Lattice of Gambles
(What do you lose when you gamble?)

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Abstract

A gambler starts with \$1. He then plays a sequence of gambles.
We pose the following questions:

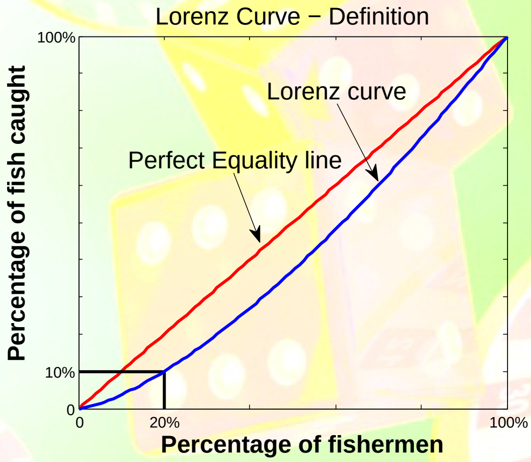
- What does he lose in a fair gamble?

$$1 \to X, X \geq 0, EX = 1$$

- Can private randomness help?
- Can he strategically combine a series of unfair gambles to reduce his loss?

Definition: Lorenz curve

- Worst 20% of fishermen catch 10% of fish.
- Best 20% of fishermen catch 30% of fish.



Defintion from wikipedia

The Lorenz curve is a graph showing the proportion of the distribution assumed by the bottom y%. It is often used to represent income distribution, where it shows for the bottom x% of households, what percentage y% of the total income they have.

Definition: Lorenz curve continued

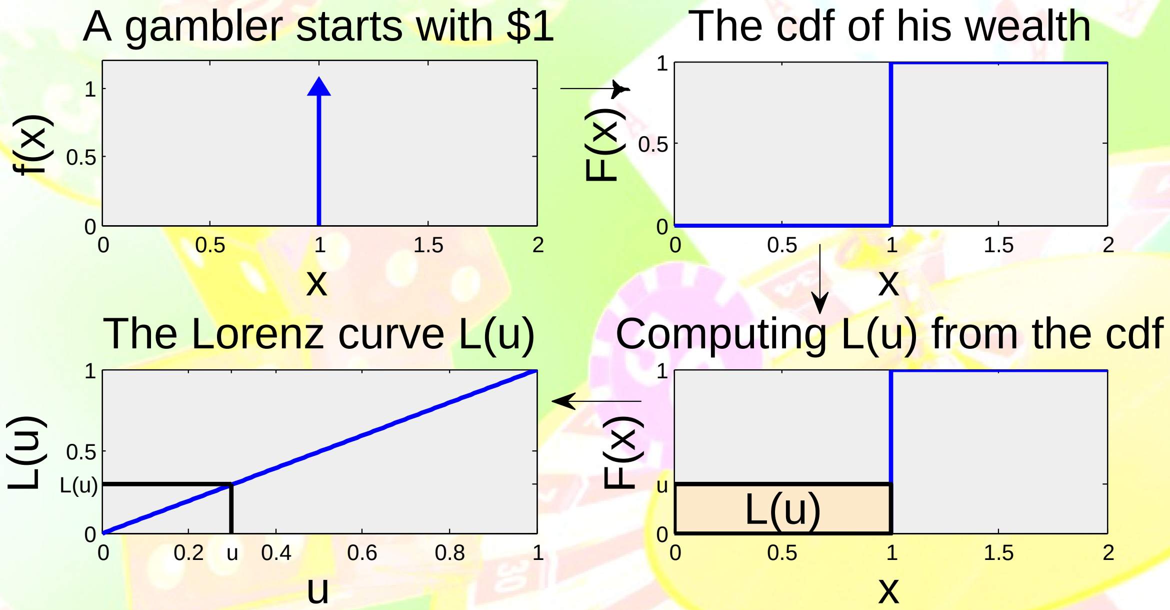
Definition

Let $X \sim F(X), X \geq 0$ be a random variable with cdf F(x). Then the Lorenz curve is

