

Spreading in Block-Fading Channels.

Muriel Médard¹ David N.C. Tse²

¹ *medard@mit.edu* Massachusetts Institute of Technology

² *dtse@eecs.berkeley.edu* University of California at Berkeley

Abstract We consider wideband fading channels which are block fading in time and in frequency (doubly block fading). We show that, as bandwidth increases to infinity, capacity goes to zero when signaling is constrained in its second moment and peak signal amplitude. This result is consistent with similar results which do not consider doubly block-fading channels. None of these results, however, offer a range for the optimal spreading bandwidth, because they consider only upper bounds to capacity. While we know that spreading over very large bandwidths is detrimental in terms of capacity, we wish to determine over what range of bandwidths spreading is beneficial. We are able to give a range for the optimal spreading bandwidth by combining our upper bound with a suitable lower bound.

1 Introduction.

Recent results in the area of wideband fading channels have shown that, as the bandwidth over which we transmit becomes arbitrarily large, capacity goes to zero if we scale the signal inversely with the bandwidth. Several models have been considered for these channels variations. In [4], a finite number of time-varying paths are considered and, if paths remain unresolvable, capacity is shown to go 0 as as bandwidth becomes arbitrarily large. In [1], a general doubly selective fading channel model is considered, in which paths never become resolvable. References [1, 3] consider DS-CDMA transmissions over channels which are block fading in frequency but continuously fading in time.

The above models all exhibit continuous variations in time. In this paper, we first show, in Section 2,

that similar results to those which apply to continuously varying doubly selective channels apply to doubly block fading channels, which are block fading in time and frequency. We are mainly interested in determining over what range of bandwidths spreading is beneficial. While all of the results mentioned above show that excessive spreading is detrimental in terms of capacity, we suspect that spreading is beneficial as long as it remains above some threshold. In Section 3, we develop a lower bound to capacity. Combining our upper and lower bounds, we show in Section 4 how we can find upper and lower bounds to the optimal spreading bandwidth.

We use a channel model where each block in frequency fades according to the model in [2]. Over each coherence bandwidth of size W , the channel experiences Rayleigh flat fading. All the channels over distinct coherence bandwidths are independent, yielding a block-fading model in frequency. We transmit over μ coherence bandwidths. The energy of the propagation coefficient $F[i]_j$ over coherence bandwidth i at sampled time j is σ_F^2 . For input $X[i]_j$ at sample time j (we sample at the Nyquist rate W), the corresponding output is $Y[i]_j = F[i]_j X[i]_j + N[i]_j$, where the $N[i]_j$ s are samples of WGN bandlimited to a bandwidth of W . The time variations are block-fading nature: the propagation coefficient of the channel remains constant for T symbols (the coherence interval), then changes to a value independent of previous values. Thus, $\underline{F}[i]_{jTW+1}^{(j+1)TW}$ is a constant vector and the $\underline{F}[i]_{jTW+1}^{(j+1)TW}$ are mutually independent for $j = 1, 2, \dots$. For the signals over each coherence bandwidth, the second moment is upper bounded by $\overline{X^2} \leq \frac{\epsilon}{\mu}$ and the amplitude is upper bounded by

$$\frac{\gamma}{\mathcal{E}} \sqrt{X^2}.$$

2 Upper bound on wideband capacity.

We first note that we can restrict ourselves, in the upper bound, to signals which satisfy the second moment constraint with equality. Indeed, let us suppose that we obtain an upper bound using a signaling scheme which does not meet the second moment constraint with equality. Then, by multiplying our signals by some $\alpha > 1$, we can achieve the second moment constraint with equality. Moreover, at the receiver we could divide the received signal by α , in effect reducing the AWGN. Thus, by the data processing theorem, the capacity of the channel with the input multiplied by α would be greater than that of the channel not multiplied by α .

The following lemma gives our upper bound.

