# Information Flow in Event-Based Stabilization of Cyber-Physical Systems

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**Abstract** The information flow required for remote stabilization of dynamical systems has been investigated extensively in the last two decades and basic results that are cornerstones for the theory of Cyber-Physical Systems (CPS) have been derived. This chapter reviews these developments with a particular emphasis on recent advancements obtained in the context of event-triggering control. In this framework, a basic realization is that, in the same way that subsequent pauses in spoken language are used to convey information, it is possible to transmit information in CPS not only by message content, but also with its timing. Exploiting this possibility, event-triggering control techniques encode information in the timing of the triggering events and can achieve system stabilization at dramatically low data-rates.

## 1 Introduction

Cyber-Physical Systems (CPS) [1] are typically designed as a network of interacting elements, including sensors, actuators, computing and communication devices connected in closed loop, with the objective of performing tasks that require interaction with the physical world. In such networked control systems a fundamental question, which is the focus of this chapter, is the following:

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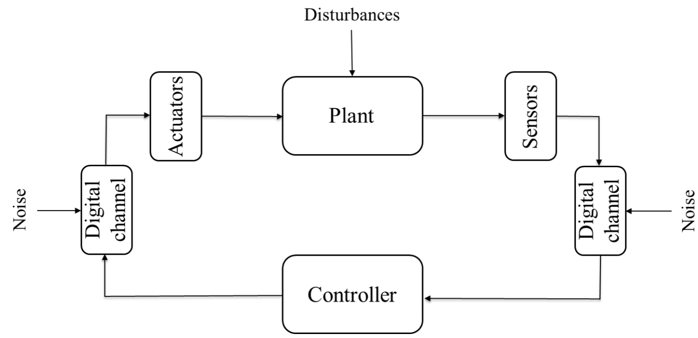



**Fig. 1** High level abstraction of a CPS from a communication perspective.

> What is the minimum amount of information transfer among the different components of the CPS that is needed to keep the overall system stable?

Immediately associated with this basic question is the engineering problem of designing encoding, decoding, and control policies that best exploit the available information flow. In the last two decades, the research community has addressed these issues by developing several mathematical abstractions that capture the interaction between communication and control in CPS. These results shed light on the behavior of real systems and provide guidelines to develop effective control policies Surveys of this literature appear in [2–6] and in the books [7–10]. This book chapter complements and extends these reviews, focusing on data-rate requirements for stabilization, and in particular on recent advancements obtained through the design of event-triggered control policies. In this case, a key point is that the information flow in the feedback loop is not only associated to data flowing through the links connecting the different devices, but it is more generally encoded in "events" that occur over time. This new point of view leads to several extensions of classic results and to a broader perspective on the information constraints associated to control systems.

To illustrate the results, we consider a high-level abstraction of a CPS as the block diagram depicted in Figure 1. A dynamical system evolves over time according to deterministic state equations, affected by stochastic disturbances. Sensors monitor the system's output and their readings are encoded and sent through a digital communication channel to a controller, whose action is fed back to the actuators through another digital communication channel. Examples of CPS that fall within this general description include autonomous and remotely controlled robots, unmanned aerial vehicles (UAVs), autonomous vehicles (AV), and several industrial control systems. We further simplify this model by assuming that the controller is co-located with the actuators and the only communication channel is between the sensors and the controller, see Figure 2. This comes at no loss of generality so long as the information available to perform encoding and decoding is the same at the sensor, controller, and actuators. In this case, Figure 2 is formally equivalent to Figure 1 [11, Proposition 2.2.], since performing decoding and re-encoding at the controller is redundant, and

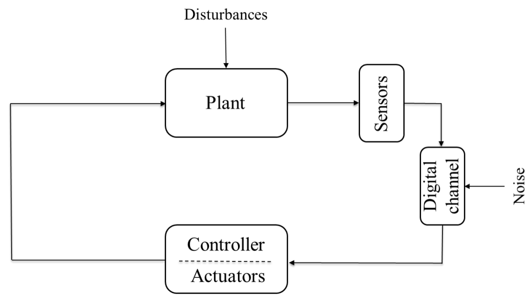



**Fig. 2** Simplified model where the controller is co-located with the actuators.

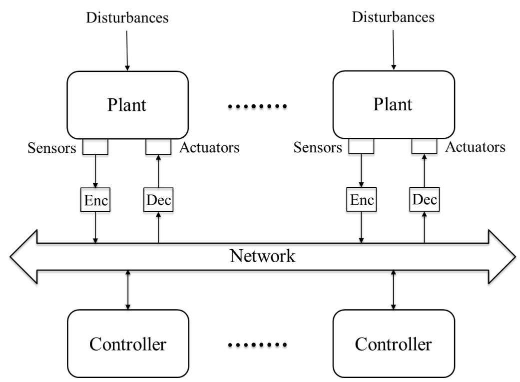



**Fig. 3** Distributed communication and control system.

the bottleneck link determines the effective data-rate. The information flow through the feedback loop can then be viewed as occurring over a single channel, which can also represent a multi-hop network connection, and in this case the effective data-rate refers to the rate available at the endpoints of the connection.

On the other hand, we point out that in practice the information available for encoding and decoding may be different at different points in the feedback loop and solutions in this case are highly dependent on the assumed information pattern. For example, it may be reasonable to assume that the encoder at the controller is uncertain of what the actuator actually received, while the encoder at the sensor is able to infer what the controller or the actuator has received using system observations. Most schemes in the literature rely on the availability of such causal feedback to perform their encoding operations, which further justifies the adoption of the model in Figure 2.

Finally, we also point out that the different blocks, including the system, sensors, controller, and actuators, can be spatially distributed, see Figure 3, and this raises additional issues regarding the distributed computation of the control policy, which we do not consider here. A key aspect of CPS to which we pay special attention is the need to use available resources such as power efficiently. In addition to being computation-aware, the architecture of the CPS should be communication-aware. The focus of this chapter is on design of systems that can leverage communication-aware solutions.

*Notation*: Random variables are displayed in sans serif, upright fonts; their realizations in serif, italic fonts. Vectors and matrices are denoted by bold lowercase and uppercase letters, respectively. For example, a random variable and its realization are denoted by $\mathsf{x}$ and $x$; a random vector and its realization are denoted by $\mathbf{x}$ and $\boldsymbol{x}$; a random matrix and its realization are denoted by $\mathbf{X}$ and $\boldsymbol{X}$, respectively.

## 2 Data-rate theorem

A fundamental result on the information flow requirements in CPS obtained in the last two decades is the so-called *data-rate theorem* [11–13]. This quantifies the effect that communication has on closed-loop stabilization of unstable systems by stating that the communication rate available in the feedback loop should be at least as large as the *intrinsic entropy rate* of the system. For continuous linear systems the intrinsic entropy rate corresponds to the sum of the unstable modes of the system and for discrete systems it corresponds to the sum of the logarithms of the unstable modes. When this condition is satisfied, the controller can compensate for the growth of the state space occurring during the communication process. To illustrate this result, consider the set of equations

$$\begin{cases} \mathbf{x}_{k+1} = \boldsymbol{A}\mathbf{x}_k + \boldsymbol{B}\mathbf{u}_k + \mathbf{v}_k & \text{(1a)} \\ \mathbf{y}_k = \boldsymbol{C}\mathbf{x}_k + \mathbf{w}_k, & \text{(1b)} \end{cases}$$

where $k = 0, 1, \ldots$ is time, $\mathbf{x}_k \in \mathbb{R}^d$ represents the state variable of the system, $\mathbf{u}_k \in \mathbb{R}^m$ is the control input, $\mathbf{v}_k \in \mathbb{R}^d$ is an additive disturbance, $\mathbf{y}_k \in \mathbb{R}^p$ is the sensor measurement, $\mathbf{w}_k \in \mathbb{R}^p$ is the measurement disturbance, and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are constant real matrixes of matching dimensions. Standard conditions on $(\boldsymbol{A}, \boldsymbol{B})$ to be reachable, $(\boldsymbol{C}, \boldsymbol{A})$ observable, are added to make the problems considered well posed. The equivalent continuous formulation is

$$\begin{cases} \dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t) + \mathbf{v}(t) & \text{(2a)} \\ \mathbf{y}(t) = \boldsymbol{C}\mathbf{x}(t) + \mathbf{w}(t). & \text{(2b)} \end{cases}$$

In a first approximation, noise and bandwidth limitations in the communication channel can be captured by modeling the channel as a rate-limited "bit pipe" capable of transmitting only a fixed number $r$ of bits in each time slot of the system's evolution, see Figure 4. In this way, the channel can represent a network connection with a limited available bit-rate, when transmitting below this rate communication errors are assumed to be negligible, and only quantization of the transmitted messages is accounted for.

