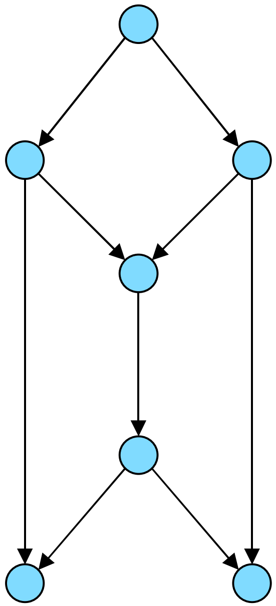
A tool oriented approach to network capacity

Ralf Koetter
ralf.koetter@tum.de

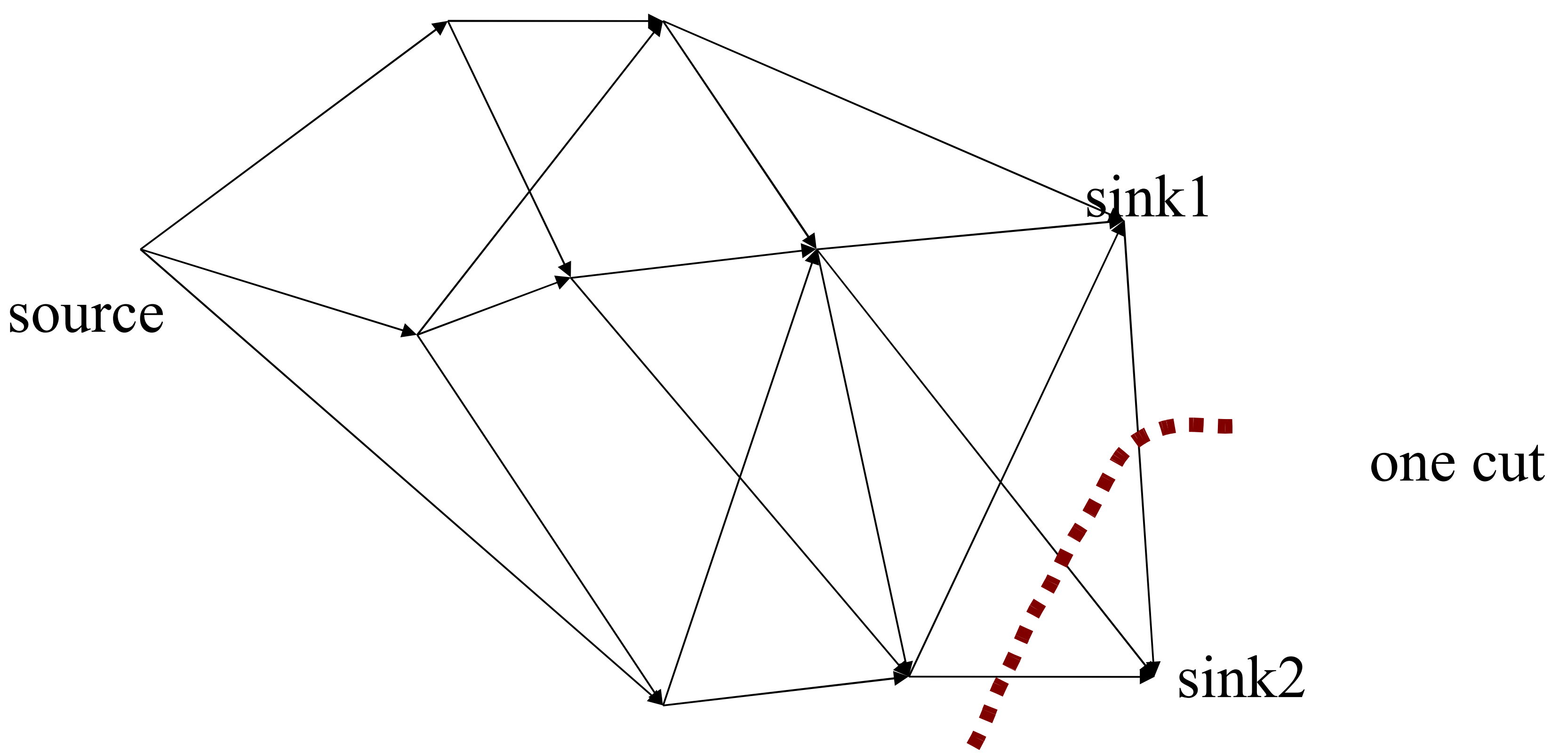
Michelle Effros
effros@caltech.edu

Muriel Medard
medard@mit.edu



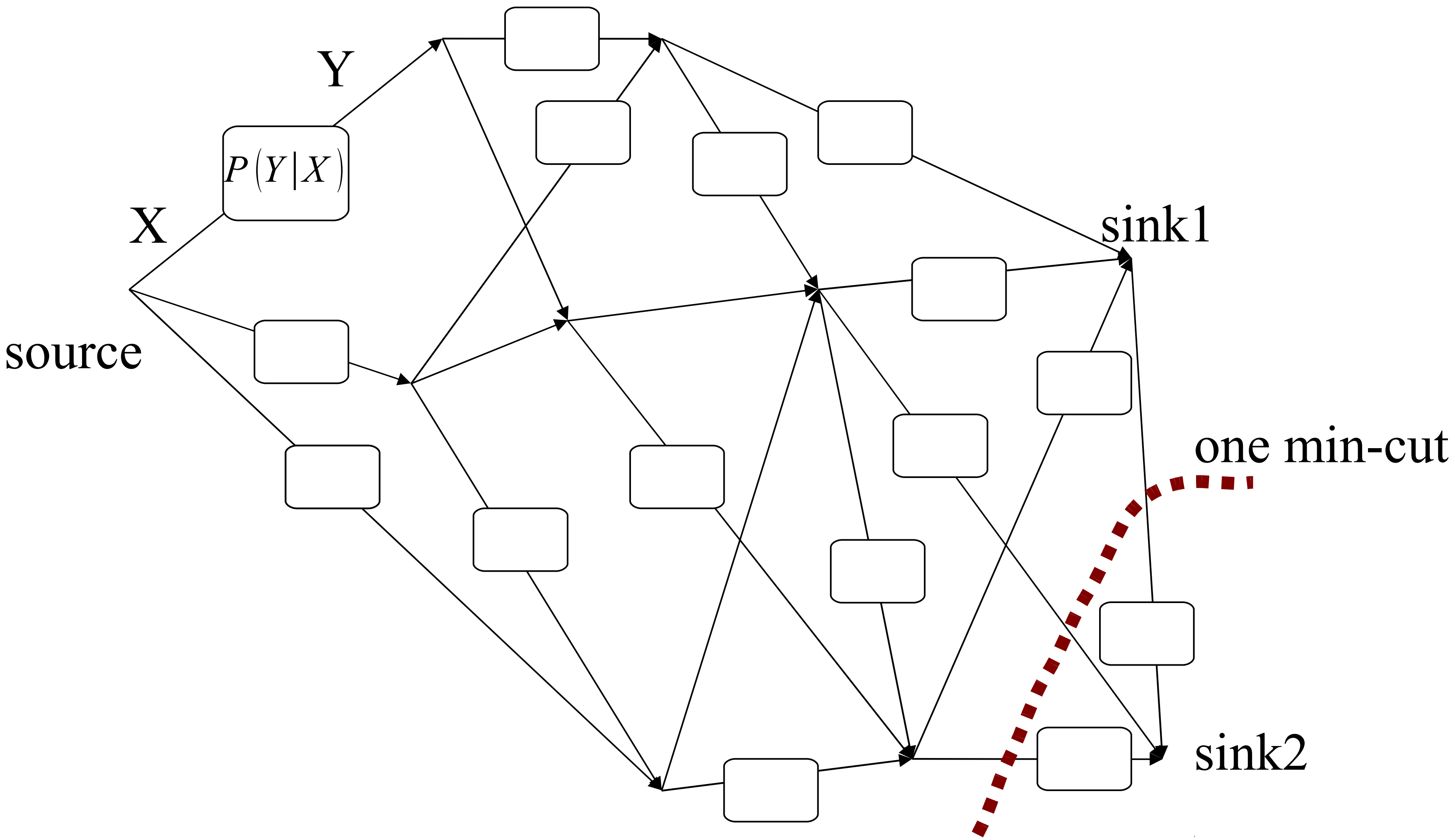
The main theorem of NC [Ahlswede, Cai, Li Young, 2001]

Links are error-free bit pipes.