Lemma 1

$$C(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma) \leq \frac{\mu W}{2} \log \left(\sigma_F^2 \frac{\mathcal{E}}{\mu} + 1 \right) - \frac{\mu \mathcal{E}^2}{2T\gamma^2} \log \left(TW \frac{\gamma^2}{\mu \mathcal{E}} \sigma_F^2 + 1 \right) \quad (1)$$

Proof of Lemma 1.

First, we express C in terms of mutual informations. From our model, we have that

$$C(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma) = \lim_{k \rightarrow \infty} \max_{p_X} \left(\frac{1}{k} \sum_{i=1}^k \sum_{j=1}^{\mu} \frac{1}{T} I(\underline{X}[j]_{iTW+1}^{(i+1)TW}; \underline{Y}[j]_{iTW+1}^{(i+1)TW}) \right) \quad (2)$$

where the fourth central moment of $X[j]_i$ is $\frac{\gamma}{\mu^2}$ and its average energy constraint is $\frac{\mathcal{E}}{\mu}$. Since we have no sender channel side information and all the bandwidth slices are independent, we may use the fact that mutual information is concave in the input distribution to determine that selecting all the inputs to be IID maximizes the RHS of (2). We first rewrite the mutual information term as:

$$\begin{aligned} & \frac{1}{T} I(\underline{X}[j]_{iTW+1}^{(i+1)TW}; \underline{Y}[j]_{iTW+1}^{(i+1)TW}) \\ &= \frac{1}{T} h(\underline{Y}[j]_{iTW+1}^{(i+1)TW}) \\ & - \frac{1}{T} h(\underline{Y}[j]_{iTW+1}^{(i+1)TW} | \underline{X}[j]_{iTW+1}^{(i+1)TW}). \end{aligned} \quad (3)$$

We may upper bound the first term of (3) by:

$$\begin{aligned} & \frac{1}{T} h(\underline{Y}[j]_{iTW+1}^{(i+1)TW}) \\ & \leq \frac{1}{2T} \log \left((2\pi e)^{TW} \left| \Lambda_{\underline{Y}[j]_{iTW+1}^{(i+1)TW}} \right| \right) \\ & \text{because entropy is maximized} \\ & \text{by a Gaussian distribution} \\ & \text{for a given autocorrelation matrix} \\ & \leq \frac{1}{2T} \log \left((2\pi e)^{TW} \prod_{i=1}^{TW} (\sigma_F^2 \sigma_{X[i]}^2 + 1) \right) \\ & \text{from Hadamard's inequality} \\ & = \frac{W}{2} \log(2\pi e) + \frac{1}{2T} \sum_{i=1}^{TW} \log(\sigma_F^2 \sigma_{X[i]}^2 + 1) \\ & \leq \frac{W}{2} \log(2\pi e) + \frac{W}{2} \log \left(\sigma_F^2 \frac{\mathcal{E}}{\mu} + 1 \right), \end{aligned} \quad (4)$$

where the last inequality uses the concavity of the log function and our average energy constraint. We now proceed to minimize the second term of (2). Note that, conditioned on $\underline{X}[j]_{iTW+1}^{(i+1)TW}$, $\underline{Y}[j]_{iTW+1}^{(i+1)TW}$ is Gaussian, since $\underline{F}[j]_{iTW+1}^{(i+1)TW}$ is Gaussian and $\underline{N}_{iTW+1}^{(i+1)TW}$ is AWGN. Hence, we have that

$$\begin{aligned} & \frac{1}{T} h(\underline{Y}[j]_{iTW+1}^{(i+1)TW} | \underline{X}[j]_{iTW+1}^{(i+1)TW}) \\ &= \frac{1}{2T} E_{\underline{X}} \left[\log \left((2\pi e)^T \left| \Lambda_{\underline{Y}[j]_{iTW+1}^{(i+1)TW}} \right| \right) \right]. \end{aligned} \quad (5)$$

$\Lambda_{\underline{Y}[j]_{iTW+1}^{(i+1)TW}}$ has k^{th} diagonal term $\sigma_F^2 x[k]^2 + 1$ and off-diagonal (k, j) term equal to $x(k)x(j)\sigma_F^2$, conditioned on $\underline{X}[j]_{iTW+1}^{(i+1)TW} = \underline{x} = [x(1), \dots, x(TW)]$. The eigenvalues λ_j of $\Lambda_{\underline{Y}}$ are 1 for $j = 1 \dots TW - 1$ and $\|\underline{x}\|^2 \sigma_F^2 + 1$ for $j = TW$.