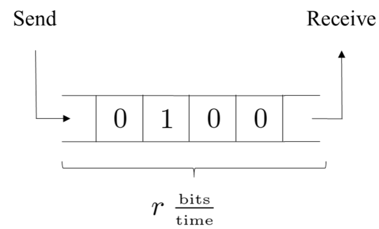



**Fig. 4** Bit-pipe channel

When the control objective is to keep its state bounded, or asymptotically drive it to zero, the control law is always a linear function of the state. Hence, for unstable linear systems under this rate-limited bit-pipe communication model, the central issue is to characterize the ability to perform a reliable estimate of the state at the receiving end of the communication channel. In order to keep the system stable, the data-rate theorem states that the information rate $r$ supported by the channel must be large enough compared to the unstable modes of the system, so that it can compensate for the expansion of the state during the communication process. Namely,

$$r > \sum_{|\lambda_i| \geqslant 1} \log_2 |\lambda_i| \quad \text{[bits/sec]}, \tag{3}$$

for discrete systems, where $\{\lambda_i\}$ are the open-loop eigenvalues raised to their corresponding algebraic multiplicities, and

$$r > \sum_{\mathrm{Re}\{\lambda_i\}>0} \lambda_i \log_2 e \quad \text{[bits/sec]}, \tag{4}$$

for continuous systems, If the real parts of all the eigenvalues of $A$ are positive (unstable), this can be written as

$$r > \mathrm{tr}(A) \log_2 e \quad \text{[bits/sec]}. \tag{5}$$

The intuition behind the data-rate theorem is evident by considering a scalar system and noticing that while the volume of the state of the open loop system increases by $|\lambda|$ in a unit time step in the discrete setting —or by $|e^{\lambda}|$ in the continuous setting— in closed loop this expansion is compensated by a factor $2^{-r}$ due to the partitioning induced by the coder providing $r$ bits of information through the communication channel. By imposing the product to be less than one and taking the logarithm base two, the results follow. Another interpretation arises if one identifies the right-hand side of (3) and (5) as a measure of the rate at which information is generated by the unstable plant, then the theorem essentially states that to achieve stability the channel must be able to transport information as fast as it is produced.

Early incarnations of this fundamental result appeared in [14–17] for undisturbed, scalar, unstable plants, when the objective is to keep the state bounded at all times. Improvement of the result from maintaining a bounded state to obtaining a state that asymptotically approaches zero are shown in [18–20] and require an adaptive "zoom-in, zoom-out" strategy that adjusts the range of the quantizer so that it increases as the plant's state approaches the target and decreases if the state diverges from the target. This follows the intuition that in order to drive the state to zero, the quantizer's resolution should become higher close to the target.

In the presence of stochastic disturbances, asymptotic stability can only be guaranteed within the range of the disturbances. The work [12] showed that for almost surely (a.s.) bounded disturbances and initial condition, the data-rate theorem holds and we can have

$$\sup_{k\in\mathbb{N}} \|\mathbf{x}_k\|^2 < \infty, \quad \text{a.s.} \tag{6}$$

On the other hand, unbounded disturbances can drive the state arbitrarily far from zero, and one can only guarantee stability in a weaker, probabilistic sense. The typical approach is to consider mean-square (m.s.) stability, namely

$$\sup_{k\in\mathbb{N}} \mathbb{E}(\|\mathbf{x}_k\|^2) < \infty. \tag{7}$$

The work [11] proved the data rate theorem using mean-square stability for systems with unbounded stochastic disturbances and bounded higher moments, namely

$$\exists \epsilon > 0 : \mathbb{E}(\|\mathbf{x}_0\|^{2+\epsilon}) < \infty, \ \sup_{k\in\mathbb{N}} \mathbb{E}(\|\mathbf{v}_k\|^{2+\epsilon}) < \infty, \ \sup_{k\in\mathbb{N}} \mathbb{E}(\|\mathbf{w}_k\|^{2+\epsilon}) < \infty. \tag{8}$$

The strict necessity of the data-rate requirement in (3) is proven in the case of bounded disturbances and a.s. stability using the Brunn-Minkowski inequality. In the case of unbounded disturbances and mean square stability it is proven using the stochastic counterpart of the inequality, namely the entropy power inequality of information theory [2]. The main difficulty in proving the sufficiency of the data-rate requirement in (3) in the unbounded support case is due to the uncertainty about the state that cannot be confined in any bounded interval. This is overcome by using an adaptive quantizer described in [11].

## 3 Extensions to noisy channels

Several generalizations of the data-rate theorem have been considered in the literature. An important body of work regards extensions to *stochastic channels*, namely channels whose behavior varies randomly due to noise. In this case, the rate available for transmission through the noisy channel must be defined in terms of information capacity. In this setting, a key result is that for undisturbed systems one can derive a data-rate theorem expressing the rate available for transmission in terms of the *Shannon capacity* of the channel [21, 22]. The Shannon capacity is the supremum of the achievable rates having arbitrarily small error probability. In contrast, when systems are subject to disturbances the standard notion of Shannon capacity turns out to be insufficient to express the ability to stabilize the system in both the a.s. and m.s.

sense. In this case, alternative notions of capacity must be used to formulate data-rate theorems, namely the *anytime capacity* must be used to express m.s. stability [23], and the *zero-error* capacity must be used to express a.s. stability [24]. The anytime capacity is the supremum of the achievable rates having exponentially decaying error probability for every transmitted bit, and the zero-error capacity is the supremum of the achievable rates when the error probability is zero. To illustrate these results, consider stabilization of a scalar system over a simple stochastic erasure channel where the rate varies randomly between the two values $\{r, 0\}$ in an i.i.d. fashion. Namely, for all $k$ we have the stochastic rate process

$$\mathbf{r}_k = \begin{cases} 0 & \text{w.p.} \quad p \\ r & \text{w.p.} \quad 1-p, \end{cases} \tag{9}$$

and we assume that both encoder and decoder have causal knowledge of the channel realization. In information-theoretic terms, this is as an $r$-bit packet erasure channel with acknowledgement of packet reception and erasure probability $p$. This channel is visually illustrated in Figure 5(a).

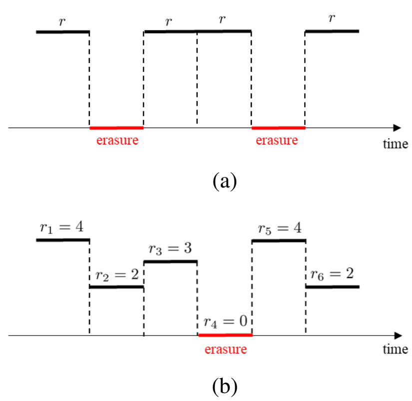



**Fig. 5** Example of stochastic channels. (a) $r$-bit packet erasure channel. (b) stochastic-rate channel, including erasures.

