Mean-field equilibrium: An approximation approach for large dynamic games

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Single agent dynamic control
Two agents: a dynamic game

Suppose two devices are trying to *coordinate* on a common position.
Multiple agents: a dynamic game
Overview

- **Stochastic dynamic games**: Dynamic models of interacting self-interested agents
- **Large scale**: The number of agents is large

Traditional game theory is impractical in this regime:
- Equilibria in dynamic games make very strong rationality assumptions
- Equilibrium strategies grow in complexity with the number of agents ("curse of dimensionality")
Mean field equilibrium (MFE)

• In large systems, it is plausible that agents can *ignore* the impact their actions have on behavior of others.

• We suppose agents treat empirical distribution of others as *fixed*.

• *Consistency check:* that distribution should arise from agents' optimal strategies.

This is the notion of *mean field equilibrium (MFE).*

[ Glynn et al.; Weintraub et al.; Huang et al.; Adlakha et al.; Yin et al.; Lasry and Lions; Bodoh-Creed; Duffie et al. ]
Survey of results

1. **Approximation:**
   *Under what conditions is MFE a good approximation to behavior in a finite system?*

2. **Existence:**
   *Under what conditions does a MFE exist?*

3. **Convergence:**
   *When do natural learning algorithms converge to MFE?*
The model

Discrete time, infinite horizon

\( x_t : \) State of an agent at time \( t \)

\( a_t : \) Action of an agent at time \( t \)

\( x_{t+1} \sim P( \cdot | x_t, a_t) : \) State transition kernel

\( f_t : \) Empirical distribution of others’ states at time \( t \)

(the population state)

\( \pi(x_t, f_t) - c(a_t) : \)

Per period payoff

\( \beta : \) Discount factor \( (0 < \beta < 1) \)
The model: example

*Distributed coordination* (*Huang et al.*):
Consider a collection of devices trying to coordinate to a common state.

**Payoff:** \( \pi(x, f) = -(x - \text{mean}(f))^2 \)

**Cost:** \( c(a) = a^2 \)

**Dynamics:** \( x_{t+1} = A \ x_t + B \ a_t + w_t \),

where \( w_t \) is i.i.d. noise
The model: generalizations

- Payoff dependent on actions of others
- Payoff, dynamics heterogeneous across agents
- Payoff nonseparable in state and action
- Dynamics coupled across agents
- Interaction with a finite subset of other players at each time step
- Nonstationarity
### The model

An agent aims to maximize expected discounted payoff:

\[
V(x|\mu', \mu) = \mathbb{E}\left[ \sum_{t \geq 0} \beta^t (\pi(x_t, f_t) - c(a_t)) \mid x_0 = x, \mu', \mu \right]
\]

\(\mu'\) = this agent’s strategy

\(\mu\) = strategy followed by all others

Here \(\mu'\) is a **cognizant strategy**: it can depend on both \(x_t\) and \(f_t\)
The mean field model

When the number of agents is large, suppose:

An agent reacts only to the long run average state distribution $f$ of other players.

Mean field expected discounted payoff:

$$V(x | \mu) = E \left[ \sum_{t \geq 0} \beta^t \left( \pi(x_t, f) - c(a_t) \right) \bigg| x_0 = x, \mu \right]$$

Here $\mu$ is a **oblivious strategy**: it depends only on $x_t$
**Mean field equilibrium**

A strategy $\mu$ and a population state $f$ constitute a mean field equilibrium (MFE) if:

1. $\mu$ is an optimal oblivious strategy given $f$ and
2. $f$ is a steady state distribution of $\mu$

A MFE population state $f$ is a fixed point of $\Phi$: $f = \Phi(f)$
Our main insights

1. MFE is a good **approximation** under **continuity** and **compactness** conditions.

2. **Continuity** and **compactness** are (essentially) necessary conditions for **existence** of MFE in **concave games and supermodular games**. Thus approximation is a **corollary** of existence!

3. Technical challenges arise when the state space is unbounded or the dynamics are coupled, but similar results hold

4. In games with **complementarities**, simple learning dynamics converge to MFE
Ongoing applications

- Multi-armed bandit games
- Queueing games
- Dynamic resource allocation games
Asymptotic equilibrium

We first ask:

Is MFE a good approximation to equilibrium behavior of cognizant players?

A MFE \((\underline{\mu}, f)\) has the **AE property** if for all \(x\), as number of players \(\to \infty\),

\[
\sup_{\mu'} V(x|\mu', \underline{\mu}) - V(x|\mu, \underline{\mu}) \to 0,
\]

where the sup is over all cognizant strategies.

[Weintraub et al.]
Asymptotic equilibrium

Suppose:

(1) State and action spaces are compact
(2) Payoff is continuous in $f$

Theorem:
The asymptotic equilibrium property holds for any MFE $(\mu, f)$.

[ Adlakha, Johari, Weintraub, Goldsmith ]
Existence

The AE property can only hold if an MFE actually exists.

A central insight of this work is that the AE property is (essentially) a corollary of existence of MFE:

Existence also uses continuity and compactness.
Concave games

Assume compactness, continuity, and:

• Action space is convex
• $\pi(x, f)$ concave in $x$
• $c(a)$ convex in $a$
• $P(\cdot | x, a)$ has countable support, and is jointly stochastically concave in $x, a$

where at least one of the previous two is strict.
Concave games

**Theorem:**
There exists a MFE, and the AE property holds.

**Note:** via a slight variation, this result applies to finite state space, finite action space games.

[Adlakha, Johari, Weintraub, Goldsmith]