Supplement to "Temperate ice in the shear margins of the Antarctic Ice Sheet: controlling processes and preliminary locations"

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1. Introduction

In this supplement, we derive the one-dimensional thermodynamic model used in the main text. We follow the same structure as used to derive the diffusion model presented in the main text, but include advection of ice due to accumulation. We also address our choice of constant vertical ice velocity over the more commonly assumed linear decrease in vertical velocity with depth, and show how this choice affects the temperate zone thicknesses across the Antarctic Ice Sheet. Finally, we present enlarged versions of the results from the main text for regions of interest.

2. Advection and diffusion problem

2.1. The case of constant vertical velocity

Conservation of energy in one-dimension including advection, diffusion, and deformation heating $\sigma_{ij}\dot{\epsilon}_{ij}$ is given by

$$-\rho c_p a \frac{dT}{dz} = K \frac{d^2 T}{dz^2} + \sigma_{ij} \dot{\epsilon}_{ij} - \overline{\lambda}, \qquad (1)$$

where we include the depth-averaged lateral advection $\overline{\lambda}$ as in Equation (12) in the main text. In this equation, K is the thermal conductivity, ρ is the ice density, c_p is the heat capacity, and we take the vertical ice velocity to be constant with depth and equal to -a, where a is the surface mass balance accumulation rate.

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Following the same obstacle problem approach as in the main text, the boundary conditions are

$$-K\frac{dT}{dz} = 0, \quad T = T_m \quad \text{on} \quad z = \xi \quad \text{and} \quad T = T_s \quad \text{on} \quad z = H,$$
(2)

where the first condition is the steady Stefan condition.

Now integrating (1) subject to (2) gives

$$T = \begin{cases} T_s + \Delta T \frac{\operatorname{Br} - \Lambda}{\operatorname{Pe}} \left[1 - \frac{z}{H} + \frac{1}{\operatorname{Pe}} \exp\left\{\operatorname{Pe}\left(\frac{\xi}{H} - 1\right)\right\} - \frac{1}{\operatorname{Pe}} \exp\left\{\operatorname{Pe}\left(\frac{\xi - z}{H}\right)\right\} \right], & \xi \le z \le H \\ T_m, & 0 \le z < \xi \end{cases}$$
(3)

where the temperature difference is $\Delta T = T_s - T_m$ and the nondimensional quantities are the Brinkman number Br, Péclet number Pe, and the lateral advection parameter Λ , defined as

$$Br = \frac{(\sigma_{ij}\dot{\epsilon}_{ij})H^2}{K\Delta T}, \quad Pe = \frac{\rho c_p a H}{K}, \quad \Lambda = \frac{\overline{\lambda}H^2}{K\Delta T}$$
(4)

The temperature field is continuous at $z = \xi$ as $T = T_m$, therefore the thickness of the temperate zone ξ is determined as a function of Br and Pe as

$$\frac{\operatorname{Pe}}{\operatorname{Br}-\Lambda} = 1 - \frac{\xi}{H} - \frac{1}{\operatorname{Pe}} + \frac{1}{\operatorname{Pe}} \exp\left\{\operatorname{Pe}\left(\frac{\xi}{H} - 1\right)\right\}.$$
(5)

We can solve this equation using the Lambert-W function as

$$\frac{\xi}{H} = \left\{ \begin{array}{c} 1 - \frac{\mathsf{Pe}}{\mathsf{Br} - \Lambda} - \frac{1}{\mathsf{Pe}} \left[1 + \mathsf{W} \left(-\exp\left\{ -\frac{\mathsf{Pe}^2}{\mathsf{Br} - \Lambda} - 1 \right\} \right) \right], & \dot{\varepsilon}_{lat} > \dot{\varepsilon}_{lat}^* \\ 0, & \dot{\varepsilon}_{lat} \le \dot{\varepsilon}_{lat}^* \end{array} \right.$$
(6)

and this solution is shown as a function of Br in Figure S2 for several values of Pe with $\Lambda = 0$ and contoured on Figure 1b of the main text as a function of Br less Λ and Pe.

As described in the main text, we can also solve for the critical strain rate required to initiate temperate ice, which is given by

$$\dot{\epsilon}_{\rm lat}^* = \left(\frac{\frac{1}{2}\mathsf{P}\mathsf{e}^2}{\mathsf{P}\mathsf{e} - 1 + \exp\{-\mathsf{P}\mathsf{e}\}} + \frac{1}{2}\Lambda\right)^{n/(n+1)} \left[\frac{K\Delta T}{A^{-1/n}H^2}\right]^{n/(n+1)}, \qquad (7)$$

and completes the derivation of the model used in the main text.

2.2. The case of linear vertical velocity

In the previous section and in the main text, we use a constant vertical velocity. A more common assumption is to a linear velocity of the form w = -az/H, so that conservation of energy is given by

$$-\rho c_p a \frac{z}{H} \frac{dT}{dz} = K \frac{d^2 T}{dz^2} + \sigma_{ij} \dot{\epsilon}_{ij}, \qquad (8)$$



Figure S1. Péclet and Brinkman numbers computed from the inputs in Figure 3 in the main text and the shear strain rates.



