Cases Where Finding the Minimum Entropy Coloring of a Characteristic Graph is a Polynomial Time Problem

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***Abstract*—In this paper, we consider the problem of finding the minimum entropy coloring of a characteristic graph under some conditions which allow it to be in polynomial time. This problem arises in the functional compression problem where the computation of a function of sources is desired at the receiver. The rate region of the functional compression problem has been considered in some references under some assumptions. Recently, Feizi et al. computed this rate region for a general one-stage tree network and its extension to a general tree network. In their proposed coding scheme, one needs to compute the minimum entropy coloring (a coloring random variable which minimizes the entropy) of a characteristic graph. In general, finding this coloring is an NP-hard problem (as shown by Cardinal et al.) . However, in this paper, we show that depending on the characteristic graph's structure, there are some interesting cases where finding the minimum entropy coloring is not NP-hard, but tractable and practical. In one of these cases, we show that, having a non-zero joint probability condition on RVs' distributions, for any desired function f, makes characteristic graphs to be formed of some non-overlapping fully-connected maximal independent sets. Therefore, the minimum entropy coloring can be solved in polynomial time. In another case, we show that if f is a quantization function, this problem is also tractable.**

***Index Terms*—Functional compression, graph coloring, graph entropy.**

I. Introduction

While data compression considers the compression of sources at transmitters and their reconstruction at receivers, functional compression does not consider the recovery of whole sources, but the computation of a function of sources at the receiver(s).

This problem has been considered in different references (e.g., [1], [2], [3], [4]). In a recent work, we showed that, for any one-stage tree network, if source nodes send ϵ-colorings of their characteristic graphs satisfying a condition called the *Coloring Connectivity Condition* (C.C.C.), followed by a Slepian-Wolf encoder, it will lead to an achievable coding scheme. Conversely, any achievable scheme induces colorings on high probability subgraphs of characteristic graphs satisfying C.C.C. An extension to a tree network is also considered in [5].

In the proposed coding scheme in [5], we need to compute the minimum entropy coloring (a coloring random variable which minimizes the entropy) of a characteristic graph. In general, finding this coloring is an NP-hard problem ([6]). However, in this paper, we show that depending on the characteristic graph's structure, there are certain cases where finding the minimum entropy coloring is not NP-hard, but tractable and practical. In one of these cases, we show that, having a non-zero joint probability condition on RVs' distributions, for any desired function f, makes characteristic graphs to be formed of some non-overlapping fully-connected maximal independent sets. Then, the minimum entropy coloring can be solved in polynomial time. In another case, we show that if f is a quantization function, this problem is also tractable.

The rest of the paper is organized as follows. Section II presents the minimum entropy coloring problem statement and the necessary technical background. In Section III, main results of this paper are expressed. An application of this paper's results in the functional compression problem is explained in Section IV. We conclude the paper in Section V.

II. Problem Setup

In this section, after expressing necessary definitions, we formulate the minimum entropy coloring problem.

We start with the definition of a characteristic graph of a random variable. Consider the network shown in Figure 1 with two sources with RVs X_1 and X_2 such that the computation of a function ($f(X_1,X_2)$) is desired at the receiver. Suppose these sources are drawn from finite sets $\mathcal{X}_1=\{x_1^1,x_1^2,...,x_1^{|\mathcal{X}_1|}\}$ and $\mathcal{X}_2=\{x_2^1,x_2^2,...,x_2^{|\mathcal{X}_2|}\}$. These sources have a joint probability distribution $p(x_1,x_k)$. We express n-sequences of these RVs as $\mathbf{X_1}=\{X_1^i\}_{i=l}^{i=l+n-1}$ and $\mathbf{X_2}=\{X_2^i\}_{i=l}^{i=l+n-1}$ with the joint probability distribution $p(\mathbf{x}_1,\mathbf{x}_2)$. Without loss of generality, we assume $l=1$, and to simplify notation, n will be implied by the context if no confusion arises. We refer to the i^{th} element of $\mathbf{x}_j$ as x_{ji}. We use $\mathbf{x}_j^1$, $\mathbf{x}_j^2$,... as different n-sequences of $\mathbf{X}_j$. We shall drop the superscript when no confusion arises. Since the sequence $(\mathbf{x}_1,\mathbf{x}_2)$ is drawn i.i.d. according to $p(x_1,x_2)$, one can write $p(\mathbf{x}_1,\mathbf{x}_2)=\prod_{i=1}^n p(x_{1i},x_{2i})$.

