

Canonical Decompositions of Games and Near Potential Games

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Background and Motivation

- **Potential games** are (noncooperative) games that are easier to analyze, have pure Nash equilibria, and natural dynamics convergences to equilibria.
- Can the properties of potential games be used to analyze games that are “close” to a potential game ?
- We present here a fundamental result: Any game has a canonical decomposition that includes three components: The **potential, harmonic, nonstrategic** components.
- This decomposition allows us to develop a new framework for studying **dynamics and equilibria** in games, by considering their potential components.

Flows and the Difference Operator

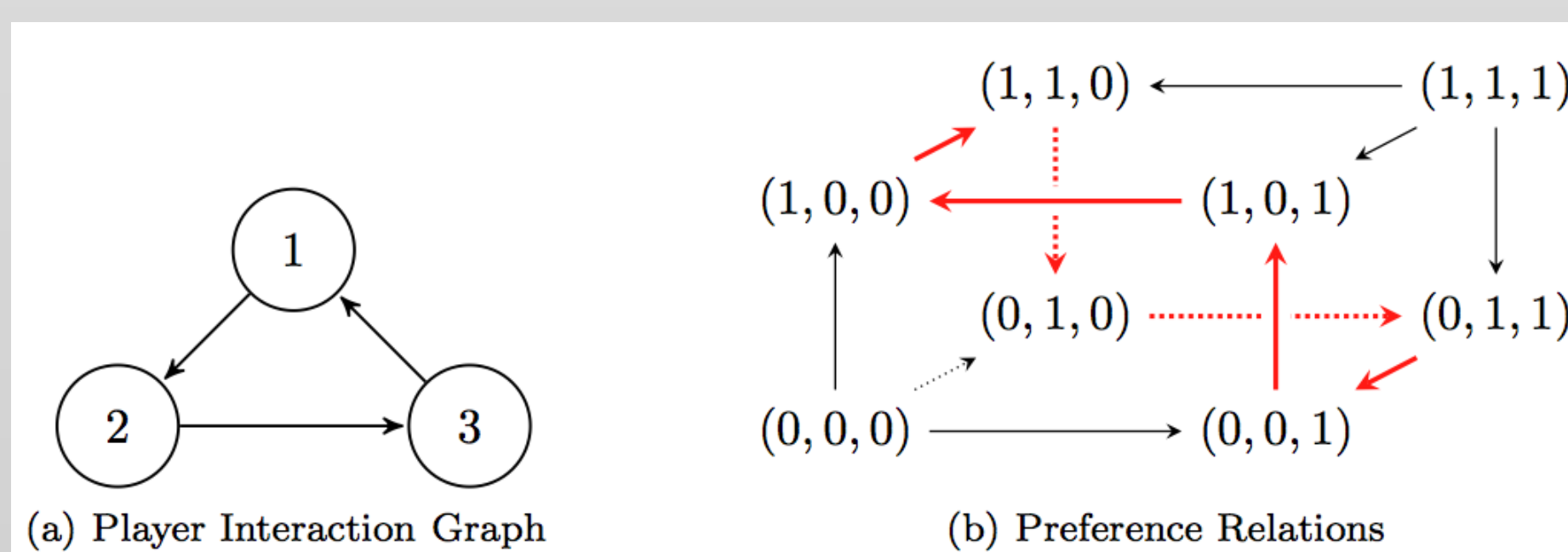
Define the **difference operator** D_m as:

$$(D_m \phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) (\phi(\mathbf{q}) - \phi(\mathbf{p})).$$

where $\mathbf{p}, \mathbf{q} \in E$, $W^m(\mathbf{p}, \mathbf{q}) = 1$ if \mathbf{p}, \mathbf{q} differ in the strategy of player m and 0 otherwise.

- A game is a **potential game** iff there exists ϕ such that $D_m u^m = D_m \phi$ for all $m \in \mathcal{M}$
- Note that $\Pi_m = D_m^* D_m$ is the projection operator to the orthogonal complement of the kernel of D_m .

The **pairwise comparisons of payoffs are similar to flows**: the tools for decompositions of flows can be used for decompositions of games.