The Shannon capacity of this channel is [25]

$$C = (1-p)r. \tag{10}$$

However, it turns out that a large Shannon capacity is not enough to achieve m.s. stabilization (7). Instead, we need a more stringent condition, that in the scalar case is expressed as

$$\mathbb{E}\left(\frac{|\lambda|^2}{2^{2\mathbf{r}_k}}\right) = p\frac{|\lambda|^2}{2^0} + (1-p)\frac{|\lambda|^2}{2^{2r}} < 1. \tag{11}$$

Using the same intuition of production and consumption of information mentioned above, this condition states that the average of the product of the open loop state expansion and compensation through $r$-bit quantization should be kept less than one to ensure stability. By rearranging terms, we obtain the anytime capacity condition [23]

$$C_A \equiv \frac{2^{2r}}{2^{2r}p + 1 - p} = \frac{2^{2r}}{p(2^{2r} - 1) + 1} > |\lambda|^2. \tag{12}$$

The notion of anytime capacity, expressed in the left-hand side of (12) for the $r$-bit erasure channel can be extended to other communication channels, and it turns out to be the correct figure of merit to express data-rate theorems describing the ability to achieve m.s. stabilization [23].

The expression in (12) shows a clear a trade-off between the reliability of the channel and the quantization rate. Namely, when the quantization rate $r \to \infty$, we obtain

$$\frac{1}{p} > |\lambda|^2, \tag{13}$$

indicating that the erasure probability $p$ must be small enough to guarantee stability. In contrast, as $r \to \infty$, the Shannon capacity (10) tends to infinity, and does not give any indication on the channel reliability needed to achieve stabilization. In general, for any finite value of $r$ both the quantization rate and the reliability of the channel play a role in determining the ability to stabilize the system. Finally, when $r \to 0$ we obtain $|\lambda| < 1$, namely the system cannot be stabilized regardless how small the erasure probability $p$ is.

There are several channels for which anytime capacity stabilization conditions similar to (12) have been obtained. These include time-varying rate channels [26,27]. In this case, the rate process $\mathbf{r}_k$ varies randomly over time in an i.i.d. fashion, taking values in a subset of the non-negative integers, see Figure 5(b). The erasure channel considered above is a special case, when the rate process takes values in $\{0, r\}$. Another extension regards stochastic channels where the rate varies according to a Markov process [28–30]. This allows arbitrary temporal correlations of the channel variations over time, and results rely on the theory of Markov Jump Linear Systems [2]. In the special case of Gaussian channels, it turns out that the Shannon capacity is indeed sufficient to characterize m.s. stability and we refer to [2, Sec 1.4.4] and references therein for a description of these results.

For general stochastic channels, and systems with bounded disturbances, if instead of m.s. stability (7), one wants to achieve the more stringent a.s. stabilization condition (6), a basic result in [24] shows that the capacity notion to use is the zero-error one. This capacity definition was originally introduced by Shannon in 1956 [31], following his celebrated 1948 paper [32], which introduced the classical Shannon capacity. Its recent application to the field of networked control has led to the development of a nonstochatsic information theory, which characterizes the zero-error capacity in terms of an information functional describing the flow of information through the feedback loop [33].

An intuitive interpretation of the results above arises considering that the Shannon capacity defines information transfer using a "weak" reliability constraint, namely it is required that the probability of decoding a codeword incorrectly tends to zero. In contrast, the zero-error capacity defines information transfer using a "strong" reliability constraint, namely it is required that the probability of decoding a codeword incorrectly is exactly zero. The anytime capacity definition sits somewhat in between the two, requiring repeated "anytime" decoding of all codewords every time a new symbol is received, and imposing that the probability of having an error in any of the decoded codewords tends to zero exponentially as more and more symbols are received. Accordingly, the Shannon capacity is too weak to express the ability to stabilize a system, the zero-error capacity can be used to express stabilization in a.s. sense, and the anytime capacity can be used to express stabilization in the weaker mean square sense. For a more extensive discussion of the relationship between the different capacity definitions, we refer the reader to [2, 30].

## 4 Event triggering

Data-rate theorems have recently been revisited in the context of event-triggering control. In this context, a more general perspective on the information flowing through the feedback loop arises by considering that information can be encoded not only in data payload but also in "events" that occur over time.

Event-triggering [34–49] is a recent control paradigm that seeks to prescribe information exchange between the controller and the plant in an opportunistic manner. The main idea can be summarized in the following quote by the greek philosopher Plato:

> Wise men speak because they have something to say; fools because they have to say something.

Rather than communicating periodically the control action, communication in event-triggering control occurs only when triggered by some events indicating the need to send fresh information to guarantee the correct execution of the task at hand (e.g., stabilization, tracking). The primary focus then is on minimizing the number of transmissions while guaranteeing the control objectives.

At a high level, one can view event triggering as *sampling in time* with the objective to identify the minimum sampling rate at which information may be transmitted through the feedback loop. Similarly, a bit-pipe communication model can be viewed as *sampling in space*, namely as quantization of the signal, and the data-rate theorem corresponds to the identification of the minimum quantization rate that can still guarantee stabilization. In the case of stochastic channels, we have both sampling in space, since we are transmitting through a digital channel messages of finite precision, and sampling in time, through errors and erasures. This view suggests that there should be a close connection between data-rate theorems and event-triggered control [47].

A first connection is revealed in [50], which presents a data-rate theorem for event-triggering strategies for systems subject to bounded disturbances and controlled over a bit-pipe communication channel. Consider the system's equations

$$\begin{cases} \dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t) + \mathbf{v}(t) & \text{(14a)} \\ \mathbf{y}(t) = \mathbf{x}(t). & \text{(14b)} \end{cases}$$

where the initial condition $\mathbf{x}(0)$ and the system disturbance $\mathbf{v}(t)$ are a.s. bounded. At each triggering event, the sensor transmits to the controller a packet of a fixed number of bits. Letting $b_s(t)$ be the total number of bits transmitted by the sensor to the controller up to time $t$, the data-rate theorem is expressed in terms of the asymptotic average transmission bit-rate of the sensor

$$r \equiv \limsup_{t\to\infty} \frac{b_s(t)}{t} \quad \text{[bits/sec]}. \tag{15}$$

This rate clearly depends on the triggering strategy. Letting $\|x_\infty\|^2$ be a deterministic bound on the the steady state and $\kappa$ be a sufficiently large constant (both depending on the range of the disturbance and the initial condition), it turns out that to obtain exponential stability at rate $\sigma$, namely requiring that for all $t \geqslant 0$

$$\|\mathbf{x}(t)\|^2 \leqslant (\kappa - \|x_\infty\|^2)e^{-2\sigma t} + \|x_\infty\|^2 \quad \text{a.s.}, \tag{16}$$

we need

$$r \geqslant (\mathrm{tr}(\boldsymbol{A}) + \sigma d)\log_2 e \quad \text{[bits/sec]}, \tag{17}$$

where $d$ indicates the dimension of the system. The expressions (16) and (17) are consistent with (5) and (6), where $\sigma d$ represent the extra bits required for exponential convergence to the steady state. It follows that the result in [50] can be viewed as being analogous to the one in [12], but obtained here in the context of event triggering for continuous systems and with exponential convergence guarantees. However, while the result in [12] is a data-rate theorem that is both necessary and sufficient for stabilization, the event-triggering controller design proposed in [50] uses an asymptotic data rate that is within a constant factor from the necessary condition (17). Nevertheless, the proposed design adjusts the communication rate in accordance with state information in an opportunistic fashion and it guarantees a uniform positive lower bound on the times between successive triggering events, thus avoiding the so called "Zeno behavior."