Definition 1. *The characteristic graph $G_{X_1}=(V_{X_1},E_{X_1})$ of X_1 with respect to X_2, $p(x_1,x_2)$, and function $f(X_1,X_2)$ is defined as follows: $V_{X_1}=\mathcal{X}_1$ and an edge $(x_1^1,x_1^2)\in\mathcal{X}_1^2$ is in E_{x_1} iff there exists a $x_2^1\in\mathcal{X}_2$ such that $p(x_1^1,x_2^1)p(x_1^2,x_2^1)>0$ and $f(x_1^1,x_2^1)\neq f(x_1^2,x_2^1)$.*

In other words, in order to avoid confusion about the

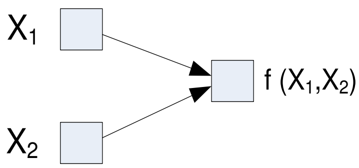

Fig. 1. A network with two sources where the computation of a function is desired at the receiver.

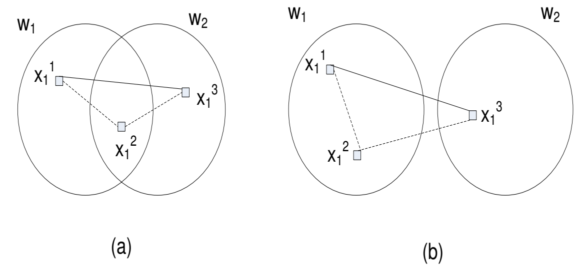

Fig. 2. Having non-zero joint probability distribution, a) maximal independent sets can not overlap with each other (this figure is to depict the contradiction) b) maximal independent sets should be fully connected to each other. In this figure, a solid line represents a connection, and a dashed line means no connection exists.

function $f(X_1, X_2)$ at the receiver, if $(x_1^1, x_1^2) \in E_{x_1}$, then descriptions of x_1^1 and x_1^2 must be different.

Next, we define a coloring function for a characteristic graph.

Definition 2. *A vertex coloring of a graph is a function $c_{G_{X_1}}(X_1) : V_{X_1} \rightarrow \mathbb{N}$ of a graph $G_{X_1} = (V_{X_1}, E_{X_1})$ such that $(x_1^1, x_1^2) \in E_{X_1}$ implies $c_{G_{X_1}}(x_1^1) \neq c_{G_{X_1}}(\binom{2}{1})$. The entropy of a coloring is the entropy of the induced distribution on colors. Here, $p(c_{G_{X_1}}(x_1^i)) = p(c^{-1}_{G_{X_1}}(c_{G_{X_1}}(x_1^i)))$, where $c^{-1}_{G_{X_1}}(x_1^i) = \{x_1^j : c_{G_{X_1}}(x_1^j) = c_{G_{X_1}}(x_1^i)\}$ for all valid j is called a color class. We also call the set of all valid colorings of a graph G_{X_1} as $C_{G_{X_1}}$.*

Sometimes one needs to consider sequences of a RV with length n. In order to deal with these cases, we can extend the definition of a characteristic graph for vectors of a RV.

Definition 3. *The n-th power of a graph G_{X_1} is a graph $G^n_{\mathbf{X}_1} = (V^n_{X_1}, E^n_{X_1})$ such that $V^n_{X_1} = \mathcal{X}_1^n$ and $(\mathbf{x}_1^1, \mathbf{x}_1^2) \in E^n_{X_1}$ when there exists at least one i such that $(x_{1i}^1, x_{1i}^2) \in E_{X_1}$. We denote a valid coloring of $G^n_{\mathbf{X}_1}$ by $c_{G^n_{\mathbf{X}_1}}(\mathbf{X}_1)$.*