$$L(u) = \int_0^u F^{-1}(v)dv$$

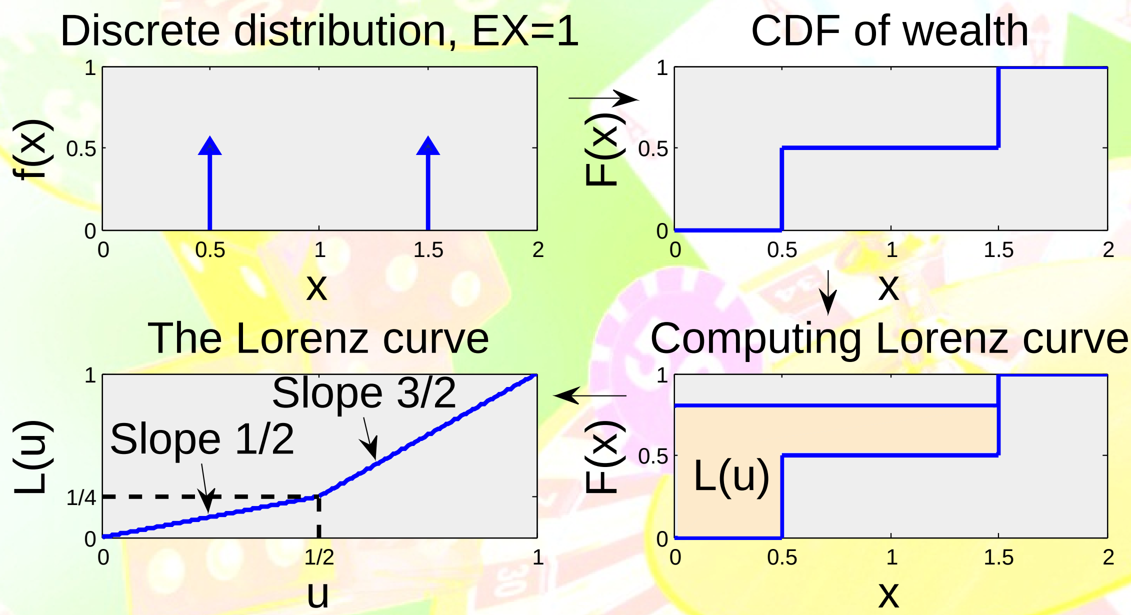
Graph to be placed.

Example 1: Perfect Equality



Examples

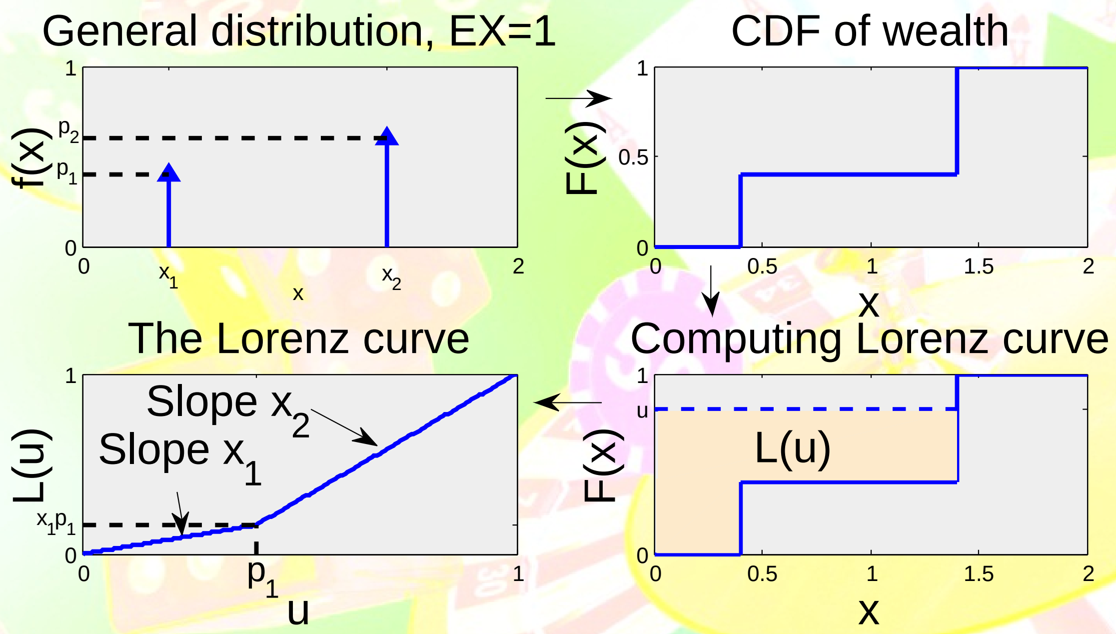
Example 2: A discrete probability distribution, EX=1.



Tom Cover Lattice of Gambles

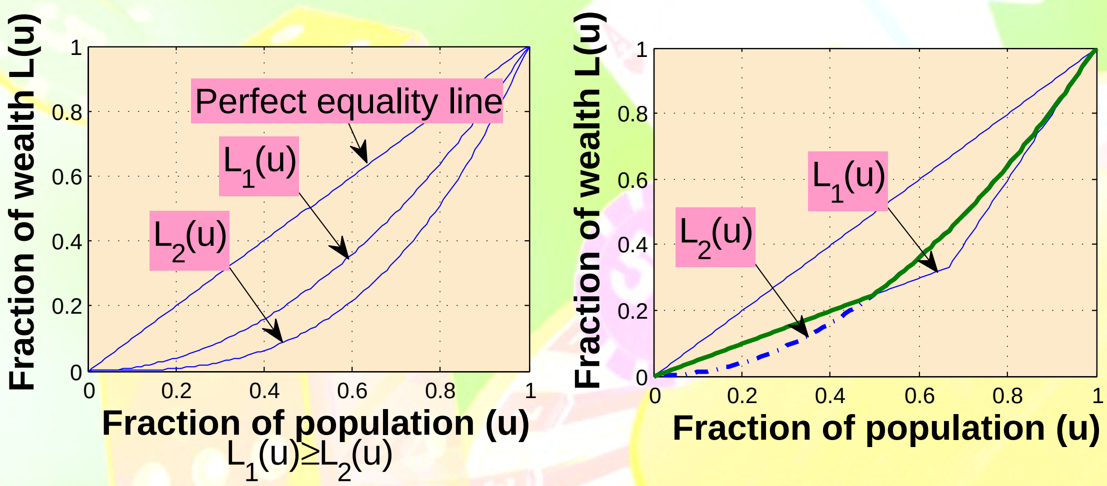
Examples

Example 3: A more general distribution, EX=1.



