Canonical Decompositions of Games and Near Potential Games

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Background and Motivation

- Potential games are (noncooperative) games that are easier to analyze, have pure Nash equilibria, and natural dynamics convergences to equilibria.
- Can the properties of potential games be used to analyze games that are "close" to a potential game ?
- We present here a fundamental result: Any game has a canonical decomposition that includes three components: The potential, harmonic, nonstrategic components.
- This decomposition allows us to develop a new framework for studying dynamics and equilibria in games, by considering their potential components.

Flows and the Difference Operator

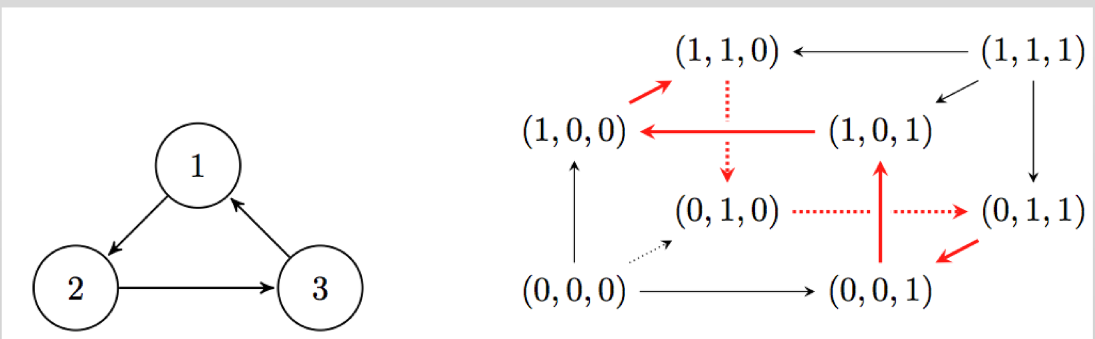
Define the difference operator D_m as:

$$(D_m\phi)(\mathbf{p},\mathbf{q}) = W^m(\mathbf{p},\mathbf{q})\,(\phi(\mathbf{q}) - \phi(\mathbf{p}))\,.$$

where $\mathbf{p},\mathbf{q} \in E$, $W^m(\mathbf{p},\mathbf{q}) = 1$ if $\mathbf{p},\mathbf{q}$ differ in the strategy of player m and 0 otherwise.

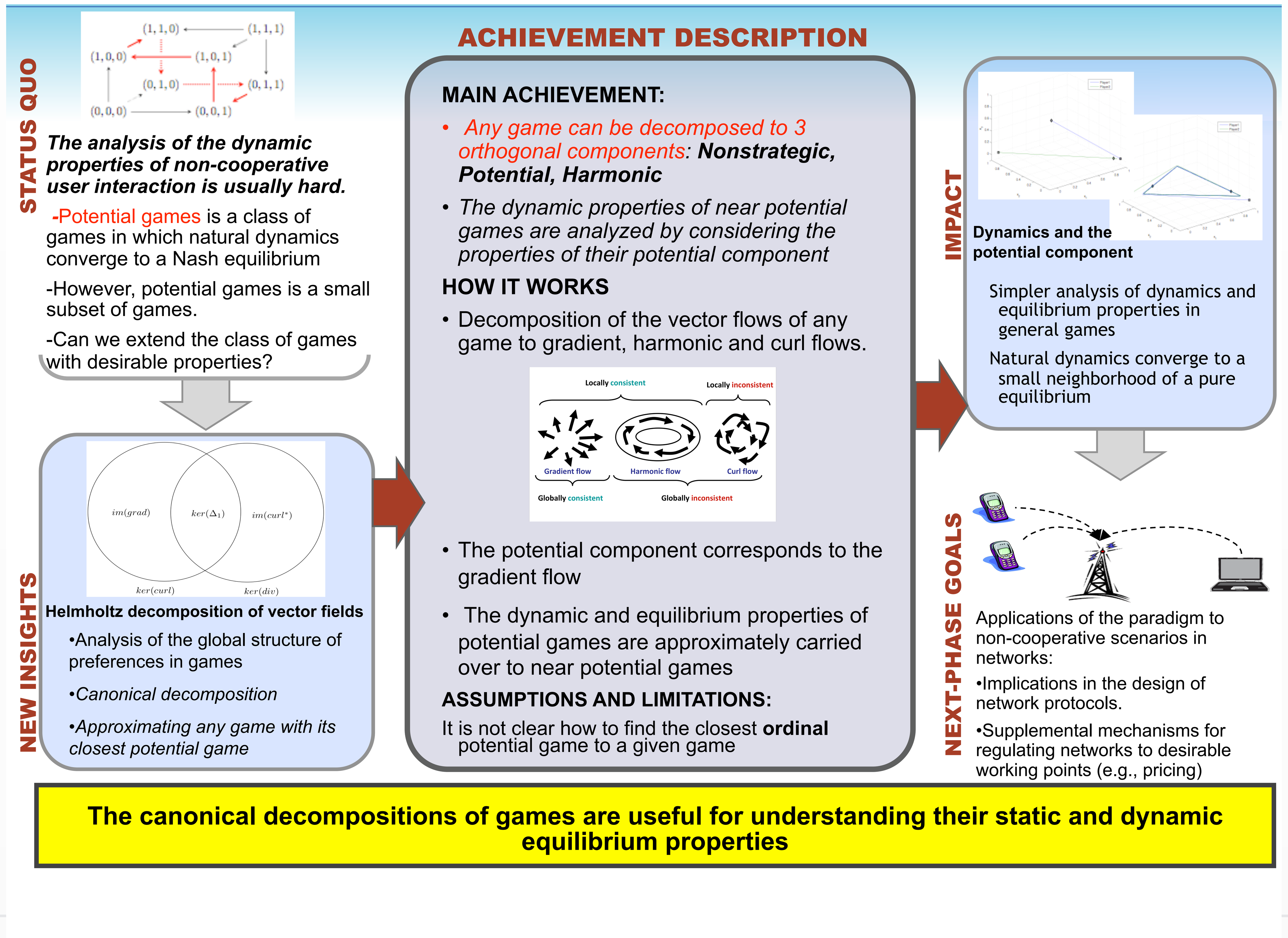
- A game is a potential game iff there exists ϕ such that $D_m u^m = D_m \phi$ for all $m \in \mathcal{M}$
- Note that $\Pi_m = D_m^* D_m$ is the projection operator to the orthogonal complement of the kernel of D_m.