In some problems such as the functional compression problem, we need to find a coloring random variable of a characteristic graph which minimizes the entropy. The problem is how to compute such a coloring for a given characteristic graph. In other words, this problem can be expressed as follows. Given a characteristic graph G_{X_1} (or, its n-th power, $G^n_{\mathbf{X}_1}$), one can assign different colors to its vertices. Suppose $C_{G_{X_1}}$ is the collection of all valid colorings of this graph, G_{X_1}. Among these colorings, one which minimizes the entropy of the coloring RV is called the minimum-entropy coloring, and we refer to it by $c^{min}_{G_{X_1}}$. In other words,

$$c^{min}_{G_{X_1}} = \arg\min_{c_{G_{X_1}} \in C_{G_{X_1}}} H(c_{G_{X_1}}). \tag{1}$$

The problem is how to compute $c^{min}_{G_{X_1}}$ given G_{X_1}.

III. Minimum Entropy Coloring

In this section, we consider the problem of finding the minimum entropy coloring of a characteristic graph. The problem is how to compute a coloring of a characteristic graph which minimizes the entropy. In general, finding $c^{min}_{G_{X_1}}$ is an NP-hard problem ([6]). However, in this section, we show that depending on the characteristic graph's structure, there are some interesting cases where finding the minimum entropy coloring is not NP-hard, but tractable and practical. In one of these cases, we show that, by having a non-zero joint probability condition on RVs' distributions, for any desired function f, finding $c^{min}_{G_{X_1}}$ can be solved in polynomial time. In another case, we show that if f is a quantization function, this problem is also tractable. For simplicity, we consider functions with two input RVs, but one can easily extend all discussions to functions with more input RVs than two.

A. Non-Zero Joint Probability Distribution Condition

Consider the network shown in Figure 1. Source RVs have a joint probability distribution $p(x_1, x_2)$, and the receiver wishes to compute a deterministic function of sources (i.e., $f(X_1, X_2)$). In Section IV, we will show that in an achievable coding scheme, one needs to compute minimum entropy colorings of characteristic graphs. The question is how source nodes can compute minimum entropy colorings of their characteristic graphs G_{X_1} and G_{X_2} (or, similarly the minimum entropy colorings of $G^n_{\mathbf{X}_1}$ and $G^n_{\mathbf{X}_2}$, for some n). For an arbitrary graph, this problem is NP-hard ([6]). However, in certain cases, depending on the probability distribution or the desired function, the characteristic graph has some special structure which leads to a tractable scheme to find the minimum entropy coloring. In this section, we consider the effect of the probability distribution.

Theorem 4. *Suppose for all $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, $p(x_1, x_2) > 0$. Then, maximal independent sets of the characteristic graph G_{X_1} (and, its n-th power $G^n_{\mathbf{X}_1}$, for any n) are some non-overlapping fully-connected sets. Under this condition, the minimum entropy coloring can be achieved by assigning different colors to its different maximal independent sets.*

Proof: Suppose $\Gamma(G_{X_1})$ is the set of all maximal independent sets of G_{X_1}. Let us proceed by contradiction. Consider Figure 2-a. Suppose w_1 and w_2 are two different non-empty maximal independent sets. Without loss of generality, assume x_1^1 and x_1^2 are in w_1, and x_1^2 and x_1^3 are in w_2. In other words, these sets have a common element x_1^2. Since w_1 and w_2 are two different maximal independent sets, $x_1^1 \notin w_2$ and $x_1^3 \notin w_1$. Since x_1^1 and x_1^2 are in w_1, there is no edge between them in G_{X_1}. The same argument holds for x_1^2 and x_1^3. But, we have an edge between x_1^1 and x_1^3, because w_1 and w_2 are two different maximal independent sets, and at least there should exist such

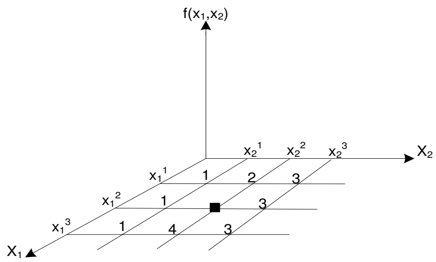

Fig. 3. Having non-zero joint probability condition is necessary for Theorem 4. A dark square represents a zero probability point.

an edge between them. Now, we want to show that it is not possible.

