.*

Proof: We start by noting that $\mathbf{y}[t] \triangleq (\mathbf{q}[t], \pi[t])$ forms a Markov Chain. Then, we study the mean drift of the following Lyapunov function of the state $\mathbf{y} = (\mathbf{q}, \pi)$:

$$\begin{aligned} V(\mathbf{y}) &= \sum_{n\in\mathcal{N}}\sum_{d\in\mathcal{N}} q_{n,d}^2 + \left(\sum_{l\in\mathcal{L}} w_l((\overset{\star}{\pi}_{\mathbf{w}})_l - \pi_l)\right)^2 \\ &= \underbrace{\mathbf{q}\cdot\mathbf{q}}_{\triangleq V_1(\mathbf{y})} + \underbrace{(\mathbf{w}\cdot(\overset{\star}{\pi}_{\mathbf{w}} - \pi))^2}_{\triangleq V_2(\mathbf{y})} \end{aligned} \qquad (5)$$

The proof uses the following two key lemmas the proofs of which are moved to Appendix A and B.

Lemma 1. *Let $\Delta_1(\mathbf{y}[t]) \triangleq \mathbb{E}[V_1(\mathbf{y}[t+1]) - V_1(\mathbf{y}[t]) \mid \mathbf{y}[t]]$, then for some $\gamma > 0$ and $c_1 < \infty$, we have*

$$\Delta_1(\mathbf{y}[t]) \quad \leq \quad -\gamma\sqrt{V_1(\mathbf{y}[t])} + 2\sqrt{V_2(\mathbf{y}[t])} + c_1.$$

Lemma 2. *Let $\Delta_2(\mathbf{y}[t]) \triangleq \mathbb{E}[V_2(\mathbf{y}[t+1]) - V_2(\mathbf{y}[t]) \mid \mathbf{y}[t]]$, then for some $\delta > 0$ and $c_2, c_3 < \infty$, we have*

$$\Delta_2(\mathbf{y}[t]) \quad \leq \quad -\delta V_2(\mathbf{y}[t]) + c_2\sqrt{V_2(\mathbf{y}[t])} + c_3.$$

Proof of Theorem 1 (Continued): Combining these lemmas, we upper-bound $\Delta(\mathbf{y}[t]) \triangleq \mathbb{E}[V(\mathbf{y}[t+1]) - V(\mathbf{y}[t]) \mid \mathbf{y}[t]]$ by

$$-\gamma\sqrt{V_1(\mathbf{y}[t])} - \delta V_2(\mathbf{y}[t]) + (2 + c_2)\sqrt{V_2(\mathbf{y}[t])} + c_1 + c_3.$$

Thus, if we consider the scenario where $V(\mathbf{y}[t]) \geq B$ for a sufficiently large $B < \infty$, then we can guarantee that

$$\Delta(\mathbf{y}[t]) \leq -\epsilon\sqrt{V_1(\mathbf{y}[t])} + \hat{B}, \qquad (6)$$

for some $\epsilon > 0$ and $\hat{B} < \infty$. This is true because in this scenario, since we have $V_1(\mathbf{y}[t]) \geq B - V_2(\mathbf{y}[t])$, we can write

$$\begin{aligned} \mathbb{E}[V(\mathbf{y}[t+1]) \mid \mathbf{y}[t]] \leq \quad & V(\mathbf{y}[t]) - \tfrac{\gamma}{2}\sqrt{V_1(\mathbf{y}[t])} \\ & -\tfrac{\gamma}{2}\sqrt{B - V_2(\mathbf{y}[t])} - \delta V_2(\mathbf{y}[t]) \quad (7) \\ & +(2 + c_2)\sqrt{V_2(\mathbf{y}[t])} + c_1 + c_3. \quad (8) \end{aligned}$$

Notice that the sum of the expressions in (7) and (8) can be made negative by choosing B large enough. Hence (6) holds when we let $\epsilon = \gamma/2$ and choose $\hat{B}$ large enough.

