Noisy Network Coding

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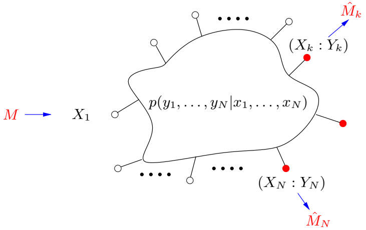
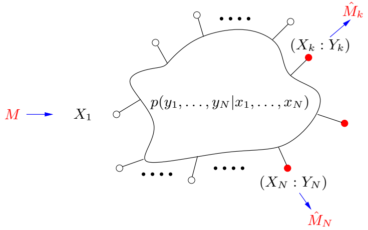
Noisy Multicast Network

- Consider an N-node discrete memoryless multicast network (DM-MN) $(\mathcal{X}_1 \times \cdots \times \mathcal{X}_N, p(y_2, \ldots, y_N | x_1, \ldots, x_N), \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_N)$



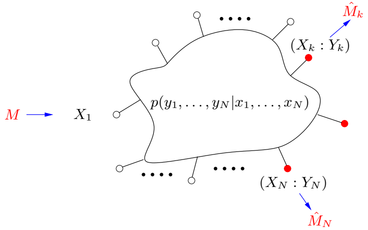
- Source node 1 wishes to send message M to destination nodes $\mathcal{D}$

Noisy Multicast Network



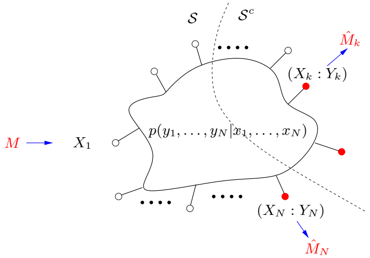
- A $(2^{nR}, n)$ code for the DM-MN:
  - Encoder: $x_1^n(m)$ for each message $m \in [1 : 2^{nR}]$
  - Relay encoder $j \in [2 : N]$: $x_{ji}(y_j^{i-1})$ for each $y_j^{i-1} \in \mathcal{Y}_j^{i-1}$, $i \in [1 : n]$
  - Decoder $k \in \mathcal{D}$: $\hat{m}_k(y_k^n)$ for each $y_k^n \in \mathcal{Y}_k^n$
- The average probability of error $P_e^{(n)} = \mathrm{P}\{\hat{M}_k \neq M \text{ for some } k \in \mathcal{D}\}$

Noisy Multicast Network



- Rate R achievable if there exists a sequence of codes with $P_e^{(n)} \to 0$
- The capacity C of the DM-MN is supremum of achievable rates
- Capacity is not known in general
- There are upper and lower bounds that coincide in some special cases

Cutset Upper Bound



- $X(\mathcal{S})$ inputs in $\mathcal{S}$; $X(\mathcal{S}^c)$, $Y(\mathcal{S}^c)$ inputs/outputs in $\mathcal{S}^c$

Cutset upper bound (EG 1981)

$$C \le \max_{p(x^N)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S},\, k \in \mathcal{S}^c} I(X(\mathcal{S}); Y(\mathcal{S}^c) | X(\mathcal{S}^c))$$

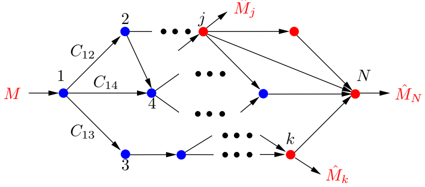
Capacity Results and Lower Bounds

- Shannon (1948) established the capacity of noisy point-to-point channel using random coding
- Ford, Fulkerson (1956) (also Elias, Feinstein, Shannon) established the capacity of noiseless unicast network using forwarding
- Ahlswede, Cai, Li, Yeung (2000) established the capacity of noiseless multicast network using network coding
- Network coding extended to multi-source multicast erasure network by Dana, Gowaikar, Palanki, Hassibi, Effros (2006)
- Network coding extended to obtain lower bound on capacity of multicast deterministic network by Avestimehr, Diggavi, Tse (2007)
- In earlier development, Cover, EG (1979) developed compress–forward scheme for the relay channel

- EG, Kim (Lecture Notes on NIT 2009) developed a noisy network coding scheme that unifies and extends above results (Lim, Kim, Chung, EG ISIT 2010, WiNC 2010)