Since there is no edge between x_1^1 and x_1^2, for any $x_2^1 \in \mathcal{X}_2$, $p(x_1^1, x_2^1)p(x_1^2, x_2^1) > 0$, and $f(x_1^1, x_2^1) = f(x_1^2, x_2^1)$. A similar argument can be expressed for x_1^2 and x_1^3. In other words, for any $x_2^1 \in \mathcal{X}_2$, $p(x_1^2, x_2^1)p(x_1^3, x_2^1) > 0$, and $f(x_1^2, x_2^1) = f(x_1^3, x_2^1)$. Thus, for all $x_2^1 \in \mathcal{X}_2$, $p(x_1^1, x_2^1)p(x_1^3, x_2^1) > 0$, and $f(x_1^1, x_2^1) = f(x_1^3, x_2^1)$. However, since x_1^1 and x_1^3 are connected to each other, there should exist a $x_2^1 \in \mathcal{X}_2$ such that $f(x_1^1, x_2^1) \neq f(x_1^3, x_2^1)$ which is not possible. So, the contradiction assumption is not correct and these two maximal independent sets do not overlap with each other.

We showed that maximal independent sets cannot have overlaps with each other. Now, we want to show that they are also fully connected to each other. Again, let us proceed by contradiction. Consider Figure 2-b. Suppose w_1 and w_2 are two different non-overlapping maximal independent sets. Suppose there exists an element in w_2 (call it x_1^3) which is connected to one of elements in w_1 (call it x_1^1) and is not connected to another element of w_1 (call it x_1^2). By using a similar discussion to the one in the previous paragraph, we may show that it is not possible. Thus, x_1^3 should be connected to x_1^1. Therefore, if for all $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, $p(x_1, x_2) > 0$, then maximal independent sets of G_{X_1} are some separate fully connected sets. In other words, the complement of G_{X_1} is formed by some non-overlapping cliques. Finding the minimum entropy coloring of this graph is trivial and can be achieved by assigning different colors to these non-overlapping fully-connected maximal independent sets.

This argument also holds for any power of G_{X_1}. Suppose $\mathbf{x}_1^1$, $\mathbf{x}_1^2$ and $\mathbf{x}_1^3$ are some typical sequences in $\mathcal{X}_1^n$. If $\mathbf{x}_1^1$ is not connected to $\mathbf{x}_1^2$ and $\mathbf{x}_1^3$, it is not possible to have $\mathbf{x}_1^2$ and $\mathbf{x}_1^3$ connected. Therefore, one can apply a similar argument to prove the theorem for $G_{\mathbf{X}_1}^n$, for some n. This completes the proof. ■

One should notice that the condition $p(x_1, x_2) > 0$, for all $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$, is a necessary condition for Theorem 4. In order to illustrate this, consider Figure 3. In this example, x_1^1, x_1^2 and x_1^3 are in $\mathcal{X}_1$, and x_2^1, x_2^2 and x_2^3 are in $\mathcal{X}_2$. Suppose $p(x_1^2, x_2^2) = 0$. By considering the value of function f at these points depicted in the figure, one can see that, in G_{X_1}, x_1^2 is not connected to x_1^1 and x_1^3. However, x_1^1 and x_1^3 are connected to each other. Thus, Theorem 4 does not hold here.

It is also worthwhile to notice that the condition used in Theorem 4 only restricts the probability distribution and it does not depend on the function f. Thus, for any function f at the receiver, if we have a non-zero joint probability distribution of source RVs (for example, when source RVs are independent), finding the minimum-entropy coloring and, therefore, the proposed functional compression scheme of Section IV and reference [5], is easy and tractable.

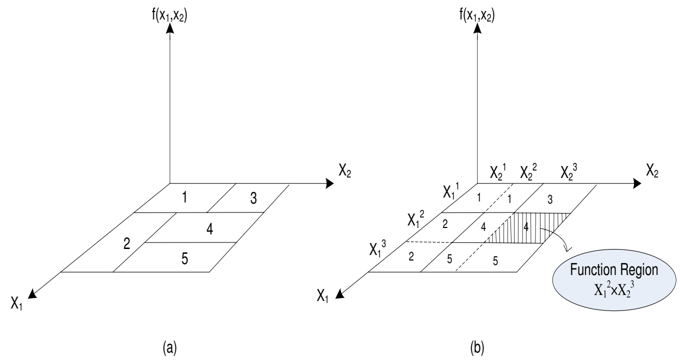

Fig. 4. a) A quantization function. Function values are depicted in the figure on each rectangle. b) By extending sides of rectangles, the plane is covered by some function regions.