## 4.1 Zeno Behavior

Event-based control is a fundamentally *hybrid phenomenon*, since we control continuous dynamics at discrete instances. In this setting the Zeno phenomenon [51–55] is an important issue that needs to be accounted for when modeling, analyzing, and

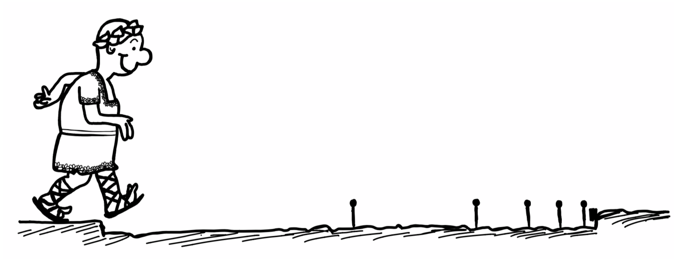


**Fig. 6** A typical Zeno paradox is that in order to reach a destination, we always need to cover half the distance to it, thus we are prevented from ever reaching it (see [56]).

controlling the system. The name Zeno refers to the ancient Greek philosopher Zeno of Elea who designed a number of paradoxes to support his view that the concepts of motion and evolution lead to contradictions. The Zeno behavior we are concerned here is related to Dichotomy paradox [56], illustrated in Figure 6, which can be stated as: “That which is in locomotion must arrive at the half-way stage before it arrives at the goal.” By applying this statement recursively, Zeno concludes that we should never be able to reach any destination. In the context of event-triggering, the Zeno phenomenon is a possible degenerate behavior of some event-triggering strategies leading to an infinite number of triggering events occurring in a finite amount of time.

Figure 7 gives an example of a possible Zeno behavior. We consider a scalar continuous-time system

$$\dot{x} = Ax(t) + Bu(t), \tag{18}$$

and we assume that at the controller the estimated state $\hat{x}$ evolves during the inter-reception times as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t). \tag{19}$$

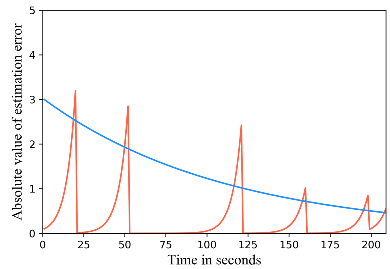
We assume the sensor can use (19) to compute $\hat{x}(t)$ for all $t \geq 0$, provided it knows $\hat{x}(0)$. Thus, the *estimation error* at the sensor is

$$z(t) = x(t) - \hat{x}(t), \tag{20}$$

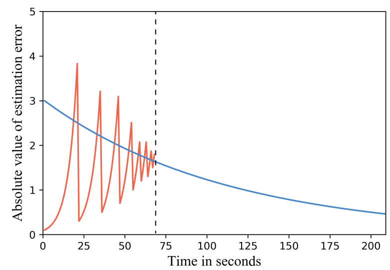
and evolves during the inter-reception times as

$$\dot{z} = Az. \tag{21}$$

This shows that during the inter-reception times the state estimation error evolves in open loop, independent of the control input. The sensor relies on this estimation error to determine the triggering events. Specifically, a triggering happens when the absolute value of the estimation error crosses a given threshold function, indicated by the blue curve in Figure 7. Figure 7(a) shows the estimation error evolution of a “good” communication strategy. In this case the estimation error after the reception




(a)




(b)

**Fig. 7** Example of Zeno phenomenon. (a) regular behavior (b) degenerate behavior. The orange line represents the absolute value of the estimation error, and the blue curve represents the triggering threshold.

of each packet jumps to a value that is far from the triggering boundary, which ensures that the time between any two consecutive triggering times has a uniform lower bound and the estimation error converges to zero. The uniform lower bound on two consecutive triggering times leads to an upper bound on the frequency at which the triggering occurs. In this example, the triggering threshold tends to zero exponentially, and a "good" communication strategy needs to ensue $|z(t)|$ tends to zero at the same rate. In contrast, Figure 7(b) shows the realization that exhibits a Zeno behavior. In this case, the frequency of the triggering events increases over time, leading to an estimation error that remains constant, while triggering occurs infinitely often. When designing event-triggering strategies one must rule out the Zeno behavior. In practice, any valid algorithm, such as the one in [50] must guarantee a uniform lower bound on the inter-transmission times between successive triggering events.

## 5 Timing information in event-triggering

The event-triggering strategy proposed in [50] adjusts the transmission rate to the state information required by the controller to guarantee stability, avoids Zeno behavior, and ensures that transmissions occur only as needed, thus saving resources. This work shows that the average transmission rate over the channel, expressed by (15), must satisfy the minimum requirement imposed by the data-rate theorem and expressed by (17). This result seems to indicate that the data-rate theorem is in complete harmony with event-triggering, in the sense that whether the transmission rate is limited by channel conditions, or it is limited opportunistically through event-triggering, the same fundamental limitation applies, which is dictated by the unstable modes of the system and by the desired convergence rate and expressed by (17).

It turns out, however, that event-triggering can also exploit an additional resource that is not accounted for in the current formulation and which allows to achieve stabilization with a dramatically lower data-rate. The work in [57] reveals that if the channel does not introduce any delay and the controller is aware of the triggering strategy used by the sensor, then one can achieve stabilization by transmitting at a rate that is arbitrarily close to zero. To illustrate this point, consider the following undisturbed system

$$\begin{cases} \dot{\mathsf{x}}(t) = \boldsymbol{A}\mathsf{x}(t) + \boldsymbol{B}\mathsf{u}(t) & (22a) \\ \mathsf{y}(t) = \boldsymbol{C}\mathsf{x}(t). & (22b) \end{cases}$$

where $\mathsf{x} \in \mathbb{R}^d$ and $\mathsf{y} \in \mathbb{R}$ (that is $\boldsymbol{C} \in \mathbb{R}^{1\times d}$) and the only uncertainty is due to the random initial condition. The channel connecting the sensor to the controller is assumed to be capable of transmitting one bit in an arbitrarily small time unit, so that communication of this binary symbol can be considered instantaneous when compared to the system dynamics. Let $\{t_s^k\}_{k\in\mathbb{N}}$ be the sequence of times at which the sensor transmits a bit to the controller. These times are set by event triggering accordingly to a level crossing strategy. Letting $h$ be a given threshold, a transmission occurs every time the difference between two successive output samples crosses the threshold, namely the triggering condition is given by

$$|\mathsf{y}(t_s^k) - \mathsf{y}(t_s^{k+1})| = h. \tag{23}$$

The system initially evolves in open-loop by letting $\boldsymbol{u} = 0$. Then, at each triggering time the sensor transmits a single bit to the controller that encodes the sign of the $h$ step change in the $\mathsf{y}$ value. By receiving at least $d + 1$ bits, and solving the following system of equations

$$\begin{bmatrix} \boldsymbol{C}(e^{At_s^1} - e^{At_s^0}) \\ \boldsymbol{C}(e^{At_s^2} - e^{At_s^1}) \\ \vdots \\ \boldsymbol{C}(e^{At_s^d} - e^{At_s^{d-1}}) \end{bmatrix} \mathsf{x}_0 = \begin{bmatrix} \mathsf{y}(t_s^1) - \mathsf{y}(t_s^0) \\ \mathsf{y}(t_s^2) - \mathsf{y}(t_s^1) \\ \vdots \\ \mathsf{y}(t_s^d) - \mathsf{y}(t_s^{d-1}) \end{bmatrix},$$

where the right-hand side is a column vector of $\pm h$ values, the controller can infer the initial condition provided that the matrix on the left-hand side is nonsingular. Once the initial condition is known, it can then stabilize the system in a closed-loop fashion. Since we have

$$\limsup_{t\to\infty} \frac{b_s(t)}{t} = \lim_{t\to\infty} \frac{(d+1)}{t} = 0, \tag{24}$$

it follows that we can stabilize the system with an arbitrarily small transmission rate.