Figure S2. Thickness of the temperate zone as a function of the Brinkman number Br for several values of the Péclet number Pe from the analytical solution from equation (6). Here we neglect lateral advection, i.e. $\Lambda = 0$.

where we ignore lateral advection (Perol & Rice, 2011, 2015). The choice of a vertical velocity that decreases linearly with depth contains the same physics as Equation (1) and therefore only superficially alters the numerical values of the results given in the main text. The advantage of this model is that it gives a more accurate representation of the temperature structure with depth, although both the constant and linear models still ignore the downstream and across-stream advection terms, which are potentially important terms (Fowler, 2013; Suckale et al., 2014). The disadvantage of using (8) is that the thickness of the temperate zone among other quantities cannot be computed in closed form and requires a numerical solution, as we will show.

Following Perol & Rice (2011, 2015) we can integrate (8) to give

$$T = C_2 + C_1 \operatorname{erf}\left(\sqrt{\frac{\operatorname{Pe}}{2}}\frac{z}{H}\right) - \Delta T \frac{\operatorname{Br}}{\operatorname{Pr}} \mathcal{I}\left(\sqrt{\frac{\operatorname{Pe}}{2}}\frac{z}{H}\right).$$
(9)

where C_1 and C_2 are constants of integration and the integral $\mathcal{I}(\chi)$ is

$$\mathcal{I}(\chi) = \int_0^1 \frac{1 - e^{-\chi^2 \lambda}}{2\lambda \sqrt{1 - \lambda}} d\lambda.$$
(10)

We now subject equation (9) to the boundary conditions (2) and find that

$$C_{2} = T_{s} - C_{1} \operatorname{erf}\left(\sqrt{\frac{\mathsf{Pe}}{2}}\right) + \Delta T \frac{\mathsf{Br}}{\mathsf{Pr}} \mathcal{I}\left(\sqrt{\frac{\mathsf{Pe}}{2}}\right) \quad \text{and} \quad C_{1} = \frac{\pi^{3/2} \Delta T}{2} \frac{\mathsf{Br}}{\mathsf{Pe}} \operatorname{erfi}\left(\sqrt{\frac{\mathsf{Pe}}{2}} \frac{\xi}{H}\right)$$
(11)

where erfi $\{\cdot\}$ is the imaginary error function. The thickness of the temperate zone ξ is then a solution to

$$\frac{\operatorname{Pe}}{\operatorname{Br}} + \frac{\pi^{3/2}}{2} \operatorname{erfi}\left(\sqrt{\frac{\operatorname{Pe}}{2}} \frac{\xi}{H}\right) \left[\operatorname{erf}\left(\sqrt{\frac{\operatorname{Pe}}{2}}\right) - \operatorname{erf}\left(\sqrt{\frac{\operatorname{Pe}}{2}} \frac{\xi}{H}\right)\right] = \mathcal{I}\left(\sqrt{\frac{\operatorname{Pe}}{2}}\right) - \mathcal{I}\left(\sqrt{\frac{\operatorname{Pe}}{2}} \frac{\xi}{H}\right),$$
(12)

which must be solved numerically using a root-finding algorithm. Thus, solving for the thickness of the temperate zone with a linear vertical advection is much more computationally expensive than the analytical expression for the constant vertical velocity in Equation (6) in the supplement (i.e. Equation (17) in the main text). Moreover, solving for the temperate zone height by using (12) is not much more efficient than directly solving (8) using an enthalpy finite-volume method (Schoof & Hewitt, 2016).

2.2.1. Enthalpy finite volume numerical method

To determine the structure of the temperature field with depth and the height of the temperate zone when the vertical velocity varies linearly with depth, we numerically solve Equation (8) and we describe the method in this section. First, we write Equation (8) in terms of the enthalpy $\mathscr{H} = \rho c_p (T - T_m)$ following (Aschwanden et al., 2012; Schoof & Hewitt, 2016; Meyer & Hewitt, 2017), which gives

$$-a\frac{z}{H}\frac{d\mathscr{H}}{dz} = K\frac{d^2T}{dz^2} + \sigma_{ij}\dot{\epsilon}_{ij} \quad \text{and} \quad T = T_m + \frac{1}{\rho c_p}\min\left\{\mathscr{H}, 0\right\}.$$
(13)

This method ensures that the temperature within the ice never exceeds the melting temperature. Additionally, we use the conservative finite volume method, which automatically enforces the internal conditions on $z = \xi$ from Equation (2). We therefore apply the boundary conditions

$$T = T_m$$
 on $z = 0$ and $T = T_s$ on $z = H$. (14)

In Figure S3(a), we show a comparison between a finite-volume numerical solution to Equation (8) and the analytical solution to the constant velocity case (6) as a function of Br for two values of Pe. Figure S3(b) shows the change in temperate zone thickness with Br for two values of Pe, comparing the constant and linear velocity forms. The key difference between the constant and linear velocity forms is that the velocity decreases with depth in the linear case. This means that the rate at which the cold ice advects down near the top of the temperate zone is lower than the surface accumulation rate. Thus, the temperate zone is thicker in the linear advection case and the constant velocity case is a

lower bound on the temperate zone thickness. We also use the finite volume numerical solution to Equation (13) to make analogous plots to the constant vertical velocity results presented in the main text for the linear vertical velocity case. These results are discussed in more detail in section 3.1 and are shown in Figure S6.