The proof of positive recurrence of the chain follows from Foster's Criteria (cf. [22] or [16]). Let us use $\mathbf{y}[\infty]$ to denote the random variable to which $\{\mathbf{y}[t]\}_t$ converges. Due to the f-$ergodic$ theorem of [16], the drift condition of (6)

[5]This definition of differential-backlog is slightly different from the ones in the literature, which is due to the assumption of undirected links here.

[6]The assumption of i.i.d. arrivals is not critical to the analysis. The same results continue to hold for processes with mild ergodicity properties ([10]).

is equivalent to $\mathbb{E}[\sqrt{V_1(\mathbf{y}[\infty]}] < \infty$. Invoking the definition of $V_1(\cdot)$ results in the intended stability result. ∎

Theorem 1 shows that under very mild conditions, if a randomized scheduler can be found that satisfies (3), and the control in (4) can be performed, then the stability will be achieved for any stabilizable incoming traffic.

IV. Fair Service of Elastic Traffic

In the previous section, we proved the throughput optimality of a randomized scheme *for inelastic traffic, provided that the exogenous arrival rates are stabilizable*. Thus, with that scheme, even if stability is achieved for a given mean arrival vector, the system may be seriously underutilized. Ideally, it would be desired to know the stability region of the network and choose the arrival rates to fully utilize the network resources. However, as noted in [26], the task of determining whether a given set of arrival rates is in the characterized stability region becomes an intractable problem as the network size grows.

In this section, we extend the throughput-optimal algorithm so that the available capacity of the network is fully utilized and fairly shared among the flows in a completely decentralized and dynamic fashion without the need to explicitly characterize the stability region. To that end, we change the considered traffic model from inelastic to elastic, where in the latter case the mean arrival rates of the sources can be modified.

To define fairness, as is standard in the recent literature (e.g. [11, 12, 15, 22, 7, 18, 13]), we use a utility function, $U_f(\cdot)$, of mean arrival rates that is a measure of flow f's preferences. Throughout, we will assume that this function is concave and non-decreasing. We call the allocation, $\mathbf{x}^\star$, that satisfies

$$\mathbf{x}^\star \in \arg\max_{\mathbf{x}\in\Lambda} \sum_{f\in\mathcal{F}} U_f(x_f) \tag{9}$$

the *fair* allocation. Notice that this is the allocation that maximizes the aggregate utility of the network. It is known that by defining $U_f(\cdot)$ appropriately different fairness criteria of interest, such as proportional or max-min fairness, can be achieved (see [22] for an extensive review).

Next, we describe the so called *Dual Congestion Control* mechanism that is implemented at the source of each flow in a completely decentralized fashion. Variations of this mechanism are studied recently in the literature [7, 18, 14, 23].

Dual Congestion Control Mechanism: Assume that every flow has access to the its entry point queue-length information, i.e. flow f knows $q_{b(f),e(f)}[t]$ for all t. Then, at the beginning of each time slot t, flow f generates $x_f[t]$ packets satisfying

$$x_f[t] \quad = \quad \min\left\{ U_f'^{-1}\left(\frac{q_{b(f),e(f)}[t]}{K}\right), M\right\}, \tag{10}$$

where M and K are positive scalars. ◇

The policy is easy to implement at each source because it only requires the queue length of the buffer at the source, and the individual utility function of the flow. Note that the only common information required at all the sources is K. Once K is determined, the flow control mechanism can operate in a completely decentralized fashion in parallel with the randomized routing-scheduling algorithm described in Section III.

Theorem 2. *For some finite constants C_1, C_2 we have*

$$\sum_{n\in\mathcal{N}}\sum_{d\in\mathcal{N}} \overline{q_{n,d}} \quad \le \quad C_1 K \tag{11}$$

$$\sum_{f\in\mathcal{F}} U_f(\bar{x}_f) \quad \ge \quad \sum_{f\in\mathcal{F}} U_f(x_f^\star) - \frac{C_2}{K} \tag{12}$$

where $\bar{x}_f \triangleq \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}[x_f[t]]$, *and similarly for* $\overline{q_{n,d}}$.