Noiseless Multicast Network

- Consider noiseless network modeled by graph $(\mathcal{N}, \mathcal{E})$

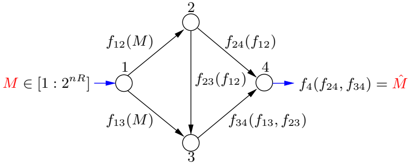


- Node 1 wishes to send M to set of destination nodes $\mathcal{D}$
- Capacity region coincides with cutset bound

Network Coding Theorem (Ahlswede, Cai, Li, Yeung 2000)

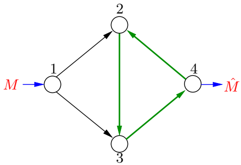
$$C \leq \min_{k \in \mathcal{D}} \min_{\mathcal{S}:\, 1 \in \mathcal{S},\, k \in \mathcal{S}^c} C(\mathcal{S})$$

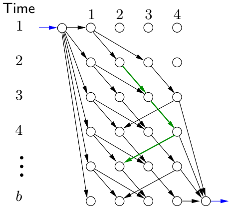
Outline of Proof: Acyclic Network



- Wolog assume zero node delay
- Use block coding (assume C_{jk} are integer valued)
- Random codebook generation:
  $f_{jk} \in [1 : 2^{nC_{jk}}]$, $(j,k) \in \mathcal{E}$, and f_4 are randomly and independently generated, each according to uniform pmf
- Key step: If $R < \min_{\mathcal{S}} C(\mathcal{S})$, $f_4(m)$ is one-to-one with high prob.
- Koetter, Medard (2003) showed that cutset bound can be achieved with zero error using linear network coding

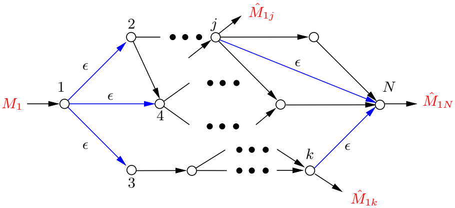
Outline of Proof: Cyclic Network





- Cannot assume zero delay nodes. Assume unit delay at each node
- Unfold to time extended (acyclic) network with b blocks
- Key step: Min-cut capacity of the new network is $\approx bC$ for b large
- By result for acyclic case, min-cut capacity for the new network is achievable
- Need to send same message b times using independent mappings

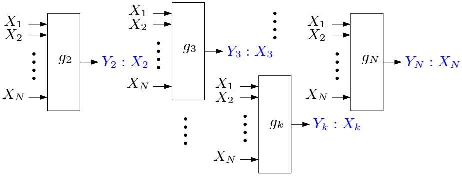
Multicast Erasure Network



- Source nodes wish to send their messages to destination nodes $\mathcal{D}$
- Link failure is observed as an erasure symbol; destination nodes have access to network erasure pattern (includes noiseless case)
- Capacity region coincides with cutset bound and achieved via network coding (Dana, Gowaikar, Palanki, Hassibi, Effros 2006)

Deterministic Multicast Network

- Generalizes noiseless multicast network with broadcast, interference



- Node 1 wishes to send message to subset of nodes $\mathcal{D}$
- Node 1 sends $x_{1i}(m)$ and node j sends $x_{ji}(y_j^{i-1})$ at time $i \in [1:n]$
- Capacity is not known in general
- Cutset upper bound reduces to

$$C \leq \max_{p(x^N)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} H(Y(\mathcal{S}^c)|X(\mathcal{S}^c))$$

Deterministic Multicast Network

Lower bound on capacity (Avestimehr, Diggavi, Tse 2007)

$$C \geq \max_{\prod_{j=1}^N p(x_j)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} H(Y(\mathcal{S}^c)|X(\mathcal{S}^c))$$

- Cutset bound:

$$C \leq \max_{p(x^N)} \min_{k \in \mathcal{D}} \min_{\mathcal{S}: 1 \in \mathcal{S}, k \in \mathcal{S}^c} H(Y(\mathcal{S}^c)|X(\mathcal{S}^c))$$

- Bounds coincide for:

  No interference (Ratnakar, Kramer 2006):

  $$Y_k = (y_{k1}(X_1), \ldots, y_{kN}(X_N)),\ k \in [2:N]$$

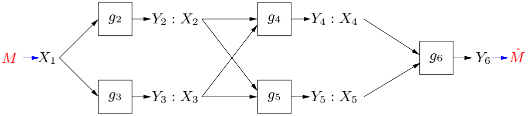
  Finite-field network (Avestimehr, Diggavi, Tse 2007):

  $$Y_k = \sum_{j=1}^N g_{jk} X_j \text{ for } g_{jk}, X_j \in \mathbb{F}_q,\ j \in [1:N],\ k \in [2:N]$$

