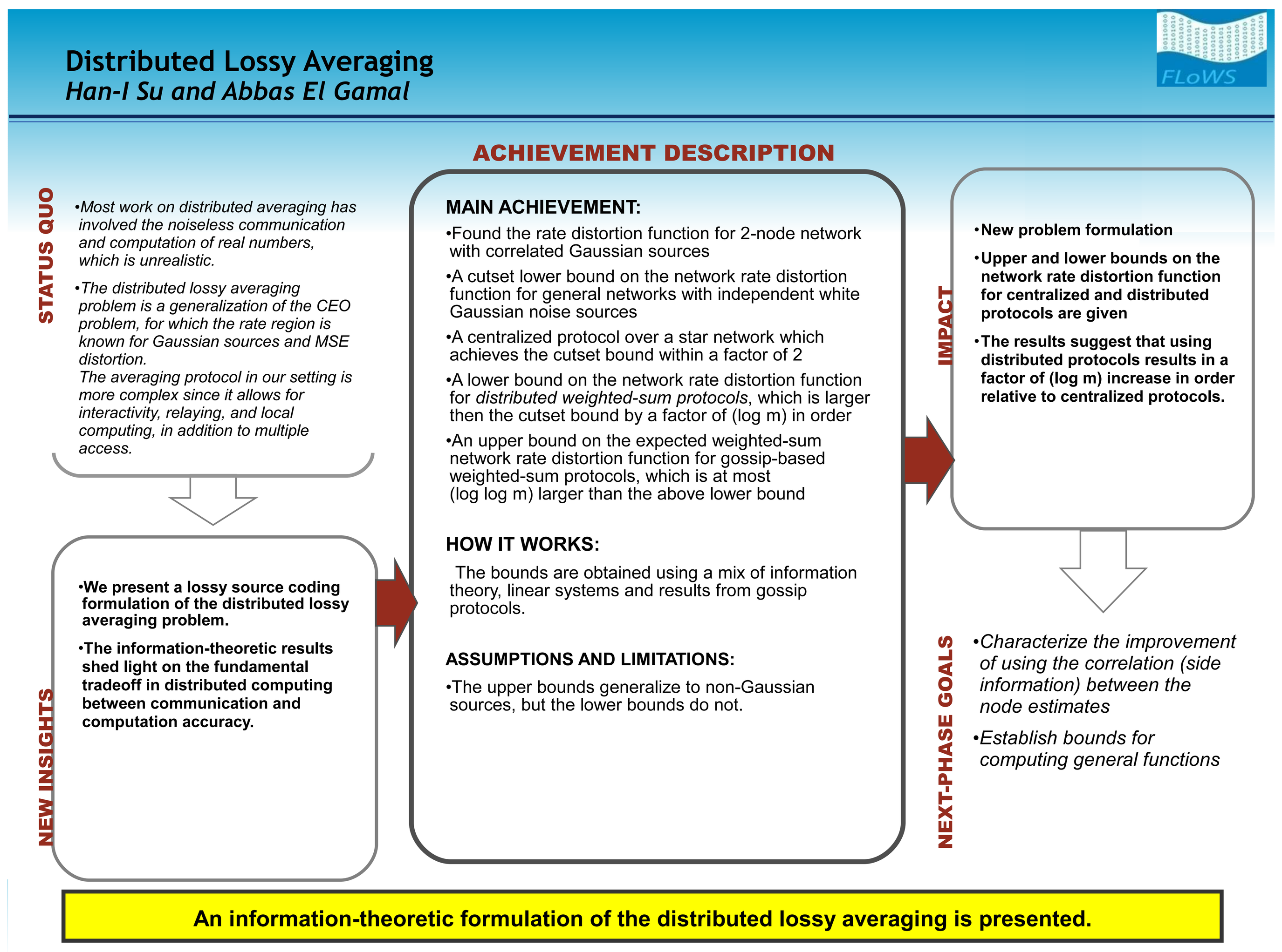
Distributed Lossy Averaging

Han-I Su and Abbas El Gamal

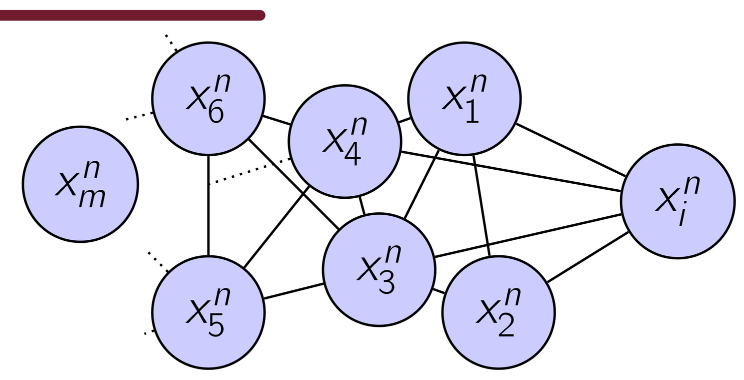
Information Systems Laboratory, Department of Electrical Engineering, Stanford University