Proof: We start by introducing a relaxed version of the optimization problem (9): for any $\varepsilon > 0$, let

$$\mathbf{x}^\star(\varepsilon) \in \arg\max_{\mathbf{x}\in\Lambda(\varepsilon)} \sum_f U_f(x_f),$$

where $\Lambda(\varepsilon) = \{\mathbf{x}\in\Lambda : [x_f+\varepsilon]_f \in \Lambda\}$. $\Lambda(\varepsilon)$ is a subset of the capacity region Λ with an ε-strip of its positive surface is deleted. It is not difficult to see that $x^\star(\varepsilon) \to x^\star$ as $\varepsilon \to 0$.

We use the same Lyapunov functions $V(\mathbf{y}), V_1(\mathbf{y})$ and $V_2(\mathbf{y})$ introduced in Section III (c.f. (5)) to prove the theorem. The following theorem studies the single-step mean drift of $V_1(\mathbf{y})$. The proof of this lemma is provided in Appendix C.

Lemma 3. *For some constants $\epsilon > 0, c_1 < \infty$,*

$$\Delta_1(\mathbf{y}[t]) \le -\varepsilon \sum_{n,d} q_{n,d}[t] + 2K\sum_f \mathbb{E}[U_f(x_f[t]) \mid \mathbf{y}[t]]$$
$$-2K\sum_f U_f(x_f^\star(\varepsilon)) - \frac{\varepsilon}{|\mathcal{N}|^2}\sqrt{V_1(\mathbf{y}[t])} + 2\sqrt{V_2(\mathbf{y}[t])} + c_1$$

Proof of Theorem 2 (Continued): In what follows, we will omit $[t]$ for convenience. By combining the result of Lemma 3 with that of Lemma 2, we can write: $\Delta(\mathbf{y}[t])$

$$\begin{aligned} \le \quad & -\varepsilon\sum_{n,d} q_{n,d} + 2K\sum_f U_f(x_f) \\ & -2K\mathbb{E}[\sum_f U_f(x_f^\star(\varepsilon)) \mid \mathbf{y}] + c_1 + c_3 \\ & -\varepsilon\frac{\sqrt{V_1(\mathbf{y})}}{|\mathcal{N}|^2} - \delta V_2(\mathbf{y}) + (2+c_2)\sqrt{V_2(\mathbf{y})} \end{aligned} \tag{13}$$

$$\overset{(a)}{\le} c_4 - \varepsilon\sum_{n,d} q_{n,d} + 2K\{\sum_f \mathbb{E}[U_f(x_f)|\mathbf{y}] - \sum_f U_f(x_f^\star(\varepsilon))\}$$

where the inequality (a) follows from the fact that (13) is upper-bounded as argued in the proof of Theorem 1.

By taking expectations of both sides of the last inequality and summing over $t = 0, 1, \cdots, T-1$, we get

$$\mathbb{E}[V(\mathbf{y}[T]) - V(\mathbf{y}[0])] \le Tc_4 - \varepsilon\sum_{t=0}^{T-1}\sum_{n,d}\mathbb{E}[q_{n,d}[t]]$$
$$+2K\sum_{t=0}^{T-1}\sum_f \mathbb{E}[U_f(x_f[t])] - 2KT\sum_f U_f(x_f^\star(\varepsilon)). \tag{14}$$

Since $V(\mathbf{y}) \geq 0$ for all $\mathbf{y}$, we can re-arrange the terms in (14) to obtain the following inequality.

$$\frac{1}{T}\sum_{t=0}^{T-1}\sum_{n,d}\mathbb{E}[q_{n,d}[t]] \quad \leq \quad \frac{V(\mathbf{y}[0])}{\varepsilon T} + \frac{c_4 + 2M|\mathcal{F}|K}{\varepsilon}.$$

The proof of (11) is complete when we let $T \to \infty$, and define $C_1 \triangleq \frac{2M|\mathcal{F}|}{\varepsilon}$.