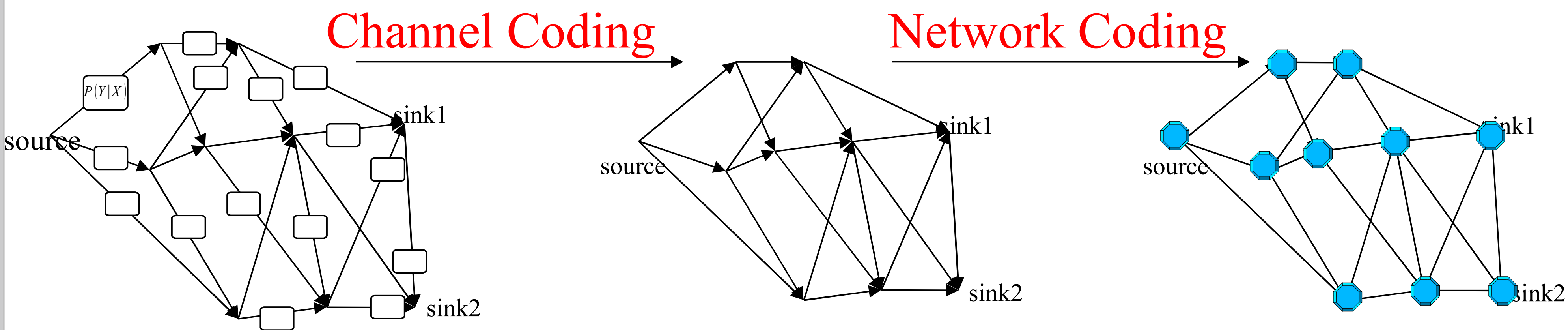
The network can transport at most the “min-cut” amount of randomness towards any sink. By sufficiently rich mixing of the data the max amount of randomness can be simultaneously achieved for all sinks. Since the only source of„randomness“ is the source, the inverse problem of infering the source data can be solved for large enough min cuts.