Hence, we may rewrite (4) as

$$\begin{aligned} & \frac{1}{T} h \left(\underline{Y}_{iTW+1}^{(i+1)TW} \mid \underline{X}_{iTW+1}^{(i+1)TW} \right) \\ &= \frac{1}{2T} E_{\underline{X}} \left[\log \left(\left\| \underline{x}_{iTW+1}^{(i+1)TW} \right\|^2 \sigma_F^2 + 1 \right) \right] \\ &+ \frac{W}{2} \log(2\pi e). \end{aligned} \quad (6)$$

We seek to minimize the RHS of (6) subject to the second moment constraint holding with equality and the subject to the peak amplitude constraint. The distribution for X which minimizes the RHS of (6) subject to our constraints can be found using the concavity of the log function. The distribution is such that the only values which $|X|$ can take are 0 and $\frac{\gamma}{\sqrt{\mu\mathcal{E}}}$ with probabilities $1 - \frac{\mathcal{E}^2}{\gamma^2}$ and $\frac{\mathcal{E}^2}{\gamma^2}$, respectively. Thus, we may lower bound (6) by

$$\begin{aligned} & \frac{1}{T} h \left(\underline{Y}_{iTW+1}^{(i+1)TW} \mid \underline{X}_{iTW+1}^{(i+1)TW} \right) \\ &\geq \frac{\mathcal{E}^2}{2T\gamma^2} \log \left(TW \frac{\gamma^2}{\mu\mathcal{E}} \sigma_F^2 + 1 \right) + \frac{W}{2} \log(2\pi e). \end{aligned} \quad (7)$$

Combining (7), (4) and (2) yields (1).

Q.E.D.

We obtain immediately from (1) that $\lim_{\mu \rightarrow \infty} C(W, \mathcal{E}, \sigma_F^2, T, \mu) \rightarrow 0$.

The RHS of (1) increases with T . Intuitively, we expect the real capacity to also increase with T , since a longer coherence time entails better measurement of the channel, and thus channel behavior which is close to that of an AWGN channel over every coherence time. Moreover, RHS of (1) increases with W . Again, a longer coherence bandwidth entails better measurement of the channel. Finally, we may note that, as γ increases, the bound in (1) converges more slowly. For any μ , the limit as $\gamma \rightarrow \infty$ is $\frac{\mu W}{2} \log \left(\sigma_F^2 \frac{\mathcal{E}}{\mu} + 1 \right)$.

3 Lower bound on wideband capacity.

Our lower bound on capacity is obtained by choosing the $X[j]$ s to be:

$$X[j] = \begin{cases} \sqrt{\frac{\mathcal{E}}{W\mu}} & \text{with probability } \frac{1}{2} \\ -\sqrt{\frac{\mathcal{E}}{W\mu}} & \text{with probability } \frac{1}{2} \end{cases}. \quad (8)$$

Moreover, we select the X s to be IID. The channel we consider for our lower bound is a BSC with cross-over probability $\epsilon = \Phi \left(-\sqrt{\frac{\mathcal{E}}{W\mu}} \right)$. Thus, the capacity of our original channel is lower bounded by $W\mu$ times the capacity of this BSC channel

$$\begin{aligned} C(W, \mathcal{E}, \sigma_N^2, T, \mu, \gamma) &\geq W\mu (1 - \mathcal{H}(\epsilon)) \\ &= W\mu (1 + \epsilon \log(\epsilon) + (1 - \epsilon) \log(1 - \epsilon)). \end{aligned} \quad (9)$$