If, on the other hand, we re-arrange the terms in (14) differently, we can get

$$\frac{1}{T}\sum_{t=0}^{T-1}\sum_{f}\mathbb{E}[U_f(x_f[t])] \geq \sum_{f} U_f(x_f^{\star}(\varepsilon)) - \frac{V(\mathbf{y}[0])}{2KT} - \frac{c_4}{2K}.$$

Hence, the proof of (12) is complete when we use Jensen's inequality to write $U_f(\bar{x}_f) \geq \frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}[U_f(x_f[t])]$, and define $C_2 \triangleq \frac{c_4}{2}$. ■

V. CONCLUSIONS

In this work, we provided a framework for the design of practical cross-layer algorithms for multi-hop wireless networks that are throughput-optimal and fair. Previously proposed low complexity implementations for multi-hop wireless networks have been greedy in nature. Despite their ease of implementation, such policies have been shown in recent works to have poor throughput performance. The greatest weakness of greedy policies is their lack of memory. In this work, we considered the generalization of a practical randomized policy first introduced in [24] for switches to multi-hop networks. We proved that with the use of a small memory unit at the nodes, this policy achieves 100% of the available capacity of the network. We also considered the case of elastic flows and provided a decentralized congestion control algorithm that works in parallel with the randomized algorithm. We showed that the resulting cross-layer algorithm allocates resources fairly across users.

We note that the extra overhead that will appear in our randomized algorithm can be reduced at the expense of larger delay, but *without sacrificing from throughput*. Also, our initial findings suggest that the additional complexity is comparable to the complexity of the greedy policies. In our companion paper [6], we show that the development and analysis of low complexity, distributed PICK and COMPARE algorithms (as described in Section III) for the two-hop interference model.

APPENDIX

A. Proof of Lemma 1:

Proof: We start by arranging the terms of $\Delta_1(\mathbf{y}[t])$.

$$\begin{aligned}\Delta_1(\mathbf{y}[t]) &= \mathbb{E}[(\mathbf{q}[t+1]-\mathbf{q}[t])\cdot(\mathbf{q}[t+1]+\mathbf{q}[t]) \mid \mathbf{y}[t]]\\ &\overset{(a)}{\leq} c_1 + 2\sum_{n,d} q_{n,d}[t](\lambda_{n,d} + \overset{\star}{\pi}{}^{d}_{into(n)}[t] - \overset{\star}{\pi}{}^{d}_{out(n)}[t]) \quad (15)\\ &\quad + 2\sum_{n,d} q_{n,d}[t](\pi^{d}_{into(n)}[t] - \pi^{d}_{out(n)}[t]\\ &\quad - \overset{\star}{\pi}{}^{d}_{into(n)}[t] + \overset{\star}{\pi}{}^{d}_{out(n)}[t]), \quad (16)\end{aligned}$$

where inequality (a) is due to the fact that $\mathbb{E}[x_f^2[t]] \leq A$, and that $\pi^d_{into(n)}[t]$ and $\pi^d_{out(n)}[t]$ are both upper-bounded by the maximum degree of $\mathcal{G}$. In (15) and (16), the optimal allocation $\overset{\star}{\pi}_{\mathbf{w}}$ is added and subtracted. For notational convenience, we use $\lambda_{n,d} \triangleq \sum_{f\in\mathcal{S}_{n,d}}\lambda_f$. Also, $\overset{\star}{\pi}{}^d_{into(n)}$ and $\overset{\star}{\pi}{}^d_{out(n)}$ are defined similarly to $\pi^d_{into(n)}$ and $\pi^d_{out(n)}$.