B. Quantization Functions

In Section III-A, we introduced a condition on the joint probability distribution of RVs which leads to a specific structure of the characteristic graph such that finding the minimum entropy coloring is not NP-hard. In this section, we consider some special functions which lead to some graph structures so that one can easily find the minimum entropy coloring.

An interesting function is a quantization function. A natural quantization function is a function which separates the $X_1 - X_2$ plane into some rectangles such that each rectangle corresponds to a different value of that function. Sides of these rectangles are parallel to the plane axes. Figure 4-a depicts such a quantization function.

Given a quantization function, one can extend different sides of each rectangle in the $X_1 - X_2$ plane. This may make some new rectangles. We call each of them *a function region*. Each function region can be determined by two subsets of $\mathcal{X}_1$ and $\mathcal{X}_2$. For example, in Figure 4-b, one of the function regions is distinguished by the shaded area.

Definition 5. *Consider two function regions $\mathcal{X}_1^1 \times \mathcal{X}_2^1$ and $\mathcal{X}_1^2 \times \mathcal{X}_2^2$. If for any $x_1^1 \in \mathcal{X}_1^1$ and $x_1^2 \in \mathcal{X}_1^2$, there exist x_2 such that $p(x_1^1, x_2)p(x_1^2, x_2) > 0$ and $f(x_1^1, x_2) \neq f(x_1^2, x_2)$, we say these two function regions are pairwise X_1-proper.*

Theorem 6. *Consider a quantization function f such that its function regions are pairwise X_1-proper. Then, G_{X_1} (and $G_{\mathbf{X}_1}^n$, for any n) is formed of some non-overlapping fully-connected maximal independent sets, and its minimum entropy coloring can be achieved by assigning different colors to different maximal independent sets.*

Proof: We first prove it for G_{X_1}. Suppose $\mathcal{X}_1^1 \times \mathcal{X}_2^1$, and $\mathcal{X}_1^2 \times \mathcal{X}_2^2$ are two X_1-proper function regions of a quantization function f, where $\mathcal{X}_1^1 \neq \mathcal{X}_1^2$. We show that $\mathcal{X}_1^1$ and $\mathcal{X}_1^2$ are two non-overlapping fully-connected maximal independent sets. By definition, $\mathcal{X}_1^1$ and $\mathcal{X}_1^2$ are two non-equal partition sets of $\mathcal{X}_1$. Thus, they do not have any element in common.

Now, we want to show that vertices of each of these partition sets are not connected to each other. Without loss of generality, we show it for $\mathcal{X}_1^1$. If this partition set of $\mathcal{X}_1$ has only one element, this is a trivial case. So, suppose x_1^1 and x_1^2 are two elements in $\mathcal{X}_1^1$. By definition of function regions, one can see that, for any $x_2 \in \mathcal{X}_2$ such that $p(x_1^1, x_2)p(x_1^2, x_2) > 0$, then $f(x_1^1, x_2) = f(x_1^2, x_2)$. Thus, these two vertices are not connected to each other. Now, suppose x_1^3 is an element in $\mathcal{X}_1^2$. Since these function regions are X_1-proper, there should exist at least one $x_2 \in \mathcal{X}_2$, such that $p(x_1^1, x_2)p(x_1^3, x_2) > 0$, and $f(x_1^1, x_2) \neq f(x_1^3, x_2)$. Thus, x_1^1 and x_1^3 are connected to each other. Therefore, $\mathcal{X}_1^1$ and $\mathcal{X}_1^2$ are two non-overlapping fully-connected maximal independent sets. One can easily apply this argument to other partition sets. Thus, the minimum entropy coloring can be achieved by assigning different colors to different maximal independent sets (partition sets). The proof for $G_{\mathbf{X}_1}^n$, for any n, is similar to the one mentioned in Theorem 4. This completes the proof. ■

It is worthwhile to mention that without X_1-proper condition of Theorem 6, assigning different colors to different partitions still leads to an achievable coloring scheme. However, it is not necessarily the minimum entropy coloring. In other words, without this condition, maximal independent sets may overlap.