Summary

STATUS QUO

The analysis of the dynamic properties of non-cooperative user interaction is usually hard.

-Potential games is a class of games in which natural dynamics converge to a Nash equilibrium

-However, potential games is a small subset of games.

-Can we extend the class of games with desirable properties?

ACHIEVEMENT DESCRIPTION

MAIN ACHIEVEMENT:

- Any game can be decomposed to 3 orthogonal components: **Nonstrategic, Potential, Harmonic**
- The dynamic properties of near potential games are analyzed by considering the properties of their potential component

HOW IT WORKS

- Decomposition of the vector flows of any game to gradient, harmonic and curl flows.

- The potential component corresponds to the gradient flow
- The dynamic and equilibrium properties of potential games are approximately carried over to near potential games

ASSUMPTIONS AND LIMITATIONS:

It is not clear how to find the closest ordinal potential game to a given game

IMPACT

Dynamics and the potential component

Simpler analysis of dynamics and equilibrium properties in general games

Natural dynamics converge to a small neighborhood of a pure equilibrium

NEW INSIGHTS

Helmholtz decomposition of vector fields

- Analysis of the global structure of preferences in games
- Canonical decomposition
- Approximating any game with its closest potential game

The canonical decompositions of games are useful for understanding their static and dynamic equilibrium properties

The Canonical Decomposition

Theorem 1. Given a game with utilities $\{u^m\}$, its orthogonal components (Nonstrategic(NS), Potential(P), Harmonic(H)) and the corresponding potential function (ϕ) are given by:

$$u_{NS}^m = (I - \Pi_m)u^m, \quad \phi = (\sum_k \Pi_k)^\dagger \sum_k \Pi_k u^k, \quad u_P^m = \Pi_m \phi, \quad u_H^m = \Pi_m(u^m - \phi).$$

- The NS component vanishes under difference operation.
- The H component is always a “zero-sum game” (i.e., $\sum_k u_H^k(\mathbf{p}) = 0$).
- The **closest** potential game (equivalently, the **projection** to the space of potential games) has utilities $\{u_P^m + u_{NS}^m\}_{m \in \mathcal{M}}$.
- We refer to games with $u_P^m = 0$ for all m , as **harmonic games**. Harmonic games generically have **no pure equilibria**.

	R	P	S		R	P	S
R	0, 0	-3x, 3x	3y, -3y	R	-2x+2y	y-z	-x+z
P	3x, -3x	0, 0	-3z, 3z	P	y-z	2x-2z	x-y
S	-3y, 3y	3z, -3z	0, 0	S	-x+z	x-y	-2y+2z

(a) Generalized RPS Game (b) Potential Function

	R	P	S		R	P	S
R	0, 0	-(x-y+z), (x+y+z)	(x+y+z), -(x+y+z)	R	0, 0	-(x+y+z), (x+y+z)	(x+y+z), -(x+y+z)
P	(2x-y-z), -(2x-y-z)	0, 0	(x+y-2z), -(x+y-2z)	P	(x+y+z), -(x+y+z)	0, 0	-(x+y+z), (x+y+z)
S	(x-2y+z), -(x-2y+z)	-(x+y-2z), (x+y-2z)	0, 0	S	-(x+y+z), (x+y+z)	(x+y+z), -(x+y+z)	0, 0

(c) The Closest Potential Game (d) Harmonic Game Component

Properties of Games by Projection

Let $\hat{\mathcal{G}}$ be the closest potential game to a given game \mathcal{G} , and let $d(\mathcal{G})$ be the distance between \mathcal{G} and $\hat{\mathcal{G}}$. The equilibria of the two games are related:

Theorem 2. Any equilibrium of $\hat{\mathcal{G}}$ is an ϵ -equilibrium of \mathcal{G} , and any equilibrium of \mathcal{G} is an ϵ -equilibrium of $\hat{\mathcal{G}}$, where $\epsilon \leq \sqrt{2} \cdot d(\mathcal{G})$.