Since the arrival rate vector λ is known to be within the stability region Λ, we can upper-bound the second expression in (15) by $-\gamma\sqrt{V_1(\mathbf{y}[t]}$ (e.g. see [5]). Next, notice that

$$(16) = 2\mathbf{w}[t]\cdot(\overset{\star}{\pi}_{\mathbf{w}}[t] - \pi[t]) \leq 2\sqrt{V_2(\mathbf{y}[t])}.$$

Combination of these results yields the proof. ■

B. Proof of Lemma 2:

Proof: We start by analyzing the conditional expectation

$$\begin{aligned}\mathbb{E}[V_2(\mathbf{y}[t+1]) \mid \mathbf{y}[t]] &\leq (1-\delta)\mathbb{E}[(\mathbf{w}[t+1]\\ &\cdot(\overset{\star}{\pi}_{\mathbf{w}}[t+1] - \pi[t+1]))^2 \mid \mathbf{y}[t], \pi[t+1] \neq \overset{\star}{\pi}_{\mathbf{w}}[t+1]].\end{aligned}$$

Next, we write the evolution of the weight vector $\mathbf{w}[t]$ as $\mathbf{w}[t+1] = \mathbf{w}[t] + \mathbf{r}[t] - \mathbf{z}[t]$, where $\mathbf{r}[t]$ denotes the vector of packets entering the queues and $\mathbf{z}[t]$ is the vector of packets leaving the queues. We can describe these vectors more precisely, but this is not necessary for the proof. The only important factor is the boundedness of them. Now, we can write $\mathbb{E}[V_2(\mathbf{y}[t+1]) \mid \mathbf{y}[t]]$

$$\begin{aligned}\leq \quad & (1-\delta)\mathbb{E}[(\mathbf{w}[t]\cdot(\overset{\star}{\pi}_{\mathbf{w}}[t+1] - \pi[t+1]) \quad (17)\\ & + (\overset{\star}{\pi}_{\mathbf{w}}[t+1] - \pi[t+1])\\ & \cdot(\mathbf{r}[t] - \mathbf{z}[t]))^2 \mid \mathbf{y}[t], \pi[t+1] \neq \overset{\star}{\pi}_{\mathbf{w}}[t+1]] (18)\end{aligned}$$

It is not difficult to upper-bound the expression in (18) with a finite constant. To bound (17), we make two observations:

(i) $\mathbf{w}[t]\cdot\overset{\star}{\pi}_{\mathbf{w}}[t+1] \leq \mathbf{w}[t]\cdot\overset{\star}{\pi}_{\mathbf{w}}[t]$ due to the definition of the optimal allocation vector $\overset{\star}{\pi}_{\mathbf{w}}$ in (2).

(ii) $\mathbf{w}[t]\cdot\pi[t+1] \geq \mathbf{w}[t]\cdot\pi[t]$ due to the update rule (3) of the randomized algorithm.

By combining these observations, we can find constants $c_2, c_3 < \infty$ that satisfy

$$\mathbb{E}[V_2(\mathbf{y}[t+1]) \mid \mathbf{y}[t]] \leq (1-\delta)V_2(\mathbf{y}[t]) + c_2\sqrt{V_2(\mathbf{y}[t])} + c_3,$$

which completes the proof. ■

C. Proof of Lemma 3:

Proof: The proof starts with the following upper bound on $\Delta_1(\mathbf{y}[t])$ (cf. (15) and (16)):

$$\begin{aligned}
\Delta_1(\mathbf{y}[t]) & \\
\leq\ & c_1 + 2\sqrt{V_2(\mathbf{y}[t])} + 2\sum_{n,d} \mathbb{E}[q_{n,d}[t] \sum_{f\in\mathcal{S}_{n,d}} x_f[t] \mid \mathbf{y}[t]] \\
& -2\sum_{(n,m)\in\mathcal{L}} \sum_d \pi^d_{(n,m)}[t](q_{n,d}[t]-q_{m,d}[t]) \\
=\ & c_1 + 2\sqrt{V_2(\mathbf{y}[t])} + 2\sum_f K\mathbb{E}[U_f(x_f[t]) \mid \mathbf{y}[t]] \\
& -2\sum_f \mathbb{E}[KU_f(x_f[t]) - q_{b(f),e(f)}x_f[t] \mid \mathbf{y}[t]] \qquad (19) \\
& -2\sum_{(n,m)\in\mathcal{L}} \sum_d \pi^d_{(n,m)}[t](q_{n,d}[t]-q_{m,d}[t]), \qquad (20)
\end{aligned}$$