It is important to note that for this scheme to work, we need to know precisely the transmission times $\{t_s^k\}$. The assumption of zero delay ensures that the time at which the controller receives a bit also reveals the time of transmission. Of course, the same is true if communication carries an arbitrary but known delay. In practice, this assumption boils down to having perfect synchronization between encoder and decoder. The second important assumption is that the system is not affected by disturbances. As we shall see below, both of these assumptions can be considerably relaxed. Before describing these additional results, we build some intuition for this zero-rate result.

The intuition behind the result follows by noting that, like pauses are used in spoken language to convey information, in the context of event-triggering control it is possible transmit information in the feedback loop not only by message content, but also with its timing. Specifically, in the absence of delay in the communication channel, the mere act of sending one bit at a given time can reveal the state of the system with arbitrary precision, and transmitting a single data payload bit at every triggering event is enough to compute the appropriate control action. In fact, we may take this intuition one step further and also notice that under the same assumptions of [57] we do not even need to transmit a bit at each triggering time. To reveal any component $\mathsf{x}$ of $\mathbf{x}(0)$, we could transmit a single arbitrary symbol ♠ at a time equal to any bijective mapping of $\mathsf{x}$ into a point of the non-negative reals. For example, we could transmit a symbol ♠ at time $t = \tan^{-1}(\mathsf{x})$, where $t \in [0, \pi]$. Since there is no choice associated to the symbol selection, in principle the reception of the symbol should not carry any information. However, its arrival time carries information and can reveal $\mathsf{x}$ with arbitrary precision. To communicate the whole vector $\mathbf{x}(0)$, we could then send $d + 1$ identical ♠ symbols at different times and encode all the components of $\mathbf{x}(0)$ in their inter-transmission times. In principle, one could even send a single ♠ symbol to encode the whole $\mathbf{x}(0)$ vector by using a $d$-dimensional space-filling curve and selecting a time of transmission for which a point on the curve is mapped onto $\mathbf{x}(0)$.

To conclude, the important message to be taken from [57] is that using event triggering information can be transmitted in the feedback loop not only by sending data but also by carefully selecting the times of transmission. A similar observation is also made in [58]. This work considers the system to be fully observable, namely $\boldsymbol{C}$ to be the identity, and the sensor to transmit, at a *fixed* sequence of transmission times, $\{t_k\}$ symbols from a finite alphabet over a delay-free and error-free communication channel. It is further assumed that a special symbol in the alphabet can be transmitted without consuming any communication resources, effectively representing the absence of an explicit transmission, while the other symbols require one unit of communication resource per transmission. From an information-theoretic perspective, this set up is related to the silence-based communication paradigm of [59]. Letting $s(t_k)$ be the total number of non-free symbols transmitted by the sensor to the controller up to time $t_k$, the asymptotic average cost per unit time is given by

$$c \equiv \limsup_{k\to\infty} \frac{s(t_k)}{t_k} \quad \text{[symbols/sec]}, \tag{25}$$

where $t_k \to \infty$ as $k$ tends to infinity. This can also be interpreted as an effective data rate, since it represents the rate accounting for only the non-free transmissions, and should be compared with (15). In both cases, the rate depends on the transmission strategy. It is shown in [58] that stabilization can be achieved with arbitrarily small values of (25) by letting the transmission times $t_k = kT$, decreasing the sampling period $T$, and transmitting non-free symbols rarely. In this regime, the transmission policy resembles an event-triggering strategy where the transmission of a non-free symbol may occur at any given time, which depends on the encoding strategy, and can be chosen with arbitrary precision as $T \to 0$. At all other times only free symbols are sent —which is analogous to sending nothing. As in the case of [57], in this setup the act of transmitting a non-free symbol now carries an amount of information that can be made arbitrarily large by decreasing the sampling time $T$. This allows to decrease the number of transmitted non-free symbols and drives the effective rate (25) arbitrarily close to zero.

## 6 The value of timing information in the presence of delay

The works we have described suggest that a more general formulation of data-rate theorems should account for two distinct information flows: one is through data payload (possibly corrupted by noise) and another is through timing (possibly corrupted by delay). Traditionally, only the data payload case has been considered, but the timing information can be very relevant especially in the context of event triggering. The work in [50] only considers communication through data payload and does not attempt to exploit timing information. As a result, it recovers the traditional data-rate theorem formulation. In contrast, the works in [57] and [58] show that stabilization can be achieved with arbitrarily small data payload rate, by exploiting the timing information implicit in event-triggering schemes, when sender and receiver are perfectly synchronized.

At this point, one may suspect that the ability to stabilize the system with zero payload rate is an artifact of the assumed perfect synchronization between the sensor and the controller achieved through a zero-delay channel. As we shall see next, this is not the case. In the presence of delay, the value of the timing information in the triggering events decreases, because in this case the sensor may only reveal the state of the system with a finite precision —which depends on the range of the unknown delay. However, as long as the amount of information supplied by timing is above what prescribed by the data-rate theorem for stabilization, it is still possible to stabilize the system with an arbitrarily small data payload rate. Next, we illustrate this point in more detail.

## 6.1 Information Access Rate vs Information Transmission Rate

The works [60, 61] make a key distinction between the *information access rate* and the *information transmission rate*. The information access rate is the rate at which the controller needs to receive information, conveyed by both data payload and timing information, and that is subject to the requirement expressed by the classic data-rate theorem. The information transmission rate is the rate at which the sensor needs to send data in the form of payload bits. This rate depends on the triggering scheme, and it can become arbitrarily small without affecting the ability to stabilize the system.

First, let us take the viewpoint of the sensor and examine the amount of information in the data payload transmissions to the controller. Let $b_s(t)$ be the number of bits in the data payload transmitted by the sensor up to time $t$, and define the information transmission rate as

$$r_s \equiv \limsup_{t\to\infty} \frac{b_s(t)}{t}.$$

Let us now consider the viewpoint of the controller and examine the amount of information that it needs to receive in order to be able to select its stabilizing policy. This includes both payload and timing information. It is also the same as the number of bits needed to construct a reliable state estimate [62, Theorem 1]. We let $b_c(t)$ be the number of bits required at the controller to perform its selection at time $t$ and define the *information access rate* as

$$r_c \equiv \limsup_{t\to\infty} \frac{b_c(t)}{t}.$$

In classic data-rate theorems $r_c$ coincides with $r_s$ because the controller uses only data payload bits to select its control law. On the other hand, as discussed above, by exploiting timing information $r_s$ and $r_c$ can be substantially different and the classic data-rate limitation applies to $r_c$ only, while we can achieve stabilization with $r_s$ arbitrarily close to zero.

To view the limitation on $r_c$, we consider the same system's equations as in (14). In this case, a necessary and sufficient condition to achieve exponential stabilization at rate $\sigma$ is given by the usual data-rate theorem formula expressed in terms of $r_c$

$$r_c \geqslant (\mathrm{tr}(\boldsymbol{A}) + \sigma d) \log_2 e \quad \text{[bits/sec]}. \tag{26}$$

This result should be compared with (17). It is important to stress that the limitation in (26) describes what is required by the controller, and it does not depend on the feedback structure — including aspects such as communication delays, information pattern at the sensor and the controller, and whether the times at which transmissions occur are state-dependent, as in event-triggered control, or periodic, as in time-triggered control. In order to obtain (26), one considers for any control input trajectory $u(t)$ the subset of initial conditions for which the plant is stabilized by such input. Then, one constructs a cover of the set of all initial conditions by stabilizing control policies. This leads to a discrete set of choices for selecting the stabilization policy for any given realization of the initial condition. It follows that the logarithm of the covering number is the number of bits needed by the controller by time $t$ to select a stabilizing control policy. A usual balance of information argument between the rate of expansion of the uncertainty in the state due to the random initial condition, and the quantization due to the covering finally leads to the lower bound in (26).