What if the links are not bit-pipes

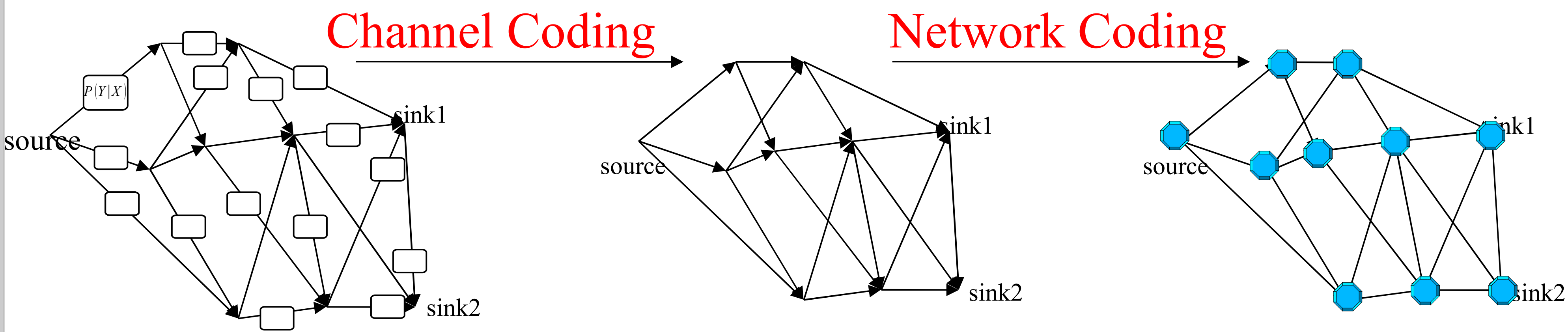


Every link is now a probabilistic channel that may operate at rates above capacity (albeit with a penalty in reliability). Does "SEPARATION" between network coding and channel coding hold?

SEPARATION: What if the links are not bit-pipes



What if the links are not bit-pipes



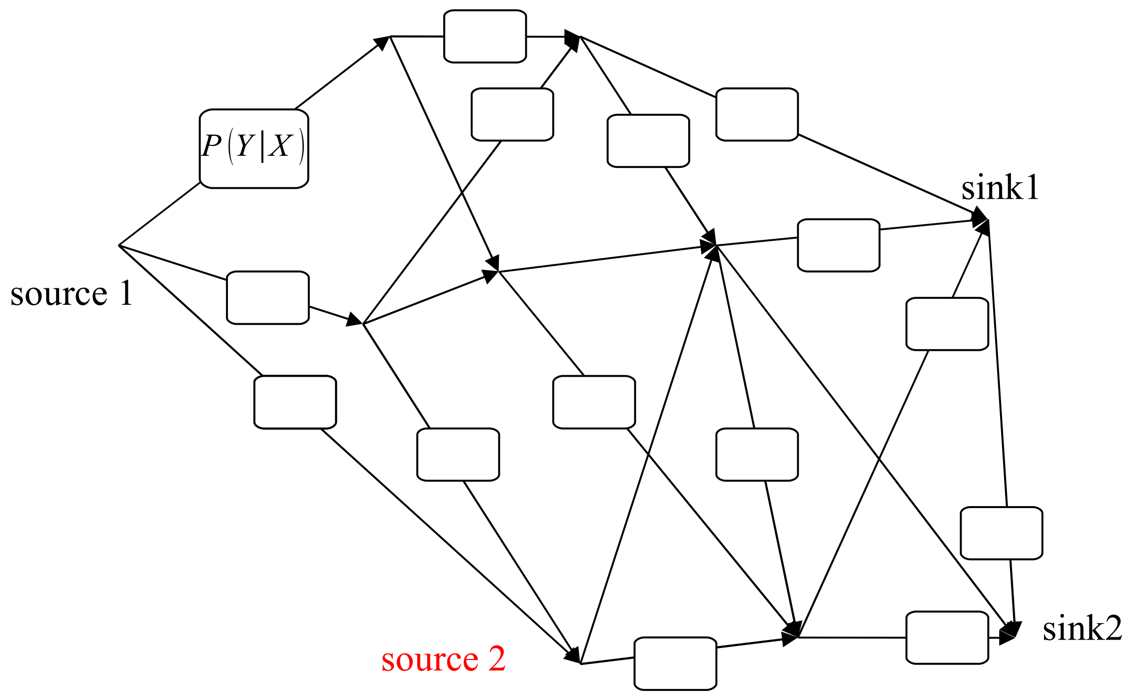
Separation holds for the multicast!

L. Song and R. W. Yeung and N. Cai, A separation theorem for single source network coding, IEEE Transactions on Information Theory, vol. 52, no. 5, pp. 1861-1871, May 2006

Shashibhushan Borade, Network Information Flow: Limits and Achievability, Proc. IEEE International Symposium on Information Theory, July 2002.,

<u>Proof outline:</u> Fano's inequality -> cutset bounds as outer bounds -> Separation approach achieves outer bound -> QED!