where we get the last equality by adding and subtracting $2K\sum_f \mathbb{E}[U_f(x_f[t]) \mid \mathbf{y}[t]]$, and by noting that

$$\sum_{n,d} q_{n,d}[t](\sum_{f\in\mathcal{S}_{n,d}} x_f[t]) = \sum_f q_{b(f),e(f)}[t]x_f[t].$$

Notice that the Dual Congestion Control mechanism is designed to optimize the expression in (19), and the Back-pressure Policy is designed to optimize (20). Bounds to these expressions are obtained in [18], which are reproduced below.

$$\sum_f \mathbb{E}[KU_f(x_f[t]) - q_{b(f),e(f)}x_f[t] \mid \mathbf{y}[t]] \geq K\sum_f U_f(x^\star_f(\varepsilon)) - \sum_f q_{b(f),e(f)}[t]x^\star_f(\varepsilon)$$

$$\sum_{(n,m)\in\mathcal{L}} \sum_d \pi^d_{(n,m)}[t](q_{n,d}[t]-q_{m,d}[t]) \geq \sum_{n,d} q_{n,d}[t](x^\star_f(\varepsilon)+\varepsilon).$$

The first inequality is straight-forward since $\mathbf{x}^\star(\varepsilon)$ is an element of the set of feasible rate vectors (i.e. vectors in $[0,M]^{|\mathcal{F}|}$) over which the Congestion Controller of (10) maximizes the the expression on the left. The second inequality follows from the fact that $\mathbf{x}^\star(\varepsilon) = (x_f(\varepsilon))_f$ is at least ε away from the boundary of Λ for all f.

By canceling common terms, we obtain

$$\begin{aligned}
\Delta_1(\mathbf{y}[t]) \leq\ & -2\varepsilon\sum_{n,d} q_{n,d}[t] + 2K\sum_f \mathbb{E}[U_f(x_f[t]) \mid \mathbf{y}[t]] \\
& -2K\sum_f U_f(x^\star_f(\varepsilon)) + 2\sqrt{V_2(\mathbf{y}[t])} + c_1.
\end{aligned}$$

Finally, we use the following lower bound $\sum_{n,d} q_{n,d}[t] \geq \frac{\sqrt{V_1(\mathbf{y}[t])}}{|\mathcal{N}|^2}$, which, when substituted into the previous expression, yields the statement of the lemma. ■

References

[1] M. Andrews, K. Kumaran, K. Ramanan, A. Stolyar, R. Vijayakumar, and P. Whiting. Scheduling in a queueing system with asynchronously varying service rates, 2000. Bell Laboratories Technical Report.

[2] D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, Belmont, MA, 1997.

[3] L. Bui, A. Eryilmaz, R. Srikant, and X. Wu. Joint asynchronous congestion control and distributed scheduling for wireless networks. Proceedings of IEEE Infocom 2006.

[4] P. Chaporkar, K. Kar, and S. Sarkar. Throughput guarantees through maximal scheduling in wireless networks. In *Proceedings of the Allerton Conference on Control, Communications and Computing*, 2005.

[5] A. Eryilmaz. Efficient and fair scheduling for wireless networks. PhD thesis, University of Illinois at Urbana, Champaign, August 2005.

[6] A. Eryilmaz, A. Ozdaglar, and E. Modiano. Polynomial complexity algorithms for full utilization of multi-hop wireless networks, 2006. LIDS Report, submitted.