Consider the following (smoothened) **best-response** dynamics:

$$\dot{x}^m = \beta_{u^m}^m(x^{-m}) - x^m, \quad \text{where}$$

$$\beta_{u^m}^m(x^{-m}) = \arg \max_{y \in \Delta E^m} \{u^m(y, x^{-m}) + H^m(y)\},$$

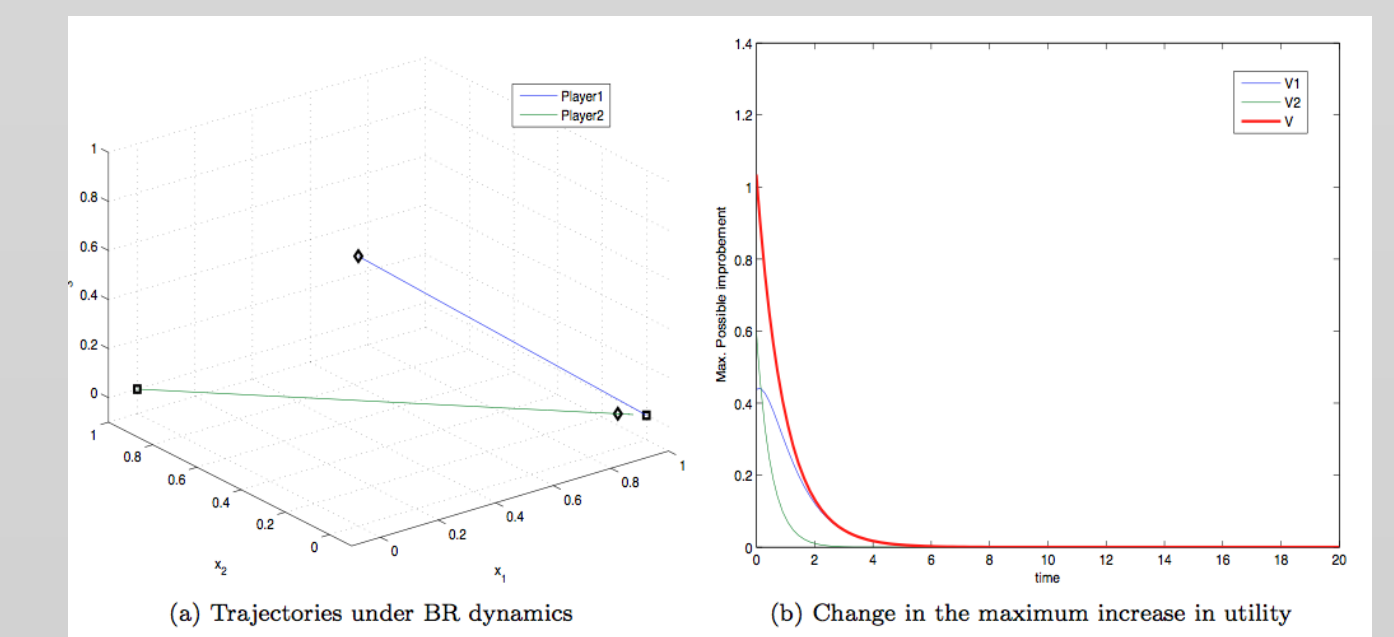
$$H^m(x^m) = -\tau \sum_{q^m \in E^m} x_{q^m} \log(x_{q^m}).$$

This dynamics is known to converge (approximately) to a Nash equilibrium for potential games. For **near** potential games,

Theorem 3. The above dynamics converges to the set of ϵ -equilibria of \mathcal{G} , where ϵ is **smaller** than

$$d(\mathcal{G}) \left(\sqrt{2} + \sqrt{h} \frac{2\bar{\phi}_c + d(\mathcal{G}) + \tau \log 2h}{4\tau} \right) + \tau \log h,$$

where $\bar{\phi}_c = \max_{m, \mathbf{p}^m, \mathbf{q}^m, \mathbf{p}^{-m}} |\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m})|$ and $h = |E|$.



Future Work

- Properties of **near-harmonic** games.
- **Applications** – Better understanding of noncooperative behavior in wired and wireless networks.