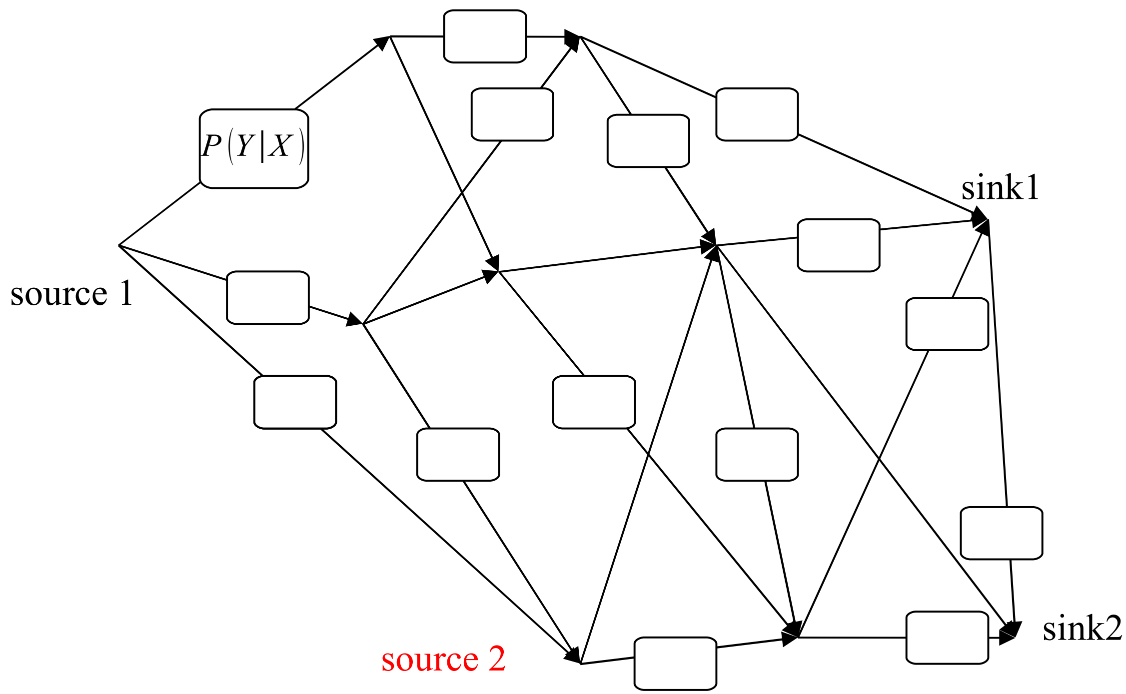
What if the links are not bit-pipes and non-multicast?



1) What is the rate region of the network for arbitrary (in particular non-multicast) demands?

2) Is this rate region achievable with channel/network-coding separation?

What if the links are not bit-pipes and non-multicast?



tough problem

1) What is the rate region of the network for arbitrary (in particular non-multicast) demands?

2) Is this rate region achievable with channel/network-coding separation?

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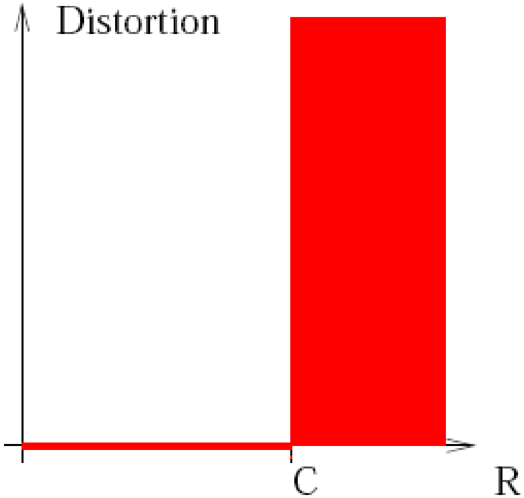
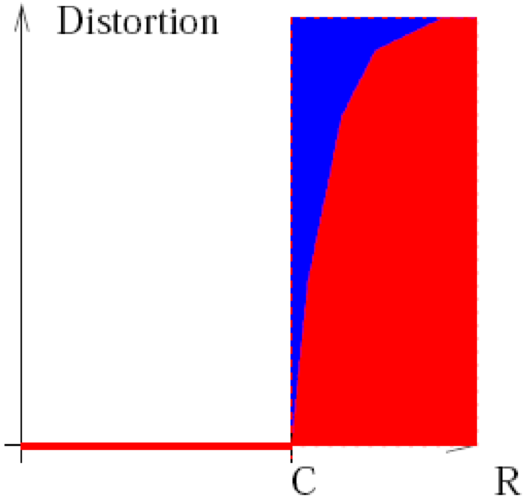
Noisy channels allow for considerably more operational modes than bit-pipes



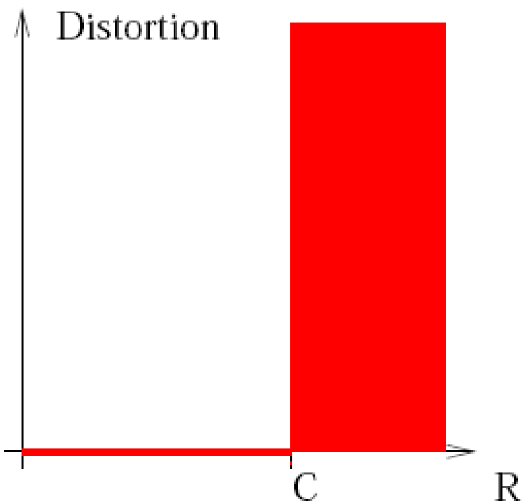
a capacitated link



a "noisy" channel

Can the blue area help?

Why is this difficult ?