Corollary 7. *If a function f is strictly increasing (or, decreasing) with respect to X_1, and $p(x_1, x_2) \neq 0$, for all $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, then, G_{X_1} (and, $G_{\mathbf{X}_1}^n$ for any n) would be a complete graph.*

Under conditions of Corollary 7, functional compression does not give us any gain, because, in a complete graph, one should assign different colors to different vertices. Traditional compression in which f is the identity function is a special case of Corollary 7.

IV. Applications in the Functional Compression Problem

As we expressed in Section II, results of this paper can be used in the functional compression problem. In this section, after giving the functional compression problem statement, we express some of previous results. Then, we explain how this paper's results can be applied in this problem.

A. Functional Compression Problem Setup

Consider the network shown in Figure 1. In this network, we have two source nodes with input processes $\{X_j^i\}_{i=1}^{\infty}$ for $j = 1, 2$. The receiver wishes to compute a deterministic function $f : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Z}$, or $f : \mathcal{X}_1^n \times \mathcal{X}_2^n \to \mathcal{Z}^n$, its vector extension.

Source node j encodes its message at a rate R_{X_j}. In other words, encoder en_{X_j} maps $\mathcal{X}_j^n \to \{1, ..., 2^{nR_{X_j}}\}$. The receiver has a decoder r, which maps $\{1, ..., 2^{nR_{X_1}}\} \times \{1, ..., 2^{nR_{X_1}}\} \to \mathcal{Z}^n$. We sometimes refer to this encoding/decoding scheme as an n-distributed functional code.

For any encoding/decoding scheme, the probability of error is defined as $P_e^n = Pr\big[(\mathbf{x}_1, \mathbf{x}_2) : f(\mathbf{x}_1, \mathbf{x}_2) \neq r(en_{X_1}(\mathbf{x}_1), en_{X_2}(\mathbf{x}_2))\big]$. A rate pair, $R = (R_{X_1}, R_{X_2})$ is achievable iff there exist encoders in source nodes operating in these rates, and a decoder r at the receiver such that $P_e^n \to 0$ as $n \to \infty$. The achievable rate region is the set closure of the set of all achievable rates.

B. Prior Results in the Functional Compression Problem

In order to express some previous results, we need to define an ϵ-coloring of a characteristic graph.

Definition 8. *Given a non-empty set $\mathcal{A} \subset \mathcal{X}_1 \times \mathcal{X}_2$, define $\hat{p}(x_1, x_2) = p(x_1, x_2)/p(\mathcal{A})$ when $(x_1, x_2) \in \mathcal{A}$, and $\hat{p}(x_1, x_2) = 0$ otherwise. $\hat{p}$ is the distribution over (x_1, x_2) conditioned on $(x_1, x_2) \in \mathcal{A}$. Denote the characteristic graph of X_1 with respect to X_2, $\hat{p}(x_1, x_2)$, and $f(X_1, X_2)$ as $\hat{G}_{X_1} = (\hat{V}_{X_1}, \hat{E}_{X_1})$ and the characteristic graph of X_2 with respect to X_1, $\hat{p}(x_1, x_2)$, and $f(X_1, X_2)$ as $\hat{G}_{X_2} = (\hat{V}_{X_2}, \hat{E}_{X_2})$. Note that $\hat{E}_{X_1} \subseteq E_{X_1}$ and $\hat{E}_{X_2} \subseteq E_{X_2}$. Finally, we say that $c_{G_{X_1}}(X_1)$ and $c_{G_{X_2}}(X_2)$ are ϵ-colorings of G_{X_1} and G_{X_2} if they are valid colorings of $\hat{G}_{X_1}$ and $\hat{G}_{X_2}$ defined with respect to some set $\mathcal{A}$ for which $p(\mathcal{A}) \geq 1 - \epsilon$.*

Among ϵ-colorings of G_{X_1}, the one which minimizes the entropy is called the minimum entropy ϵ-coloring of G_{X_1}. Sometimes, we refer to it as the minimum entropy coloring when no confusion arises.