Gambling increases inequality

$$X \sim F(X),\ X \geq 0,\ L(u) = \int_0^u F^{-1}(v)dv$$



Gambling increases inequality

- Let F_1 and F_2 correspond to the laws of the random variables X_1 and X_2 respectively, $X_1, X_2 \geq 0, EX_1 = EX_2 = 1$.
- Let the corresponding Lorenz Curves be denoted L_1 and L_2 respectively.

Theorem

There is a sequence of fair binary gambles that starts from distribution X_1 and ends up with a distribution X_2 iff

$$L_1(u) \geq L_2(u) \quad \forall u \in [0, 1]. \tag{1}$$

Achievability proof

- Let $U_1 = F_1(X_1)$. Also, let $X_1' = F_2^{-1}(U_1)$.
- If $X_1' = X_1$ then do nothing.
- If $X_1' < X_1$, find the minimum $u > U_1$ such that $L_1(u) - L_2(u) = L_1(U_1) - L_2(U_2)$ and label it U_1''.
- Let $X_1'' = F_2^{-1}(U_1'')$.
- Gamble to the two points X_1' and X_1'' with the proper weights such that it is a fair gamble.
- In the opposite case where $X_1' > X_1$, do the same thing using the maximum $u < U_1$ meeting the named condition.

Lattice of Gambles

Lorenz curve gives partial ordering:

$F_1(x) \preceq F_2(x) \Leftrightarrow L_1(u) \geq L_2(u)$

(Graph to be placed)

definition

We define

$$L_1 \cup L_2 = \max\{L_1(u), L_2(u)\}$$
$$L_1 \cap L_2 = \text{Convex Hull of } L_1, L_2$$

Lattice of Gambles continued

Thus $X \equiv 1 \quad (L(u) = u)$ is the maximal element.
Example:
(Insert figure showing $\underline{f}$ and $\bar{f}$.)

Some randomness is free

- Suppose the casino offers binary fair bets:

$$X \to \begin{cases} 2X, & \text{prob}\frac{1}{2} \\ 0, & \text{prob}\frac{1}{2} \end{cases}$$

- Suppose a gambler starts with \$1.
- Suppose that he wants to achieve a uniform distribution unif$[0,2]$ on his wealth after gambling.

There is more than one possible gambling strategy to achieve this.

Some randomness is free

Method 1:

- Bet $\$\frac{1}{2}$, then bet $\$\frac{1}{4}$, then $\$\frac{1}{8}$, ...
- After infinite bets the distribution of wealth is unif$[0,2]$.

Here, the gambler must place on the casino's table a total of

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$$

Method 2:

- Generate $Y \sim$ unif$[0,1]$. ("Air" bet)
- Bet \$Y. ("physical" bet)

Here, the gambler places on the table a total amount

$$E(Y)=\frac{1}{2}<1$$

Fair Prices of Unfair Gambles

Let $f : [0,1] \to [0,1]$ be a binary gamble menu offered by a casino, that is, for each $x \in [0,1]$,

$$x \to \begin{cases} 1, & \text{prob} \quad f(x) \\ 0, & \text{prob} \quad 1 - f(x) \end{cases}$$

The gambling menu is fair if $f(x) = x$.

Definition

A gambling menu is **stable** if it cannot be gamed to produce a more attractive menu.

Fair Prices of Unfair Gambles Continued

Theorem

Necessary and sufficient condition for a stable gambling menu:

$$\forall\ a, b, \theta \in [0,1] \text{ where } b > a$$
$$f(a + (b-a)\theta) \geq f(a) + (f(b) - f(a))f(\theta).$$

Example:

- Suppose $f(x) = px,\ p < 1$, that is, the casino charges a fixed fraction p in expected value. This is not price stable.
- A clever gambler would use an infinite sequence of very low probability and low cost gambles to achieve $f(x) = 1 - (1-x)^p$. This is price stable!

Martingale Convergence

Let S_n be the sequence of R.V's generated by fair gambles from $S_0 = 1$.

Theorem

$S_n \to S$ in distribution.

proof

$$L_n(u) \searrow L(u)$$
$$F_S(s) = L'^{-1}(s)$$

Conclusions

- Lattice of fair gambles
- Wealth from gambles converges in distribution
- Gambles divide into "air" gambles and "physical" gambles.
- What do you lose when you gamble?
  Answer: Position you can never get back.

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