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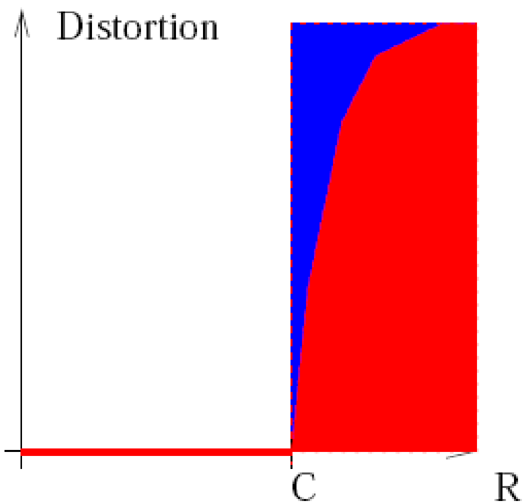
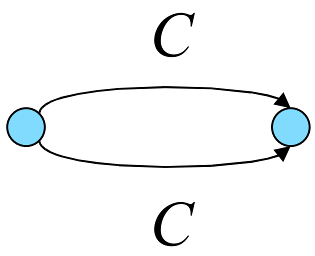


a "noisy" channel



Capacity *2C*
(best to use each independent link at rate 2C and to correlate the inputs)

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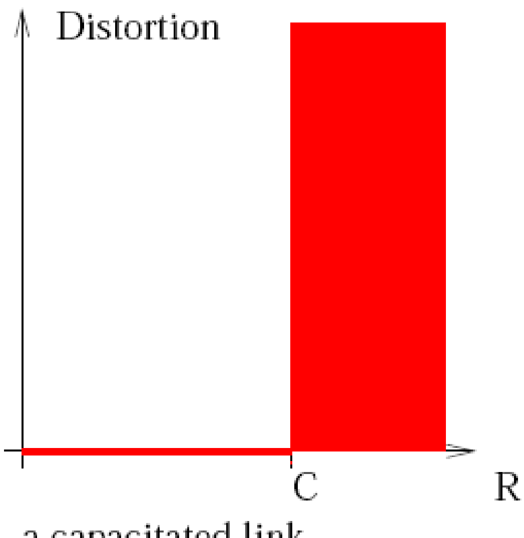
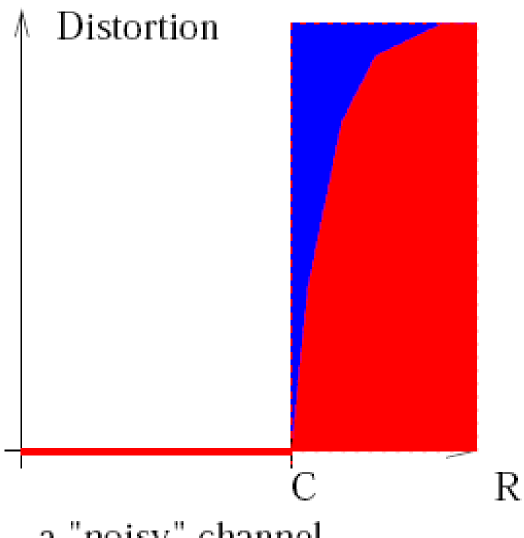
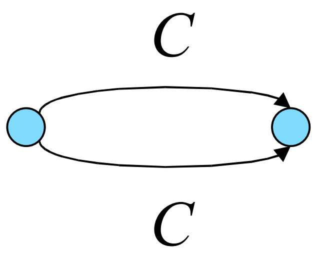
Why is this difficult ?







Capacity *2C*
(best to use each link at rate 2C and to correlate the inputs)

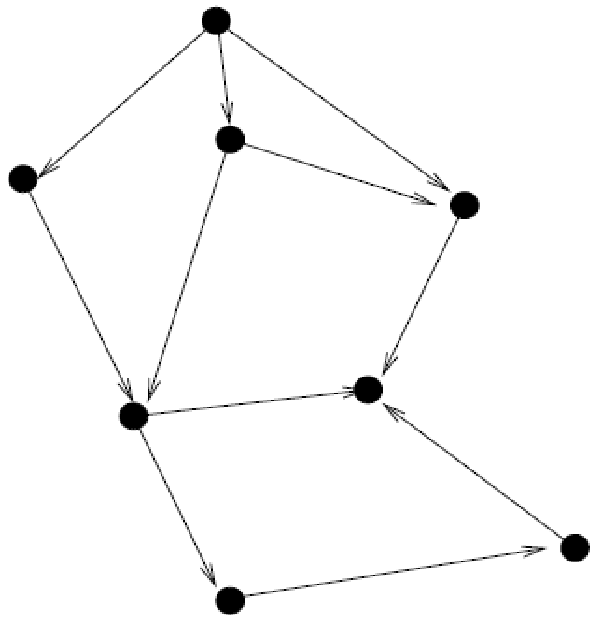
Can the blue area help?

S. Shamai and S. Verdu, Capacity of channels with uncoded side information, Europ. Trans. Telecommun., vol. 6, no. 5, pp. 587-600, Sept.-Oct. 1995.