Consider the network shown in Figure 1. The rate region of this network has been considered in different references under some assumptions. Reference [3] considered the case where X_2 is available at the receiver as the side information, while reference [4] computed a rate region under some conditions on source distributions called the zigzag condition. The case of having more than one desired function at the receiver with the side information was considered in [7].

However, the rate region for a more general case has been considered in [5]. Reference [5] shows that if source nodes send ϵ-colorings of their characteristic graphs satisfying a condition called the *Coloring Connectivity Condition* (C.C.C.), followed by a Slepian-Wolf encoder, it will lead to an achievable coding scheme. Conversely, any achievable scheme induces colorings on high probability subgraphs of characteristic graphs satisfying C.C.C.

In the following, we briefly explain C.C.C. for the case of having two source nodes. An extension to the case of having more source nodes than one is presented in [5].

Definition 9. *A path with length m between two points $Z_1 = (x_1^1, x_2^1)$ and $Z_m = (x_1^2, x_2^2)$ is determined by m points Z_i, $1 \leq i \leq m$ such that,*

i) $P(Z_i) > 0$, for all $1 \leq i \leq m$.

ii) Z_i and Z_{i+1} only differ in one of their coordinates.

Definition 9 can be expressed for two n-length vectors $(\mathbf{x}_1^1, \mathbf{x}_2^1)$ and $(\mathbf{x}_1^2, \mathbf{x}_2^2)$.

Definition 10. *A joint-coloring family J_C for random variables X_1 and X_2 with characteristic graphs G_{X_1} and G_{X_2}, and any valid colorings $c_{G_{X_1}}$ and $c_{G_{X_2}}$, respectively is defined*

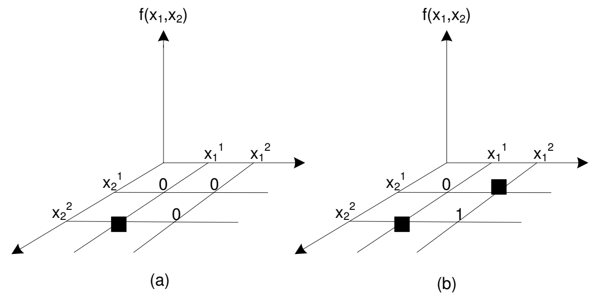

Fig. 5. Two examples of a joint coloring class: a) satisfying C.C.C. b) not satisfying C.C.C. Dark squares indicate points with zero probability. Function values are depicted in the picture.

as $J_C = \{j_c^1, ..., j_c^{n_{j_c}}\}$ *where* $j_c^i = \left\{(x_1^{i_1}, x_2^{j_1}), (x_1^{i_2}, x_2^{j_2}) : c_{G_{X_1}}(x_1^{i_1}) = c_{G_{X_1}}(x_1^{i_2}), c_{G_{X_2}}(x_2^{j_1}) = c_{G_{X_2}}(x_2^{j_2})\right\}$ *for any valid* i_1, i_2, j_1 *and* j_2, *where* $n_{j_c} = |c_{G_{X_1}}| \times |c_{G_{X_2}}|$. *We call each* j_c^i *as a joint coloring class.*

Definition 10 can be expressed for RVs $\mathbf{X}_1$ and $\mathbf{X}_2$ with characteristic graphs $G^n_{\mathbf{X}_1}$ and $G^n_{\mathbf{X}_2}$, and any valid ϵ-colorings $c_{G^n_{\mathbf{X}_1}}$ and $c_{G^n_{\mathbf{X}_2}}$, respectively.

Definition 11. *Consider RVs* X_1 *and* X_2 *with characteristic graphs* G_{X_1} *and* G_{X_2}, *and any valid colorings* $c_{G_{X_1}}$ *and* $c_{G_{X_2}}$. *We say these colorings satisfy the Coloring Connectivity Condition (C.C.C.) when, between any two points in* $j_c^i \in J_C$, *there exists a path that lies in* j_c^i, *or function* f *has the same value in disconnected parts of* j_c^i.