Summary



Lossy Averaging Problem

- Complete graph with m nodes. Node i observes i.i.d. source X_i
- Each node wishes to estimate the average $S^n := (1/m)\sum_{i=1}^m X_i^n$ to a prescribed MSE distortion



- Averaging protocol:
  - T rounds of node-pair, two-way communication/ computing
  - $(R_1, \dots, R_m, n)$ block code: In round t, node i transmits at total rate $r_i(t) \geq 0$ if it is selected; otherwise, $r_i(t) = 0$. $R_i = \sum_{t=1}^T r_i(t)$
  - Per-node average rate $R = (1/m)\sum_{i=1}^m R_i$
  - Distributed protocols: *Not* depend on node identities
  - Gossip protocols: Random node subset selections [Hedetniemi et al., 88]
- For fixed T and fixed sequence of node-pair selections:
  - (R, D) is *achievable* if there is a sequence of $(R_1, \dots, R_m, n)$ codes such that $R = (1/m)\sum_{i=1}^m R_i$ and

$$\limsup_{n\to\infty} \frac{1}{mn}\sum_{i=1}^m\sum_{k=1}^n \mathrm{E}\left[(S_{ik} - S_i)^2\right] \leq D$$

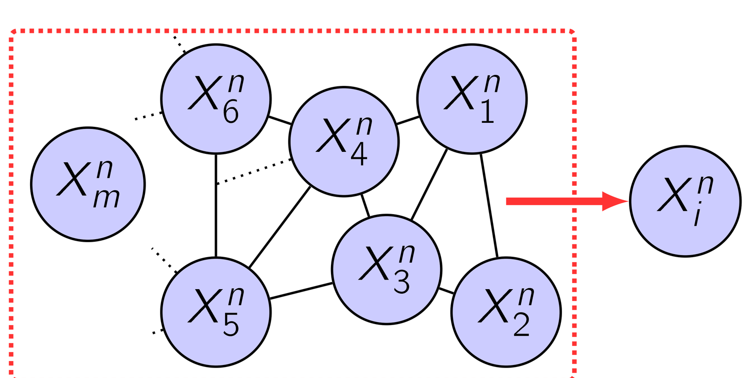
  - $R(D) = \inf\{R : (R, D) \text{ is achievable}\}$
- Network rate distortion function

$$R^*(D) = \inf\{R(D) : \text{all node-pair selection sequences and } T\}$$

Cutset Lower Bound on $R^*(D)$

- Independent WGN sources with average power 1

> Cutset Lower Bound:
>
> $$R^*(D) \geq \frac{1}{2}\log\left(\frac{m-1}{m^2 D}\right)$$
>
> for $D < (m-1)/m^2$



Super-node: $P = \frac{m-1}{m^2}$

- Tight for 2-node network with correlated Gaussian sources

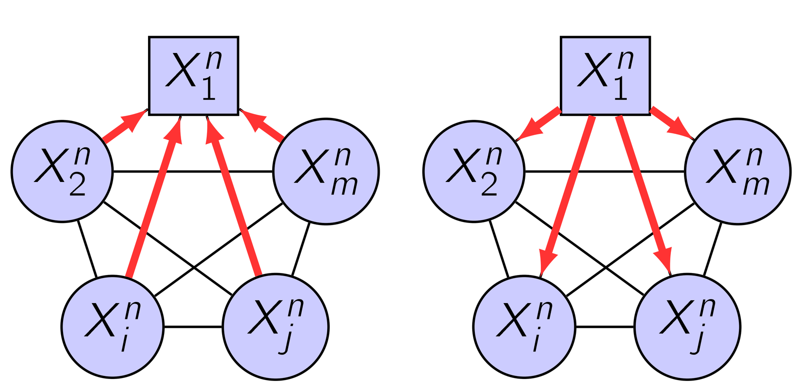
Upper Bound on $R^*(D)$

- Use centralized protocol; node 1 acts as cluster-head
- Round $t = 1, \dots, (m-1)$:

$$r_i(i-1) = \frac{1}{2}\log\left(\frac{1}{d}\right), d = \frac{mD}{2}$$

- Round $t = m, \dots, (2m-2)$:

$$r_1(t) = \frac{1}{2}\log\left(\frac{2}{mD}\right)$$



> Upper bound:
>
> $$R^*(D) \leq \frac{m-1}{m}\log\left(\frac{2}{mD}\right)$$

Distributed Weighted-Sum Protocol

- T rounds of node-pair, two-way communication/ computing
- Estimate of node i at $t = 0$: $S_i^n(0) = X_i^n$
- Assume (i, j) selected in round $t > 0$:
  - Node i: Sends description $\hat{S}_i^n(t)$
    - ∗ *Distortion* $d\,\mathrm{E}(S_{ik}^2(t))$ and *rate* $(1/2)\log(1/d)$
  - Similarly, node j sends description $\hat{S}_j^n(t)$
  - Nodes update their estimates

$$S_v^n(t+1) = \frac{1}{2}S_v^n(t) + \frac{1}{2(1-d)}\hat{S}_{i+j-v}^n(t) \text{ for } v = i, j$$