  Used to approximate capacity of Gaussian networks in high SNR

Outline of Proof

- Layered networks:

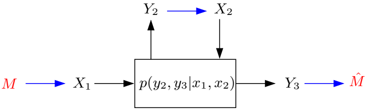


  - Random codebook generation:
    Randomly and independently generate $x_j^n(y_j^n)$ for each sequence y_j^n
  - Key step: If R satisfies lower bound, end-to-end mapping is one-to-one with high probability

- Non-layered network:
  - Construct time extended (layered) network with b blocks
  - Key step: If R satisfies lower bound, end-to-end mapping is one-to-one with high probability
  - Again send the same message b times using independent mappings

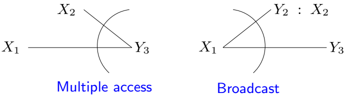
Relay Channel

- The relay channel (van der Meulen 1971) is a 3-node DMN

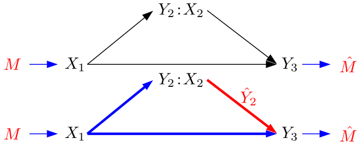


- Node 1 wishes to send M to node 3 with help of node 3 (relay)
- Capacity is not known in general
- Cutset upper bound reduces to (Cover, EG 1979)

$$C \leq \max_{p(x_1,x_2)} \min \{I(X_1, X_2; Y_3), I(X_1; Y_2, Y_3|X_2)\}$$



Compress–Forward Lower Bound



- The relay compresses its received signal and forwards it to receiver

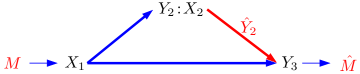
Compress–forward lower bound (Cover, EG 1979)

$$C \geq \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)} I(X_1; \hat{Y}_2, Y_3|X_2)$$

subject to $I(X_2; Y_1) \geq I(Y_2; \hat{Y}_2|X_2, Y_3)$

- Cutset bound:
$$C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3), I(X_1; Y_2, Y_3|X_2)\}$$

Outline of Proof



- Send $b-1$ independent messages over b, n-transmission blocks
- At the end of block j, relay chooses description $\hat{y}_2^n(j)$ of $y_2^n(j)$
- Since the receiver has side information $y_3^n(j)$ about $\hat{y}_2^n(j)$, we use Wyner–Ziv coding to reduce rate necessary to send $\hat{y}_2^n(j)$
  The bin index is sent to the receiver in block $j+1$ via $x_2^n(j+1)$
- At the end of block $j+1$, the receiver first decodes $x_2^n(j+1)$ from which it finds $\hat{y}_2^n(j)$
  It then finds unique $\hat{m}_j$ such that $(x_1^n(\hat{m}_j), x_2^n(j), \hat{y}_2^n(j)), y_3^n(j)$ are jointly typical typical

Equivalent Compress–Forward Lower Bound

Compress–forward lower bound (EG, Mohseni, Zahedi 2006)

$$C \geq \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)} \min\{I(X_1, X_2; Y_3) - I(Y_2; \hat{Y}_2|X_1, X_2, Y_3), \; I(X_1; \hat{Y}_2, Y_3|X_2)\}$$

- Compress–forward lower bound (Cover, EG 1979):
  $$C \geq \max_{p(x_1)p(x_2)p(\hat{y}_2|y_2,x_2)} I(X_1; \hat{Y}_2, Y_3|X_2)$$
  subject to $I(X_2; Y_3) \geq I(Y_2; \hat{Y}_2|X_2, Y_3)$
- Cutset bound:
  $$C \leq \max_{p(x_1,x_2)} \min\{I(X_1, X_2; Y_3), I(X_1; Y_2, Y_3|X_1)\}$$
- This characterization generalizes naturally to networks
- Generalization yields strictly higher rates than extension of original characterization by Kramer, Gastpar, Gupta (2005)

Main Result: Noisy Network Coding Lower Bound

- The alternative characterization of compress–forward lower bound for relay channel generalizes to discrete memoryless multicast network

Theorem (EG, Kim Lecture on NIT 2009)