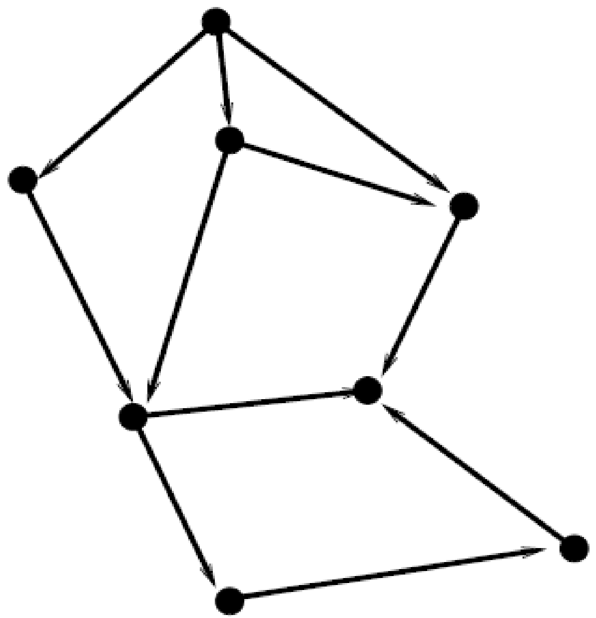
Sichao Yang and R.K., “Network coding over a noisy relay : a belief propagation approach”, in Proceedings of IEEE International Symposium on Information Theory, 2007

(here a network with five independent links isoperated so that each linkis used at a rate above it's capacity)

The main result of this talk!



Network A of DMCs (i, j) with capacity C_{ij}

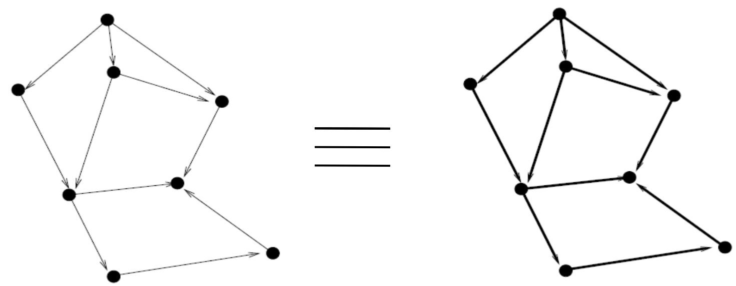


Network B of error-free bit-pipes with throughput C_{ij}

For any set of demands the rate regions of networks A and B coincide!

Theorem Let a network of independent memoryless channels be given with arbitrary demands on source and sink pairs. The achievable rate region is not changed if any channel in the network is replaced by another memoryless channel with the same capacity.

An equivalence theory of network capacity



Network A of DMCs (i, j) with capacity C_{ij}

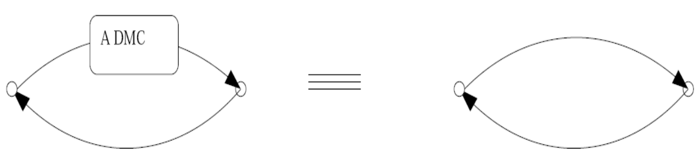
Network B of error-free bit-pipes with throughput C_{ij}

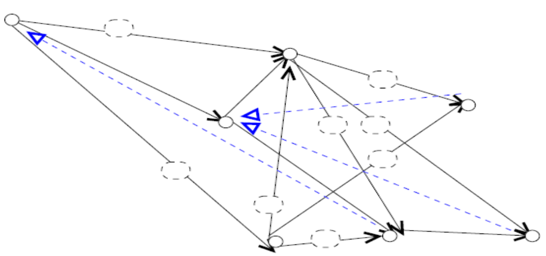
some consequences:

--- Separation holds in the general case

--- determining the rate-region of a network is essentially a network coding problem

--- feedback questions are fairly easy to answer





--- the difficulty in characterizing networks is not of statistical but of combinatorial nature (and NP- hard)

An equivalence theory of network capacity

some consequences:

--- Deciding if a rate tuple restricted to scalar, linear network coding lies within the rate region of a network is essentially solved. (non-linear network coding is wide open)

--- error exponent considerations are quite different

--- extensions to broadcast, MAC, dependent channels etc are possible. This leads to a general approach: Given an arbitrary network it is possible to construct a network of error free bitpipes so that the achievability of a rate point in one network implies the achiveability of a rate point in the other etc. So without solving the combinatorial problem we can investigate the rate-region of networks within an equivalence theory.

So far so good ...... how about interference?

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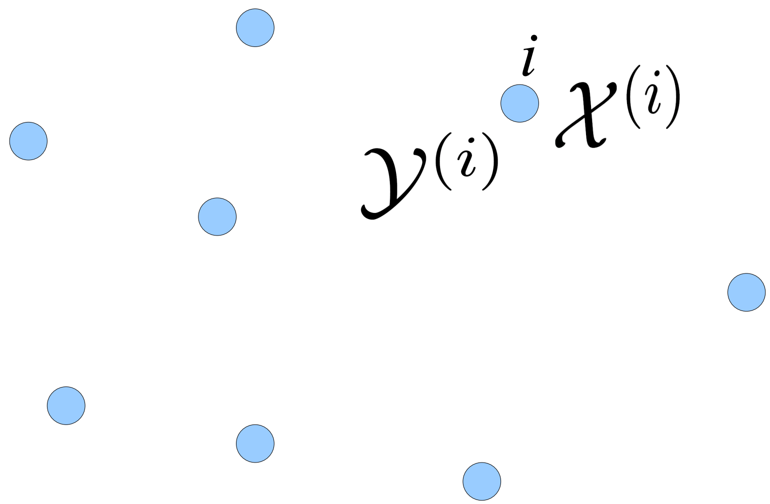
The information theoretic view of a network with *m* nodes:



Every node receives a signal that is influenced by all other nodes in every time step. At node i:

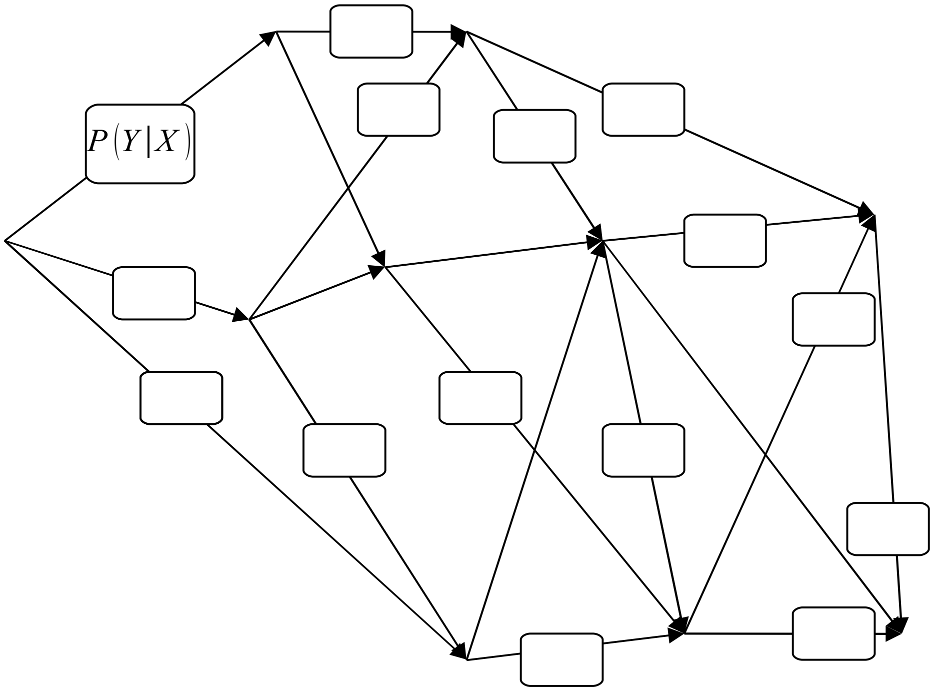
Input alphabet: $\mathcal{X}^{(i)}$

Output alphabet: $\mathcal{Y}^{(i)}$

A network:

$$(\mathcal{X}^{(1)} \times \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(m)}, p(y^{(1)}, y^{(2)}, \ldots, y^{(m)} | x^{(1)}, x^{(2)}, \ldots, x^{(m)}), \mathcal{Y}^{(1)} \times \mathcal{Y}^{(2)}, \ldots, \mathcal{Y}^{(m)})$$

The non-information theoretic view of a network



This is equivalent to a **factorization** of the transition probabilities:

$$\left(\mathcal{X}^{(1)} \times \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(m)}, \prod_{e \in E} p(Y^{(V_2(e))} | X^{(V_1(e))}), \mathcal{Y}^{(1)} \times \mathcal{Y}^{(2)}, \ldots, \mathcal{Y}^{(m)} \right)$$

Two goals:

1) Understanding the „best“ and sparse factorization of a network.

2) Understanding the behavior of the factors within the network.

A library of factors:

--- single link

--- broadcast factor

--- multiple access factor

--- interference factors

--- and so on....

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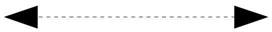
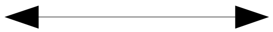
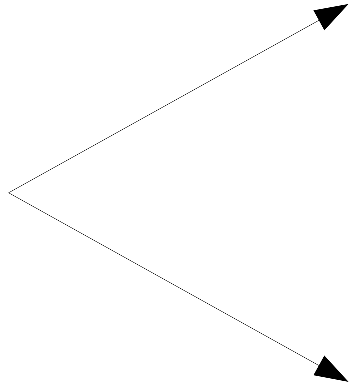
Single Link:  is equivalent to: 

Broadcast:

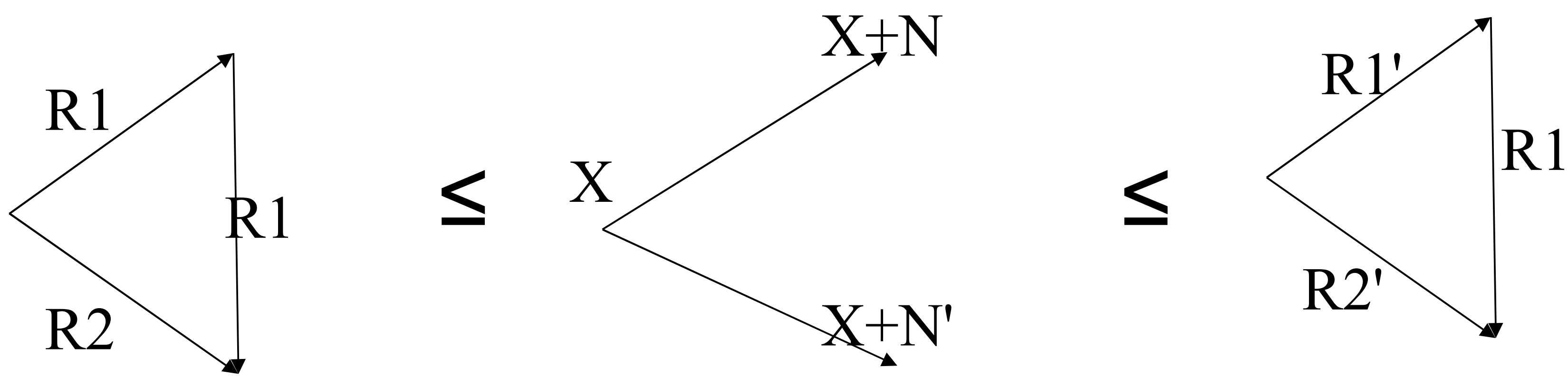


Is equivalent to: ?????

physically degraded, statistically degraded, non-degraded
are all quite different. ?????