The pairwise comparisons of payoffs are similar to flows: the tools for decompositions of flows can be used for decompositions of games.



(a) Player Interaction Graph (b) Preference Relations

Summary



The Canonical Decomposition

Theorem 1. *Given a game with utilities $\{u^m\}$, its orthogonal components (Nonstrategic(NS), Potential(P), Harmonic(H)) and the corresponding potential function (ϕ) are given by:*

$$u^m_{NS} = (I - \Pi_m)u^m, \quad \phi = \left(\textstyle\sum_k \Pi_k\right)^{\dagger} \textstyle\sum_k \Pi_k u^k, \quad u^m_P = \Pi_m \phi, \quad u^m_H = \Pi_m(u^m - \phi).$$

- The NS component vanishes under difference operation.
- The H component is always a "zero-sum game" (i.e., $\sum_k u^k_H(\mathbf{p}) = 0$).
- The closest potential game (equivalently, the projection to the space of potential games) has utilities $\{u^m_P + u^m_{NS}\}_{m\in\mathcal{M}}$.
- We refer to games with $u^m_P = 0$ for all m, as harmonic games. Harmonic games generically have no pure equilibria.

	R	P	S
R	0, 0	-3x, 3x	3y, -3y
P	3x, -3x	0, 0	-3z, 3z
S	-3y, 3y	3z, -3z	0, 0

(a) Generalized RPS Game

	R	P	S
R	-2x+2y	y-z	-x+z
P	y-z	2x-2z	x-y
S	-x+z	x-y	-2y+2z

(b) Potential Function

	R	P	S
R	0, 0	-(2x-y-z), (2x-y-z)	-(x-2y+z), (x-2y+z)
P	(2x-y-z), -(2x-y-z)	0, 0	(x+y-2z), -(x+y-2z)
S	(x-2y+z), -(x-2y+z)	-(x+y-2z), (x+y-2z)	0, 0

(c) The Closest Potential Game

	R	P	S
R	0, 0	-(x+y+z), (x+y+z)	(x+y+z), -(x+y+z)
P	(x+y+z), -(x+y+z)	0, 0	-(x+y+z), (x+y+z)
S	-(x+y+z), (x+y+z)	(x+y+z), -(x+y+z)	0, 0

(d) Harmonic Game Component

Properties of Games by Projection

Let $\hat{\mathcal{G}}$ be the closest potential game to a given game $\mathcal{G}$, and let $d(\mathcal{G})$ be the distance between $\mathcal{G}$ and $\hat{\mathcal{G}}$. The equilibria of the two games are related:

Theorem 2. *Any equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$, and any equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$, where $\epsilon \leq \sqrt{2}\cdot d(\mathcal{G})$.*

Consider the following (smoothened) best-response dynamics:

$$\dot{x}^m = \beta^m_{u^m}(x^{-m}) - x^m, \quad \text{where}$$

$$\beta^m_{u^m}(x^{-m}) = \arg\max_{y\in\Delta E^m} \left\{u^m(y, x^{-m}) + H^m(y)\right\},$$

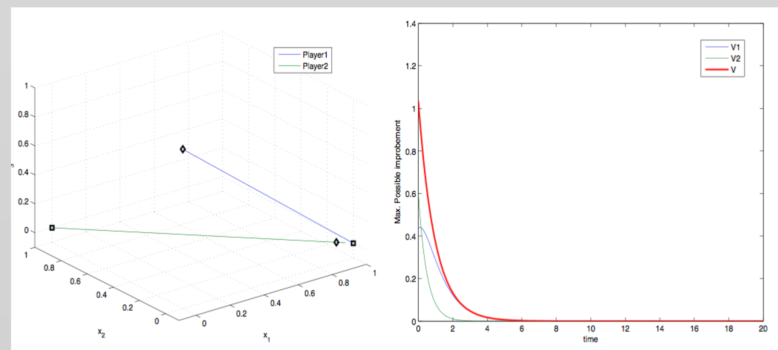
$$H^m(x^m) = -\tau \sum_{q^m\in E^m} x_{q^m}\log(x_{q^m}).$$

This dynamics is known to converge (approximately) to a Nash equilibrium for potential games. For near potential games,

Theorem 3. *The above dynamics converges to the set of ϵ-equilibria of $\mathcal{G}$, where ϵ is smaller than*

$$d(\mathcal{G})\left(\sqrt{2} + \sqrt{h}\frac{2\bar{\phi}_c + d(\mathcal{G}) + \tau\log 2h}{4\tau}\right) + \tau \log h,$$

where $\bar{\phi}_c = \max_{m,\mathbf{p}^m,\mathbf{q}^m,\mathbf{p}^{-m}} |\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m})|$ *and* $h = |E|$.


(a) Trajectories under BR dynamics (b) Change in the maximum increase in utility

Future Work

- Properties of near-harmonic games.
- Applications – Better understanding of noncooperative behavior in wired and wireless networks.