2.2.2. Onset of temperate ice with linear vertical velocity

For large values of the Brinkman number and a small Péclet number, the form of the velocity, i.e constant or linear, does not matter much to the solution as the dominant balance in Equation (1) or (8) is between diffusion and deformational heat. If the Péclet number is large, then the dominant balance is primarily between advection and deformation heating, which is a singular perturbation in that it neglects diffusion, the term with the highest derivative. In this limit, we have constructed matched asymptotic solution, which we don't include here as the constant velocity gives a uniform description that we can apply to the entire ice sheet.

Interestingly, however, the critical Br^* required to initiate temperate ice can be determined analytically from (12) and is given by

$$Br^* = \frac{Pe}{\mathcal{I}\left(\sqrt{\frac{Pe}{2}}\right)},\tag{15}$$

where $\mathcal{I}\left(\sqrt{Pe/2}\right)$ is easy to evaluate using Gauss-Jacobi quadrature. Additionally, if we expand $\mathcal{I}\left(\sqrt{Pe/2}\right)$ asymptotically for small Pe we find that

$$\mathcal{I}\left(\sqrt{\frac{\mathsf{Pe}}{2}}\right) \sim \frac{\mathsf{Pe}}{2} \tag{16}$$

which gives $Br^* \approx 2$, the correct minimum Brinkman number to initiate temperate ice for a small Péclet number. In Figure S4, we compare the critical Br as a function of Pe for both constant and linear cases. In Figure S6, we show the results from a finite volume numerical solution for the Antarctic Ice Sheet.

3. Main text maps enlarged

In the main text, we apply the constant velocity model to all of Antarctica and show regions of interest in the insets. In Figure S5, we show the insets at full size to better highlight the fine-scale detail.

3.1. Results with linear vertical velocity

To examine the continental-scales patterns in the case where the vertical velocity varies linearly with depth, we use our enthalpy finite volume numerical solution to Equation (13) in order to determine the temperate zone thickness



Figure S3. Temperate zone thickness as a function of the Brinkman number: comparison between the constant vertical velocity analytical solution and the linear vertical velocity finite-volume numerical solution.



Figure S4. Minimum Brinkman number required to initiate a temperate zone as a function of the Péclet number for the constant and vertical velocity cases.

throughout the Antarctic Ice Sheet. To reduce the computational cost, we identified regions that may have temperate ice by using the analytical expression for the constant vertical velocity critical strain rate $\dot{\epsilon}_{lat}^*$, so that we only consider regions where $\dot{\epsilon}_{lat} > 0.1\dot{\epsilon}_{lat}^*$. We then compute the critical shear strain rate for the linear vertical velocity case using Equation (15) and the temperate zone thickness by solving Equation (13). The results for the ratio of the observed strain rates to the critical strain rate and the temperate zone thicknesses are shown in Figure S6. The spatial patterns visible in Figures S5 and S6 are similar and the main difference is that the temperate zones are more extensive in the linear vertical velocity case. No new temperate zones emerge in the linear vertical velocity case. Thus, the results for the linear vertical advection are interesting and important as they are quantitatively more realistic, yet, at the same time, they show the same patterns as the constant velocity results and are more computationally expensive to run.

3.2. Effect of cold-air ponding

To estimate the influence of cold-air ponding (see Discussion section of the main text), we subtracted 15 °C from the mean annual surface temperatures (Fig. 3b of the main text) and reran the advection-diffusion model with the assumption of constant vertical velocity. The results, shown in Figs. S7 and S8 are qualitatively similar to the results shown in the main text, indicating a limited influence of cold-air ponding on the broad locations and approximate thicknesses of the temperate ice zones.



Figure S5. Enlarged region-of-interest maps from the main text for constant vertical velocity. (a) Ratio of lateral shear strain rate to critical shear strain rate and (b) normalized temperate zone thickness.



Figure S6. Results from the case of lingar vertical velocity. (a) Ratio of lateral shear strain rate to critical shear strain rate and (b) normalized temperate zone thickness in the regions of interest across the Antarctica Ice Sheet.



Figure S7. The estimated influence of cold-air ponding on the development of temperate ice zones. The main map shows the difference between Br (divided by Pe) values calculated using the mean annual surface temperature shown in Fig. 3b in the main text and surface air temperatures that are everywhere 15 °C colder. Note that colder surface temperature always decrease Br, so the difference is always positive. Inset images show the relative inferred thicknesses of the temperate ice zones within the regions of interest.



Figure S8. The estimated influence of cold-air ponding on the development of temperate ice zones. (a) Ratio of lateral shear strain rate to critical shear strain rate and (b) normalized temperate zone thickness in the regions of interest across the Antarctica Ice Sheet. These parameters are calculated as in Figure S6 and in the main text but using mean annual air temperatures that have been reduced everywhere by 15 °C.

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