We may bound ϵ as follows:

$$\begin{aligned} \epsilon &= \frac{1}{2} - \int_{-\sqrt{\frac{\mathcal{E}}{W\mu}}}{0} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx \\ &\leq \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \\ &\leq \frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}. \end{aligned} \quad (10)$$

Hence, we may upper bound $\mathcal{H}(\epsilon)$ in the following manner:

$$\begin{aligned} \mathcal{H}(\epsilon) &\leq \\ &- \left(\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \right) \log \left(\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \right) \\ &- \left(\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \right) \log \left(\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \right) \\ &= \frac{-1}{2} \log \left(\frac{1}{4} - \frac{1}{16} \frac{\mathcal{E}}{W\mu} e^{-2\frac{\mathcal{E}}{W\mu}} \right) \\ &+ \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \log \left(\frac{\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}}{\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} \right) \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{-1}{2} \log \left(1 - \frac{1}{4} \frac{\mathcal{E}}{W\mu} e^{-2\frac{\mathcal{E}}{W\mu}} \right) \\
&+ \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}} \log \left(\frac{\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}}{\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} \right) \quad (11)
\end{aligned}$$

Using the fact that $\ln(1-x) \geq -x$, we can write the bounds:

$$\begin{aligned}
&\frac{1}{2} \log \left(1 - \frac{1}{4} \frac{\mathcal{E}}{W\mu} e^{-2\frac{\mathcal{E}}{W\mu}} \right) \\
&\geq -\frac{\log(e)}{8} \frac{\mathcal{E}}{W\mu} e^{-2\frac{\mathcal{E}}{W\mu}} \\
&\geq -\frac{1}{4} \frac{\mathcal{E}}{W\mu} e^{-2\frac{\mathcal{E}}{W\mu}}. \quad (12)
\end{aligned}$$

Thus, from (9, 11, 12), we obtain

Lemma 2

$$\begin{aligned}
C(W, \mathcal{E}, \sigma_N^2, T, \mu, \gamma) &\geq \frac{-\mathcal{E}}{4} e^{-2\frac{\mathcal{E}}{W\mu}} \\
- \frac{\sqrt{\mathcal{E}W\mu}}{4} e^{-\frac{\mathcal{E}}{W\mu}} \log \left(\frac{\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}}{\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} \right) &\quad (13)
\end{aligned}$$

We may examine the behavior of the RHS of (13) as $\mu \rightarrow \infty$. We have $\lim_{\mu \rightarrow \infty} \left(\frac{\mathcal{E}}{4} e^{-2\frac{\mathcal{E}}{W\mu}} \right) = \frac{\mathcal{E}}{4}$. The limit of the second term of the RHS of (13) can be examined using L'Hospital's rule:

$$\begin{aligned}
&\lim_{\mu \rightarrow \infty} \frac{\sqrt{\mathcal{E}W\mu}}{4} e^{-\frac{\mathcal{E}}{W\mu}} \log \left(\frac{\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}}{\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} \right) \\
&= \frac{\sqrt{\mathcal{E}W}}{4} \lim_{\mu \rightarrow \infty} \left(\left(\frac{\mathcal{E}}{W\mu^2} e^{-\frac{\mathcal{E}}{W\mu}} \right. \right. \\
&\times \log \left(\frac{\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}}{\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} \right) + e^{-\frac{2\mathcal{E}}{W\mu}} \\
&\times \left(\frac{\sqrt{\mathcal{E}}}{4\sqrt{W}} \mu^{-\frac{3}{2}} + \frac{\mathcal{E}^{\frac{3}{2}}}{W} \frac{1}{\mu^2} e^{-\frac{\mathcal{E}}{W\mu}} \right) \\
&\times \frac{1}{\frac{1}{2} - \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} + e^{-\frac{2\mathcal{E}}{W\mu}}
\end{aligned}$$