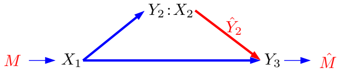
$$C \geq \max \min_{k \in \mathcal{D}} \min_{\substack{\mathcal{S} \subseteq [1:N] \\ 1 \in \mathcal{S},\, k \in \mathcal{S}^c}} \big(I(X(\mathcal{S}); \hat{Y}(\mathcal{S}^c), Y_k | X(\mathcal{S}^c)) - I(Y(\mathcal{S}); \hat{Y}(\mathcal{S}) | X^N, \hat{Y}(\mathcal{S}^c), Y_k) \big),$$

where the maximum is over $\prod_{k=1}^{N} p(x_k)p(\hat{y}_k|y_k, x_k)$

- Includes as special cases:
  - Capacity of noiseless multicast networks
  - Lower bound on deterministic multicast networks
  - Capacity of wireless erasure muticast networks
- Shows that network coding is a special case of compress–forward ☺
- Simpler and more general proof (deals directly with cyclic networks)

Proof Outline

- Source node sends same message b times; relays use compress–forward; decoders use simultaneous decoding
- No binning; don't require decoding compression indices correctly!
- For simplicity, consider proof for relay channel



- The relay uses independently generated compression codebooks:
$$\mathcal{B}_j = \{\hat{y}_2^n(l_j|l_{j-1}) : l_j, l_{j-1} \in [1 : 2^{nR_2}]\},\ j \in [1 : b]$$
l_{j-1} is compression index of $\hat{Y}_2^n(j-1)$ sent by relay in block j
- The senders use independently generated transmission codebooks:
$$\mathcal{C}_j = \{(x_1^n(j, m), x_2^n(l_{j-1})) : m \in [1 : 2^{nbR}],\ l_{j-1} \in [1 : 2^{nR_2}]\}$$
- Encoding: Sender transmits $X_1^n(j, m)$ in block $j \in [1 : b]$
Upon receiving $Y_2^n(j)$ and knowing $X_2^n(l_{j-1})$, the relay finds jointly typical $\hat{Y}_2^n(l_j|l_{j-1})$, and sends $X_2^n(l_j)$ in block $j + 1$

Outline of Proof

Block	1	2	3	...	$b-1$	b
X_1	$x_1^n(1,m)$	$x_1^n(2,m)$	$x_1^n(3,m)$	...	$x_1^n(b-1,m)$	$x_1^n(b,m)$
Y_2	$\hat{y}_2^n(l_1\|1)$	$\hat{y}_2^n(l_2\|l_1)$	$\hat{y}_2^n(l_3\|l_2)$	...	$\hat{y}_2^n(l_{b-1}\|l_{b-2})$	$\hat{y}_2^n(l_b\|l_{b-1})$
X_2	$x_2^n(1)$	$x_2^n(l_1)$	$x_2^n(l_2)$	...	$x_2^n(l_{b-2})$	$x_2^n(l_{b-1})$
Y_3	$\emptyset$	$\emptyset$	$\emptyset$	...	$\emptyset$	$\hat{m}$

- Decoding: After receiving $Y_3^n(j)$, $j \in [1 : 2^{nR}]$, the receiver finds unique $\hat{m}$ such that:
$$(x_1^n(j, \hat{m}), \hat{y}_2^n(l_j|l_{j-1}), x_2^n(l_{j-1}), y_3^n(j)) \text{ are jointly typical}$$
for all $j \in [1 : b]$ and for some $l_1, l_2, \ldots, l_b$

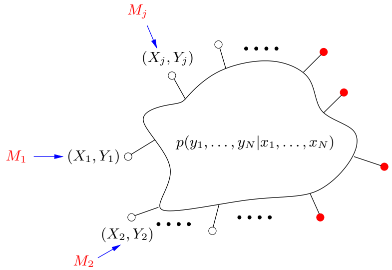
Analysis of the Probability of Error

- Assume $M = 1$, $L_1 = L_2 = \cdots = L_b = 1$, and let $\mathcal{E}_j(m, l_{j-1}, l_j) = \{(X_1^n(j,m), \hat{Y}_2^n(l_j|l_{j-1}), X_2^n(l_{j-1}), Y_3^n(j)) \in \mathcal{T}_\epsilon^{(n)}\}$
- The average probability of error

$$\begin{aligned} \mathrm{P}(\mathcal{E}) &\le \mathrm{P}(\cup_{j=1}^b \mathcal{E}_j^c(1,1,1)) + \mathrm{P}(\cup_{m\neq 1} \cup_{l^b} \cap_{j=1}^b \mathcal{E}_j(m, l_{j-1}, l_j)) \\ &\le \sum_{j=1}^b \mathrm{P}(\mathcal{E}_j^c(1,1,1)) + \sum_{m\neq 1}\sum_{l^b} \prod_{j=2}^b \mathrm{P}(\mathcal{E}_j(m, l_{j-1}, l_j)) \end{aligned}$$