- Define $R_{\mathrm{WS}}^*(D) \geq R^*(D)$

Lower Bound on $R_{\mathrm{WS}}^*(D)$

> $$R_{\mathrm{WS}}^*(D) \geq (\text{minimum T}) \times (\text{minimum average rate per round})$$
> $$\geq \left(\frac{m}{2}\log\frac{1}{\sqrt{D} + 1/m}\right)\left(\frac{1}{m}\log\frac{1}{4mD}\right)$$

Gossip-Based Weighted-Sum Protocol

- Class of distributed weighted-sum protocols
- Node-pair selected independently at random in each round
- Expected weighted-sum network rate distortion function

$$\mathrm{E}(R_{\mathrm{WS}}(D)) = \inf\{\mathrm{E}(R) : (R, \Delta) \text{ is achievable}, \mathrm{E}(\Delta) < D\}$$

- $R_{\mathrm{WS}}^*(D) \leq \mathrm{E}(R_{\mathrm{WS}}(D))$

> Upper bound on $\mathrm{E}(R_{\mathrm{WS}}(D))$:
>
> $$\mathrm{E}(R_{\mathrm{WS}}(D)) \leq \frac{m-1}{m}\left(\ln\frac{2}{D}\right)\left(\log\frac{(m-1)\ln(2/D)}{m^2 D}\right)$$

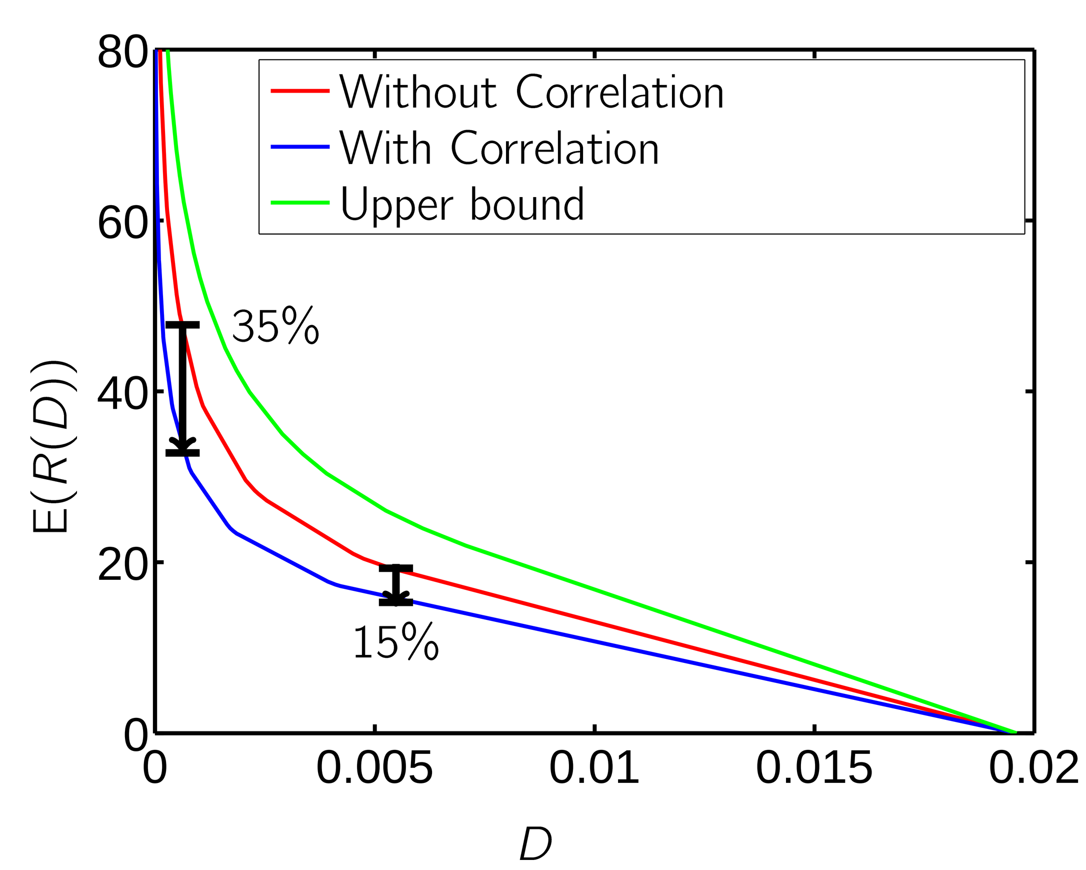
Summary of Bounds

	Bounds	$D = \Theta(1/m)$	$D = \Theta(1/m^2)$
Cutset	$R^*(D)$	$\Omega(1)$	$\Omega(\log m)$
Centralized	$R^*(D)$	$O(1)$	$O(\log m)$
Distributed	$R_{\mathrm{WS}}^*(D)$	$\Omega(\log m)$	$\Omega((\log m)^2)$
	$\mathrm{E}(R_{\mathrm{WS}}(D))$	$O((\log m)(\log\log m))$	$O((\log m)^2)$

Price of using a distributed protocol is roughly $\log m$

Effect of Using Correlation

- We ignored the build up in correlation
- Can achieve better rate using Wyner-Ziv coding
- Very difficult to analyze; using simulations ($m = 50$):



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