Having established the classic data-rate theorem result for the information access rate, we can now ask what is the data-rate requirement on the information transmission rate $r_s$, assuming that the sensor has access to causal feedback regarding what has been received by the controller and for different ranges of the possible delay. While we have already established that $r_s$ can be arbitrarily close to zero in the absence of delay, the presence of unknown delay decreases the amount of information that can be communicated by timing, and this may require $r_s$ to become positive.

To illustrate the results, we denote by $\{t_s^k\}_{k\in\mathbb{N}}$ the sequence of times when the sensor transmits a packet of a certain number $g(t_s^k)$ of bits to the controller. We assume the packet is delivered to the controller without error and entirely but with an unknown delay. Letting $\{t_c^k\}_{k\in\mathbb{N}}$ be the sequence of times when the controller receives the packets transmitted at times $\{t_s^k\}_{k\in\mathbb{N}}$, we assume that for all $k \in \mathbb{N}$ the communication delay $\Delta_k = t_c^k - t_s^k$ satisfies

$$\Delta_k \leqslant \gamma, \tag{27}$$

where $\gamma \in \mathbb{R}_{\geq 0}$, and that both $t_s^k$ and $t_c^k$ tend to infinity as $k \to \infty$. We can then study how the rate $r_s$ required for stabilization using an event-triggering varies as a function of $\gamma$. The work [60] considers the case of systems without disturbances, where the objective is to drive the state to zero at an exponential rate $\sigma$. The work [61] considers the case of systems with disturbances using a notion of input to state stability, which guarantees that the state is bounded at all times and this bound, as usual, depends on the range of the disturbance. To illustrate the results, we focus on scalar systems in the following.

A plot of the rate required to keep the state bounded when using any threshold-based event-triggering policy based on the value of the state estimation error is depicted in Figure 8. This shows that the required rate for stabilization undergoes a *phase transition*: for small values of the delay upper bound $\gamma$ the system can be stabilized with an arbitrarily small information transmission rate. However, when $\gamma$ reaches the critical threshold

$$\gamma_c = \frac{\ln 2}{A} \tag{28}$$

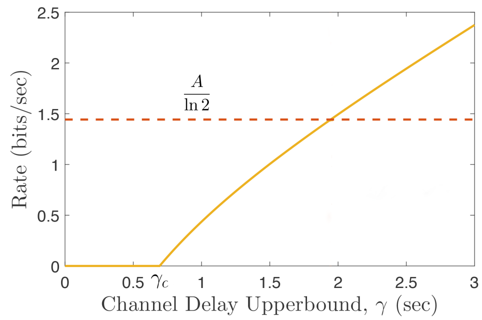



**Fig. 8** Phase transition of the necessary information transmission rate for stabilization. The graph is valid for any generic system. In this example we have a scalar system with no disturbance, $A = 1$, $\gamma_c = \ln 2/A = 0.6931$ and the rate dictated by the data-rate theorem is $R_c \geq A/\ln 2 = 1.4427$.

the required rate begins to increase, eventually surpassing the data-rate theorem requirement $A/\ln 2$. This indicates that for $\gamma < \gamma_c$ the amount of information contained in the timing of the triggering events is large enough that the rate that must be supplied by data-payload to guarantee stability is zero. On the other hand, when $\gamma > \gamma_c$ the information contained in the timing of the triggering events is not enough to guarantee stability and the rate must begin to increase. One way to interpret this result is that in the presence of delay the value of the timing information supplied by event triggering "deteriorates" and eventually becomes insufficient to be used alone for stabilization. On the other hand, increasing the the delay also affects the rate at which the transmitted payload bits are received, which results in a higher transmission rate requirement that can surpass the data-rate theorem requirement.

It is interesting to point out that the critical value $\gamma_c$ at which the information transmission rate becomes positive equals the inverse of the entropy rate of the system, namely (28) is the inverse of the critical rate in the data-rate theorem formula (5). Recalling the production and consumption of information analogy that we discussed in Section 2, we have that for $\gamma = \gamma_c$ the entropy of the system expands by one new bit at every delay occurrence and this amount of information cannot be counter-balanced by the information carried by the event triggering times. In other words, the information supplied by the triggering events can always be "one bit short" due to the uncertainty introduced by the delay, and this bit must be supplied by the data payload to ensure stabilization. For this reason, the rate $r_s$ begins to increase once $\gamma$ reaches the critical value $\gamma_c$. Figure 9 also shows a sufficient condition for stabilization obtained using a given triggering strategy described in [61] that employs a fixed threshold policy and compares it with the necessary condition and with the data-rate theorem requirement.

Finally, we now turn our attention to the results for exponential stabilization, driving the state to zero at rate $\sigma \geqslant 0$, for systems without disturbances. The work in [60] shows that in this case an analogous phase transition behavior occurs. Figure 10 shows the required rate for exponential stabilization when using any threshold-based triggering strategy that guarantees the state estimation error to jump

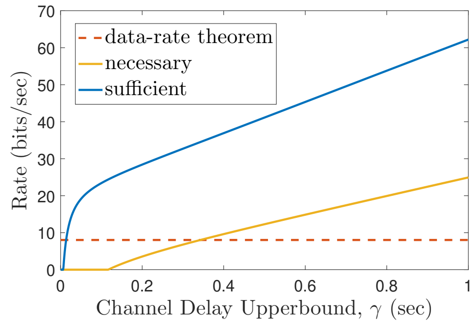
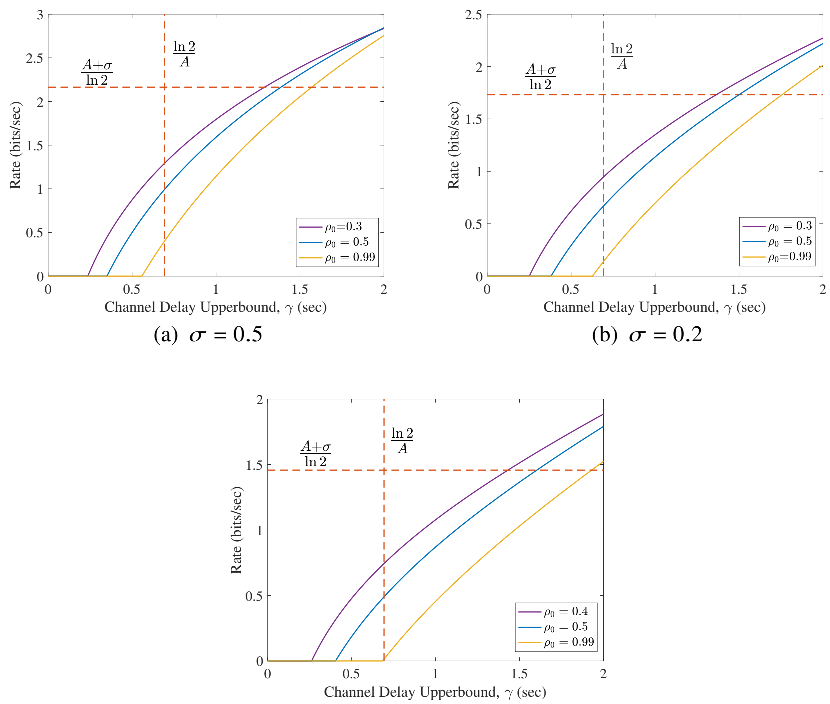



**Fig. 9** Sufficient and necessary transmission rates for stabilization. Here we have a scalar system with disturbances bounded by 0.4, with $A = 5.5651$. The rate dictated by the data-rate theorem is $R_c \geq A/\ln 2 = 8.02874$.

below the threshold at every triggering time. The parameter $0 \leqslant \rho_0 \leqslant 1$ in the figure regulates the magnitude of the jump below the triggering threshold. As $\sigma \longrightarrow 0$ we have no exponential convergence guarantees, and as $\rho_0 \longrightarrow 1$ we have no constraints on the magnitude of the jumps below the threshod. The plots show that when these limiting cases are considered jointly, we recover the result in [61]. Namely, the necessary condition becomes the same as the one depicted in Figure 8, and the critical channel delay upper bound tends once again to the inverse of the entropy rate of the plant.