- If $m \neq 1$ and $l_{j-1} = 1$,
$$\mathrm{P}(\mathcal{E}_j(m, l_{j-1}, l_j)) \le 2^{-n(I(X_1;\hat{Y}_2,Y_3|X_2)-\delta(\epsilon))}$$
- If $m \neq 1$ and $l_{j-1} \neq 1$,
$$\mathrm{P}(\mathcal{E}_j(m, l_{j-1}, l_j)) \le 2^{-n(I(X_1,X_2;Y_3)+I(\hat{Y}_2;X_1,Y_3|X_2)-\delta(\epsilon))}$$
- The rest of the proof is just algebra ☺

Extension: Noisy Multi-source Multicast Network



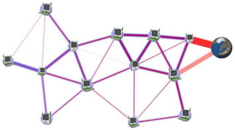
- Noisy network coding generalizes to this case (Lim, Kim, El Gamal, Chung ITW 2010)
- Extends result on erasure networks (Dana, Gowaikar, Palanki, Hassibi, Effros 2006) and deterministic networks (Perron 2009)

Extension: Noisy Multi-unicast Network

- Noisy network coding extends naturally to multi-unicast networks
- For example, consider an N-node DMN where node 1 wishes to send M_j to node 3, and node 2 wishes to send M_j to node 4
- Using noisy network coding, we view the network as an interference channel with senders X_1 and X_2 and respective receivers $(Y_3, \hat{Y}_5, \hat{Y}_6, \ldots, \hat{Y}_N)$ and $(Y_4, \hat{Y}_5, \hat{Y}_6, \ldots, \hat{Y}_N)$
- We use coding strategies for interference channel (ISIT 2010):
  - Each receiver decodes only its message (treats interference as noise)
  - Each receiver decodes both messages
  - One receiver uses the former strategy, the other uses the later
- Each relay can generate different $\hat{Y}$ for each destination node (WiNC 2010)

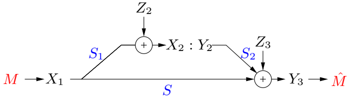
How Good Is Noisy Network Coding

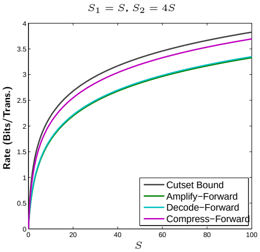
- In noisy network coding we compress signals and retransmit them
- This has the advantage that relays don't need to know codebooks
- It is also a good strategy in "high SNR" (e.g., noiseless links)
- However, noisy network coding is not always a good strategy
  Consider cascade of noisy channels:
  - Optimal strategy is for each relay to use decode–forward
- This is also what we do in wireless mesh networks

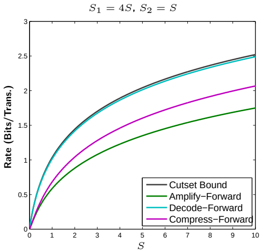


- Another strategy is amplify–forward (analog-to-analog)

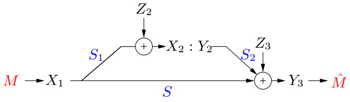
Example 1: AWGN Relay Channel







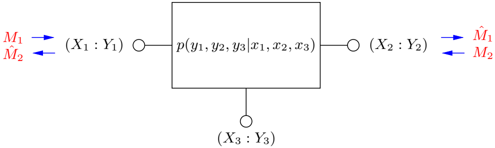
Example 1: AWGN Relay Channel



- Decode–forward: within $1/2$ bit of cutset bound
- Compress–forward: within $1/2$-bit of cutset bound (Chang, Chung, Lee 2008)
- Amplify–forward: within 1-bit of cutset bound (Chang, Chung, Lee 2008)