[7] A. Eryilmaz and R. Srikant. Fair resource allocation in wireless networks using queue-length based scheduling and congestion control. In *Proceedings of IEEE Infocom*, Miami, FL, March 2005.

[8] A. Eryilmaz and R. Srikant. Resource allocation of multi-hop wireless networks. In *Proceedings of International Zurich Seminar on Communications*, February 2006.

[9] A. Eryilmaz, R. Srikant, and J. Perkins. Stable scheduling policies for fading wireless channels. In *Proceedings of IEEE International Symposium on Information Theory*, 2003. A longer version appeared in the *IEEE/ACM Transactions on Networking, 2005*.

[10] A. Eryilmaz, R. Srikant, and J. R. Perkins. Stable scheduling policies for fading wireless channels. *IEEE/ACM Transactions on Networking*, 13:411–425, April 2005.

[11] F. P. Kelly. Charging and rate control for elastic traffic. *European Transactions on Telecommunications*, 8:33–37, 1997.

[12] F. P. Kelly, A. Maulloo, and D. Tan. Rate control in communication networks: Shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49:237–252, 1998.

[13] X. Lin and N. Shroff. Joint rate control and scheduling in multihop wireless networks. In *Proceedings of IEEE Conference on Decision and Control*, Paradise Island, Bahamas, December 2004.

[14] X. Lin and N. Shroff. The impact of imperfect scheduling on cross-layer rate control in multihop wireless networks. In *Proceedings of IEEE Infocom*, Miami, FL, March 2005.

[15] S. H. Low and D. E. Lapsley. Optimization flow control, I: Basic algorithm and convergence. *IEEE/ACM Transactions on Networking*, 7:861–875, December 1999.

[16] S. Meyn and R. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, 1993.

[17] E. Modiano, D. Shah, and G. Zussman. Maximizing throughput in wireless networks via gossiping. In *ACM SIGMETRICS/IFIP Performance*, 2006.

[18] M.J. Neely, E. Modiano, and C. Li. Fairness and optimal stochastic control for heterogeneous networks. In *Proceedings of IEEE Infocom*, pages 1723–1734, Miami, FL, March 2005.

[19] M.J. Neely, E. Modiano, and C.E. Rohrs. Dynamic power allocation and routing for time varying wireless networks. In *Proceedings of IEEE Infocom*, pages 745–755, April 2003.

[20] G. Sasaki and B. Hajek. Link scheduling in polynomial time. *IEEE Transactions on Information Theory*, 32:910–917, 1988.

[21] S. Shakkottai and A. Stolyar. Scheduling for multiple flows sharing a time-varying channel: The exponential rule. *Translations of the AMS, Series 2*, A volume in memory of F. Karpelevich, 207:185–202, 2002.

[22] R. Srikant. *The Mathematics of Internet Congestion Control*. Birkhäuser, Boston, MA, 2004.

[23] A. Stolyar. Maximizing queueing network utility subject to stability: Greedy primal-dual algorithm. *Queueing Systems*, 50(4):401–457, 2005.

[24] L. Tassiulas. Linear complexity algorithms for maximum throughput in radio networks and input queued switches. In *Proceedings of IEEE Infocom*, pages 533–539, 1998.

[25] L. Tassiulas and A. Ephremides. Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. *IEEE Transactions on Automatic Control*, 36:1936–1948, December 1992.

[26] L. Tassiulas and A. Ephremides. Dynamic server allocation to parallel queues with randomly varying connectivity. *IEEE Transactions on Information Theory*, 39:466–478, March 1993.

[27] X. Wu and R. Srikant. Regulated maximal matching: A distributed scheduling algorithm for multi-hop wireless networks with node-exclusive spectrum sharing, 2005. Submitted to *IEEE Conference on Decision and Control*.

[28] X. Wu and R. Srikant. Bounds on the capacity region of multi-hop wireless networks under distributedgreedy scheduling. In *Proceedings of IEEE Infocom*, 2006.