$$\begin{aligned}
&\times \left(\frac{\sqrt{\mathcal{E}}}{4\sqrt{W}} \mu^{-\frac{3}{2}} + \frac{\mathcal{E}^{\frac{3}{2}}}{W} \frac{1}{\mu^2} e^{-\frac{\mathcal{E}}{W\mu}} \right) \\
&\times \frac{1}{\frac{1}{2} + \frac{1}{4} \sqrt{\frac{\mathcal{E}}{W\mu}} e^{-\frac{\mathcal{E}}{W\mu}}} \left(\frac{-2}{\mu^{-\frac{3}{2}}} \right) \\
&= \frac{-\mathcal{E}}{4} \quad (14)
\end{aligned}$$

Hence, the limit of the RHS of (13) as $\mu \rightarrow \infty$ is 0, and Lemmas 1 and 2 provide tight bounds.

4 Upper and lower bounds to the optimal spreading bandwidth

We may now use our upper and lower bounds to find a range for μ^* , the optimal value of μ to which to spread.

First note that spreading from μ to $n\mu$ is beneficial iff $C(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma) \leq C(W, \mathcal{E}, \sigma_F^2, T, n\mu, \gamma)$. We also note that, given the convexity of capacity, we have that $C(W, \mathcal{E}, \sigma_F^2, T, \mu, \gamma) = nC\left(W, \frac{\mathcal{E}}{n}, \sigma_F^2, T, \mu, \gamma\right)$.

Let us fix $\mu = 1$. Our lower and upper bounds from Lemmas 1 and 2 are denoted, respectively, by the functions \mathbf{C}_{low} and \mathbf{C}_{up} . Figure 1 shows \mathbf{C}_{low} and \mathbf{C}_{up} as a function of \mathcal{E} , with all other parameters kept constant. These functions have the property that they are convex for small values of \mathcal{E} and concave for large values of \mathcal{E} . They each exhibit a single saddle point.

The line tangent to \mathbf{C}_{low} emanating from the origin is tangent to \mathbf{C}_{low} at E_1 . It intersects \mathbf{C}_{up} at E_0 and E_2 , where $E_2 > E_0$. Let us assume that $E_2 = nE_1$, for n integer. Then, we have that

$$\begin{aligned}
&\frac{1}{n} \mathbf{C}_{up} \left(W, E_2, \sigma_F^2, T, 1, \gamma \right) \\
&= \mathbf{C}_{low} \left(W, E_1, \sigma_F^2, T, 1, \gamma \right) \quad (15)
\end{aligned}$$

Let us now assume that $mE_0 = E_1$. Then, we have that

$$\begin{aligned}
&\mathbf{C}_{up} \left(W, E_0, \sigma_F^2, T, 1, \gamma \right) \\
&= \frac{1}{m} \mathbf{C}_{low} \left(W, E_1, \sigma_F^2, T, 1, \gamma \right). \quad (16)
\end{aligned}$$

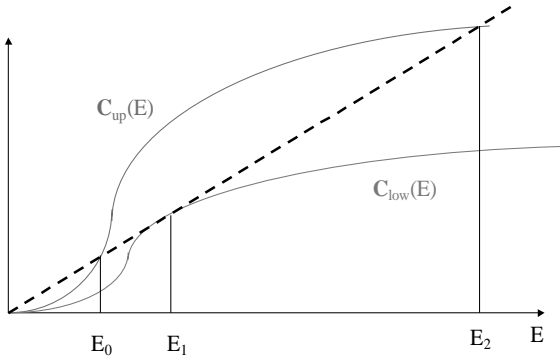


Figure 1: Upper and lower bounds for capacity and the graphical interpretation of the upper and lower bounds to the optimal μ

For $\mu = 1$, E_1 is an upper bound to the optimal \mathcal{E} and E_0 is a lower bound. If we fix $\mathcal{E} = \xi$, then μ^* must be in the range $\left[\frac{\xi}{E_1}, \frac{\xi}{E_0}\right]$.

References

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