Example 2: AWGN Two-Way Relay

- Two-way relay channel is a 3-node DMN



- Node 1 wishes to send message M_1 to node 2
  Node 2 wishes to send message M_2 to node 1
- AWGN two-way relay:

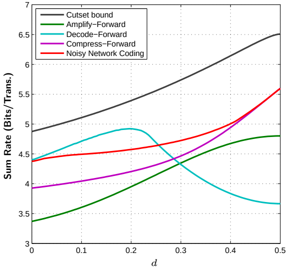
$$Y_k = \sum_{j \neq k} g_{jk} X_j + Z_k \text{ for } k = 1, 2, 3,$$

where $Z_k \sim \mathrm{N}(0, 1)$. Power constraint P on every sender

Example 2: AWGN Two-Way Relay Channel

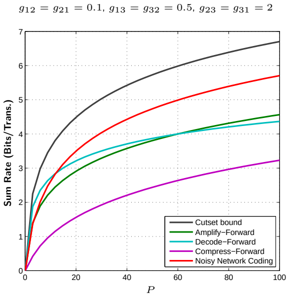
- Extensions of decode–forward, compress–forward, and amplify–forward compared by Rankov, Wittneben (2006) and Katti, Maric, Goldsmith, Katabi, Medard (2007) among others

Node 1 to 2 distance: 1; node 1 to 3 distance: $d \in [0, 1]$; $g_{13} = g_{31} = d^{-3/2}$, $g_{23} = g_{32} = (1 - d)^{-3/2}$



Example 2: AWGN Two-Way Relay Channel

- NNC is within 1 bit of cutset bound for all channel gains
- Gap unbounded for all other schemes



Example 3: Multi-source Multicast Gaussian Network

- Channel model: $Y^N = GX^N + Z^N$
- The cutset bound yields

$$\sum_{j\in\mathcal{S}} R_j \le \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G(\mathcal{S})^T\right| + \frac{\min\{|\mathcal{S}|,|\mathcal{S}^c|\}}{2}\log(2|\mathcal{S}|)$$

- With $\hat{Y}_j = Y_j + \hat{Z}_j$, $\hat{Z}_j \sim \mathrm{N}(0,1)$, noisy network coding bound yields

$$\sum_{j\in\mathcal{S}} R_j < \frac{1}{2}\log\left|I + \frac{P}{2}G(\mathcal{S})G(\mathcal{S})^T\right| - \frac{|\mathcal{S}|}{2}$$

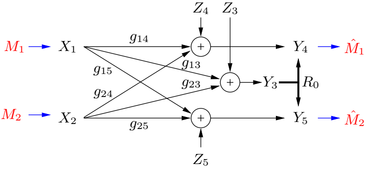
- Thus, noisy network coding is optimal within $(N/4)\log(2N)$ bits/trans. for $N > 3$
- This generalizes and improves single-source result by Avestimehr, Diggavi, Tse (2007)
- 2-way relay: Unbounded gap for decode–forward / amplify–forward

Example 4: AWGN Interference Relay Channel

- Consider a 2-user pair interference channel with a relay:

$$Y_k = g_{1k}X_1 + g_{2k}X_2 + Z_k \text{ for } k = 3,4,5$$

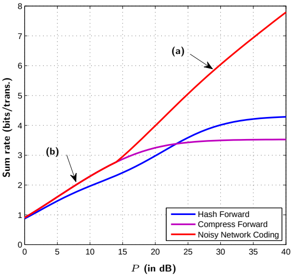
$Z_k \sim \mathrm{N}(0,1)$ and power constraint P on each sender



Example 4: AWGN Interference Relay Channel

- Razaghi–Yu (2010) compared compress–forward, hash–forward (Cover, Kim 2007)

$$g_{14} = g_{25} = 1,\ g_{15} = g_{24} = g_{13} = 0.5,\ g_{23} = 0.1,\ R_0 = 1$$



(a) treating interference as noise (b) decoding both messages

Conclusion

- Network coding and its generalizations to erasure and deterministic networks are special cases of compress–forward
- Noisy network coding lower bound can be extended to non-multicast messaging requirements
- Many interesting areas to explore
  - Applications of noisy network coding to wireless networks, ...
  - Combining noisy network coding with (partial) decode (compute)–forward
- To learn more:
  Lecture notes on NIT at: http://arxiv.org/abs/1001.3404
  Papers at: http://arxiv.org/abs/1002.3188, ISIT 2010, WiNC 2010