C.C.C. can be expressed for RVs $\mathbf{X}_1$ and $\mathbf{X}_2$ with characteristic graphs $G^n_{\mathbf{X}_1}$ and $G^n_{\mathbf{X}_2}$, and any valid ϵ-colorings $c_{G^n_{\mathbf{X}_1}}$ and $c_{G^n_{\mathbf{X}_2}}$, respectively.

In order to illustrate this condition, we borrow an example from [5].

Example 12. *For example, suppose we have two random variables* X_1 *and* X_2 *with characteristic graphs* G_{X_1} *and* G_{X_2}. *Let us assume* $c_{G_{X_1}}$ *and* $c_{G_{X_2}}$ *are two valid colorings of* G_{X_1} *and* G_{X_2}, *respectively. Assume* $c_{G_{X_1}}(x_1^1) = c_{G_{X_1}}(x_1^2)$ *and* $c_{G_{X_2}}(x_2^1) = c_{G_{X_2}}(x_2^2)$. *Suppose* j_c^1 *represents this joint coloring class. In other words,* $j_c^1 = \{(x_1^i, x_2^j)\}$, *for all* $1 \leq i, j \leq 2$ *when* $p(x_1^i, x_2^j) > 0$. *Figure 5 considers two different cases. The first case is when* $p(x_1^1, x_2^2) = 0$, *and other points have a non-zero probability. It is illustrated in Figure 5-a. One can see that there exists a path between any two points in this joint coloring class. So, this joint coloring class satisfies C.C.C. If other joint coloring classes of* $c_{G_{X_1}}$ *and* $c_{G_{X_2}}$ *satisfy C.C.C., we say* $c_{G_{X_1}}$ *and* $c_{G_{X_2}}$ *satisfy C.C.C. Now, consider the second case depicted in Figure 5-b. In this case, we have* $p(x_1^1, x_2^2) = 0$, $p(x_1^2, x_2^1) = 0$, *and other points have a non-zero probability. One can see that there is no path between* (x_1^1, x_2^1) *and* (x_1^2, x_2^2) *in* j_c^1. *So, though these two points belong to a same joint coloring class, their corresponding function values can be different from each other. For this example,* j_c^1 *does not satisfy C.C.C. Therefore,* $c_{G_{X_1}}$ *and* $c_{G_{X_2}}$ *do not satisfy C.C.C.*

In Section III, we did not consider C.C.C. in finding the minimum entropy coloring. For the non-zero joint probability case, under the condition mentioned in Theorem 4, any valid colorings (and therefore, minimum entropy colorings) of characteristic graphs satisfy C.C.C mentioned in Definition 11. It is because we have at least one path between any two points. However, for a quantization function, after achieving minimum entropy colorings, C.C.C. should be checked for colorings. In other words, despite the non-zero joint probability condition where C.C.C. always holds, for the quantization function case, one should check C.C.C. separately.

V. Conclusion

In this paper we considered the problem of finding the minimum entropy coloring of a characteristic graph under some conditions which make it to be done in polynomial time. This problem has been raised in the functional compression problem where the computation of a function of sources is desired at the receiver. Recently, [5] computed the rate region of the functional compression problem for a general one stage tree network and its extension to a general tree network. In the proposed coding schemes of [5] and some other references such as [4] and [7], one needs to compute the minimum entropy coloring of a characteristic graph. In general, finding this coloring is an NP-hard problem ([6]). However, in this paper, we showed that, depending on the characteristic graph's structure, there are some interesting cases where finding the minimum entropy coloring is not NP-hard, but tractable and practical. In one of these cases, we show that, by having a non-zero joint probability condition on RVs' distributions, for any desired function f, finding the minimum entropy coloring can be solved in polynomial time. In another case, we show that, if f is a type of a quantization function, this problem is also tractable.

In this paper, we considered two cases such that finding the minimum entropy coloring of a characteristic graph can be done in polynomial time. Depending on different applications, one can investigate other conditions on either probability distributions or the desired function to have a special structure of the characteristic graph which may lead to polynomial time algorithms to find the minimum entropy coloring.

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