## 7 A practical demonstration of the value of timing

While control under communication constraints has been mainly a subject of theoretical investigation, we wish to complete our treatment with a practical example. We consider an inverted pendulum, which is a classic example of an inherently unstable nonlinear system with numerous practical applications. This example is based on the work in [63], demonstrating stabilization using event-triggering over a communication channel subject to random packet delays. The pendulum is controlled by two propellers as shown in Figure11, and is constructed using off-the-shelf components. Specifically, the frame is built with plywood sheets, and we employ an InvenSense MPU6050 MEMS sensor (which consists of a 3-axis accelerometer and a 3-axis gyroscope), using a complementary filter to estimate the pendulum's angle and angular velocity, and we have a Raspberry Pi Model 3 acting as the computation unit as well as the controller. Finally, two small DC motors equipped with two identical propellers are used as actuators. The architecture of the system is illustrated in Figure 12. The reader is referred to [63] for further details of the implementation and




(a) $\sigma = 0.5$ (b) $\sigma = 0.2$

(c) $\sigma = 0.01$

**Fig. 10** Illustration of the phase transition behavior for system without disturbances for different values of $\rho_0$ and convergence rate $\sigma$. The graph is valid for any generic system. In this example we have a scalar system with $A = 1$, and $\ln 2/A = 0.6931$.

associated experiments [1]. Here, we limit to point out that results confirm that for small values of the delay the payload transmission rate can be lower than the entropy rate of the plant. On the other hand, by increasing the upper bound on the delay in the communication channel, higher data payload transmission rates are required to satisfy the requirements of the proposed control strategy.

## 8 Conclusion

Information-flow requirements in networked control systems have been intensively investigated in the last two decades and have become extremely relevant today in

[1] The code can also be found at `https://github.com/mkhojas/Event-Triggered-Firmware`. A video that illustrates the main ideas and demonstrates the experimental results can be found at `https://youtu.be/1P0i-tWsPoA`

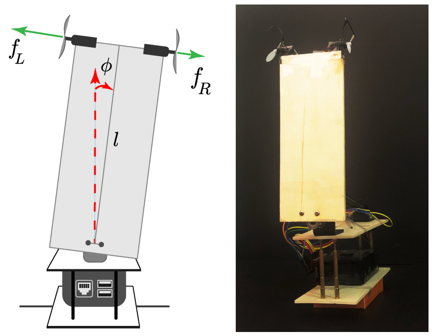



**Fig. 11** An inverted pendulum controlled by thrust force of two propellers. The angle $\phi$ of the pendulum from the vertical line and its rate of change, measured by the sensor and transmitted to the controller over a digital channel with bounded unknown delay, are used to determine the left and right thrust forces of the propellers.

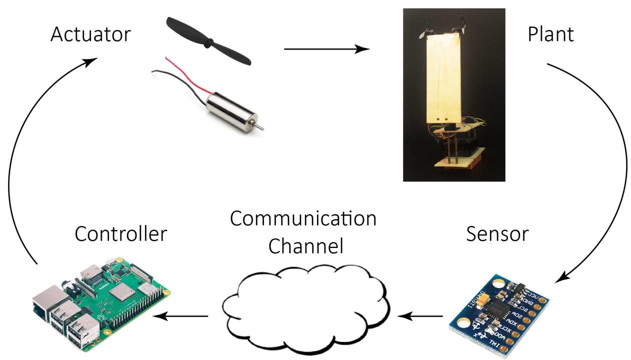



**Fig. 12** Architecture and components of the prototype.

the context of CPS. Results point out a balance of information principle: to ensure stability, information should be delivered to the controller at a rate that compensates for the uncertainty produced by the open-loop system. This basic principle appears as the leitmotiv of various data-rate theorems that can be applied in different practical scenarios. In the context of event-triggering the same principle applies, provided that when measuring the information flow one accounts for the additional information that can be communicated by choosing the times of transmission. While the information carried by data payload can be corrupted by noise, the information carried by timing can be corrupted by delay. Theoretical results as well as practical demonstrations have shown that for low values of the communication delay, the information carried by timing can be used to stabilize the system at dramatically low data rates. Moving forward, we expect that a complete theory of communication over feedback loops can be constructed by considering encoding and decoding strategies accounting for both communication by timing and data payload, as well as accounting for the distributed nature of many system implementations. The focus of this chapter was on the design of systems that can leverage communication-aware solutions. Depending on the application, a communication-aware design can lead to higher or lower computation cost. Studying architectures that aim to simultaneously reduce the computation and communication costs is a research venue that is left for future study.

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## References

1. K.-D. Kim and P. R. Kumar, "Cyber-physical systems: A perspective at the centennial," *Proc. IEEE*, vol. 100 (Special Centennial Issue), pp. 1287–1308, 2012.
2. M. Franceschetti and P. Minero, "Elements of information theory for networked control systems," in *Information and Control in Networks*. Springer, 2014, pp. 3–37.
3. B. G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proc. IEEE*, vol. 95, no. 1, pp. 108–137, 2007.
4. D. Liberzon, "Nonlinear control with limited information," *Communications in Information & Systems*, vol. 9, no. 1, pp. 41–58, 2009.
5. F. Colonius, U. Helmke, J. Jordan, C. Kawan, R. Sailer, and F. Wirth, "Analysis of networked systems," in *Control Theory of Digitally Networked Dynamic Systems*. Springer, 2014, pp. 31–79.
6. J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, 2007.
7. S. Yüksel and T. Başar, *Stochastic Networked Control Systems: Stabilization and Optimization under Information Constraints*. Springer Science & Business Media, 2013.
8. A. S. Matveev and A. V. Savkin, *Estimation and control over communication networks*. Springer Science & Business Media, 2009.
9. S. Fang, J. Chen, and H. Ishii, *Towards integrating control and information theories*. Springer, 2017.
10. C. Kawan, "Invariance entropy for deterministic control systems," *Lecture Notes in Mathematics*, vol. 2089, 2013.
11. G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM J Control Optim*, vol. 43, no. 2, pp. 413–436, 2004.
12. S. Tatikonda and S. K. Mitter, "Control under communication constraints," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1056–1068, Jul. 2004.
13. J. Hespanha, A. Ortega, and L. Vasudevan, "Towards the control of linear systems with minimum bit-rate," in *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*, 2002.
14. J. Baillieul, "Feedback designs for controlling device arrays with communication channel bandwidth constraints," in *ARO Workshop on Smart Structures*, Penn. State U., USA, 1999, pp. 16–18.
15. ——, "Feedback designs in information-based control," in *B. Pasik-Duncan (ed.), Proceedings of the Workshop on Stochastic Theory and Control*, Lawrence, Kansas, Springer, 2001.
16. W. S. Wong and R. Brockett, "Systems with finite communication bandwidth constraints. I. State estimation problems," *IEEE Trans. Autom. Control*, vol. 42, no. 9, pp. 1294–1299, Sept. 1997.
17. W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints. II. stabilization with limited information feedback," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 1049–1053, 1999.

18. R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, July 2000.
19. N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 46, no. 9, pp. 1384–1400, Sep. 2001.
20. D. Liberzon, "On stabilization of linear systems with limited information," *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 304–307, Feb. 2003.
21. S. Tatikonda and S. K. Mitter, "Control over noisy channels," *IEEE Trans. Autom. Control*, vol. 49, no. 7, pp. 1196–1201, Jul. 2004.
22. A. S. Matveev and A. V. Savkin, "An analogue of shannon information theory for detection and stabilization via noisy discrete communication channels," *SIAM J Control Optim*, vol. 46, no. 4, pp. 1323–1367, 2007.
23. A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link. Part I: Scalar systems," *IEEE Trans. Inf. Theory*, vol. 52, no. 8, pp. 3369–3395, 2006.
24. A. S. Matveev and A. V. Savkin, "Shannon zero error capacity in the problems of state estimation and stabilization via noisy communication channels," *Int. J. Control*, vol. 80, no. 2, pp. 241–255, 2007.
25. T. A. Cover and J. A. Thomas, *Elements of Information Theory*, 1st ed. New York, NY, 10158: John Wiley & Sons, Inc., 1991.
26. N. C. Martins, M. A. Dahleh, and N. Elia, "Feedback stabilization of uncertain systems in the presence of a direct link," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 438–447, 2006.
27. P. Minero, M. Franceschetti, S. Dey, and G. N. Nair, "Data rate theorem for stabilization over time-varying feedback channels," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 243–255, 2009.
28. K. You and L. Xie, "Minimum data rate for mean square stabilizability of linear systems with markovian packet losses," *IEEE Trans. Autom. Control*, vol. 56, no. 4, pp. 772–785, 2010.
29. P. Minero, L. Coviello, and M. Franceschetti, "Stabilization over Markov feedback channels: the general case," *IEEE Trans. Autom. Control*, vol. 58, no. 2, pp. 349–362, 2013.
30. P. Minero and M. Franceschetti, "Anytime capacity of a class of Markov channels," *IEEE Trans. Autom. Control*, vol. 62, no. 3, pp. 1356–1367, 2017.
31. C. Shannon, "The zero error capacity of a noisy channel," *IRE Trans. on Inf. Theory*, vol. 2, no. 3, pp. 8–19, 1956.
32. C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, no. 7, pp. 379–423, 1948.
33. G. Nair, "A non-stochastic information theory for communication and state estimation," *IEEE Trans. Autom. Control*, vol. 58, pp. 1497–1510, 2013.
34. K. J. Astrom and B. M. Bernhardsson, "Comparison of riemann and lebesgue sampling for first order stochastic systems," in *Proc. IEEE Conf. Decision and Control*, vol. 2. IEEE, 2002, pp. 2011–2016.
35. P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Trans. Autom. Control*, vol. 52, no. 9, pp. 1680–1685, 2007.
36. B. A. Khashooei, D. J. Antunes, and W. Heemels, "A consistent threshold-based policy for event-triggered control," *IEEE Control Syst. Lett*, vol. 2, no. 3, pp. 447–452, 2018.
37. A. Girard, "Dynamic triggering mechanisms for event-triggered control," *IEEE Trans. Autom. Control*, vol. 60, no. 7, pp. 1992–1997, 2014.
38. X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Trans. Autom. Control*, vol. 56, no. 3, p. 586, 2011.
39. D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Trans. Autom. Control*, vol. 57, no. 5, pp. 1291–1297, 2012.
40. B. Demirel, V. Gupta, D. E. Quevedo, and M. Johansson, "On the trade-off between communication and control cost in event-triggered dead-beat control," *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 2973–2980, 2017.
41. S. Trimpe and R. D'Andrea, "Event-based state estimation with variance-based triggering," *IEEE Trans. Autom. Control*, vol. 59, no. 12, pp. 3266–3281, 2014.

42. A. Seuret, C. Prieur, S. Tarbouriech, and L. Zaccarian, "Lq-based event-triggered controller co-design for saturated linear systems," *Automatica*, vol. 74, pp. 47–54, 2016.
43. J. Umlauft and S. Hirche, "Feedback linearization based on gaussian processes with event-triggered online learning," *IEEE Trans. Autom. Control*, vol. 65, no. 10, pp. 4154–4169, 2019.
44. D. Maity and J. S. Baras, "Optimal event-triggered control of nondeterministic linear systems," *IEEE Trans. Autom. Control*, vol. 65, no. 2, pp. 604–619, 2019.
45. W. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Proc. IEEE Conf. Decision and Control*. IEEE, 2012, pp. 3270–3285.
46. A. Tanwani and A. Teel, "Stabilization with event-driven controllers over a digital communication channel with random transmissions," in *Proc. IEEE Conf. Decision and Control*, 2017, pp. 6063–6068.
47. M. Lemmon, *Event-Triggered Feedback in Control, Estimation, and Optimization*. London: Springer London, 2010, pp. 293–358.
48. M. Miskowicz, *Event-based control and signal processing*. CRC press, 2018.
49. D. Tolić and S. Hirche, *Networked control systems with intermittent feedback*. Crc Press, 2017.
50. P. Tallapragada and J. Cortés, "Event-triggered stabilization of linear systems under bounded bit rates," *IEEE Trans. Autom. Control*, vol. 61, no. 6, pp. 1575–1589, 2015.
51. K. H. Johansson, M. Egerstedt, J. Lygeros, and S. Sastry, "On the regularization of zeno hybrid automata," *Syst. Control. Lett*, vol. 38, no. 3, pp. 141–150, 1999.
52. J. Zhang, K. H. Johansson, J. Lygeros, and S. Sastry, "Zeno hybrid systems," *INT J ROBUST NONLIN*, vol. 11, no. 5, pp. 435–451, 2001.
53. M. Heymann, F. Lin, G. Meyer, and S. Resmerita, "Analysis of zeno behaviors in a class of hybrid systems," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 376–383, 2005.
54. E. A. Lee, "Constructive models of discrete and continuous physical phenomena," *IEEE Access*, vol. 2, pp. 797–821, 2014.
55. A. D. Ames, H. Zheng, R. D. Gregg, and S. Sastry, "Is there life after zeno? taking executions past the breaking (zeno) point," in *American Control Conference*. IEEE, 2006, pp. 6–pp.
56. L. C. Epstein, *Thinking physics: Practical lessons in critical thinking*. Inside Press, 2002.
57. E. Kofman and J. H. Braslavsky, "Level crossing sampling in feedback stabilization under data-rate constraints," in *Proc. IEEE Conf. Decision and Control*. IEEE, 2006, pp. 4423–4428.
58. J. Pearson, J. P. Hespanha, and D. Liberzon, "Control with minimal cost-per-symbol encoding and quasi-optimality of event-based encoders," *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2286–2301, 2017.
59. A. K. Dhulipala, C. Fragouli, and A. Orlitsky, "Silence-based communication," *IEEE Trans. Inf. Theory*, vol. 56, no. 1, pp. 350–366, 2009.
60. M. J. Khojasteh, P. Tallapragada, J. Cortés, and M. Franceschetti, "The value of timing information in event-triggered control," *IEEE Trans. Autom. Control*, vol. 65, no. 3, pp. 925–940, 2019.
61. M. J. Khojasteh, M. Hedayatpour, J. Cortés, and M. Franceschetti, "Exploiting timing information in event-triggered stabilization of linear systems with disturbances," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 1, pp. 15–27, 2021.
62. M. J. Khojasteh, P. Tallapragada, J. Cortés, and M. Franceschetti, "Time-triggering versus event-triggering control over communication channels," in *Proc. IEEE Conf. Decision and Control*. IEEE, 2017, pp. 5432–5437.
63. M. J. Khojasteh, M. Hedayatpour, and M. Franceschetti, "Theory and implementation of event-triggered stabilization over digital channels," in *Proc. IEEE Conf. Decision and Control*. IEEE, 2019, pp. 4183–4188.