The broadcast channels

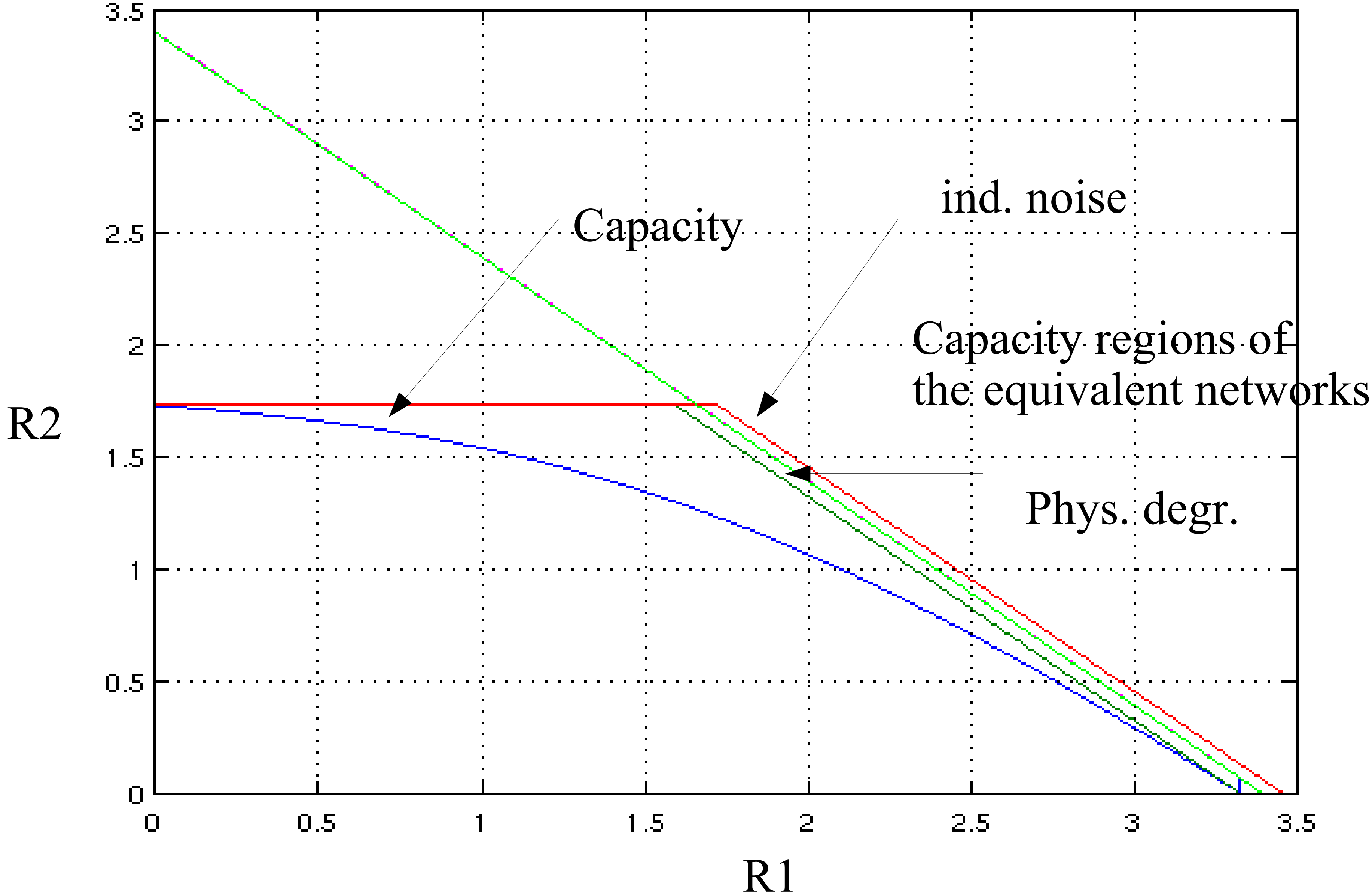
We only have upper and lower bounds (only possible):



bit-pipe network   gaussian network   bit-pipe network

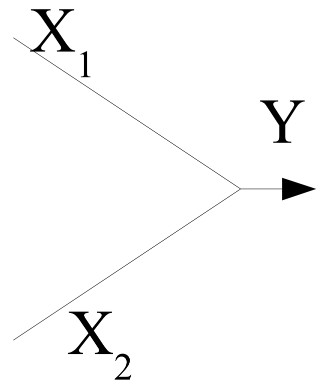
We can sandwich the behavior of e.g. gaussian broadcast networks inside two bit pipe networks.

$R_1 > I(X,Y_1) \quad R_2 > I(X,Y_2|Y_1)$

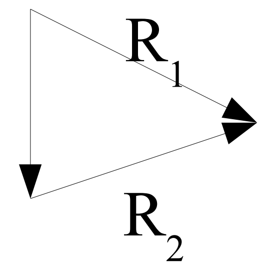


Capacity
ind. noise
Capacity regions of
the equivalent networks
Phys. degr.
R2
R1
0
0.5
1
1.5
2
2.5
3
3.5

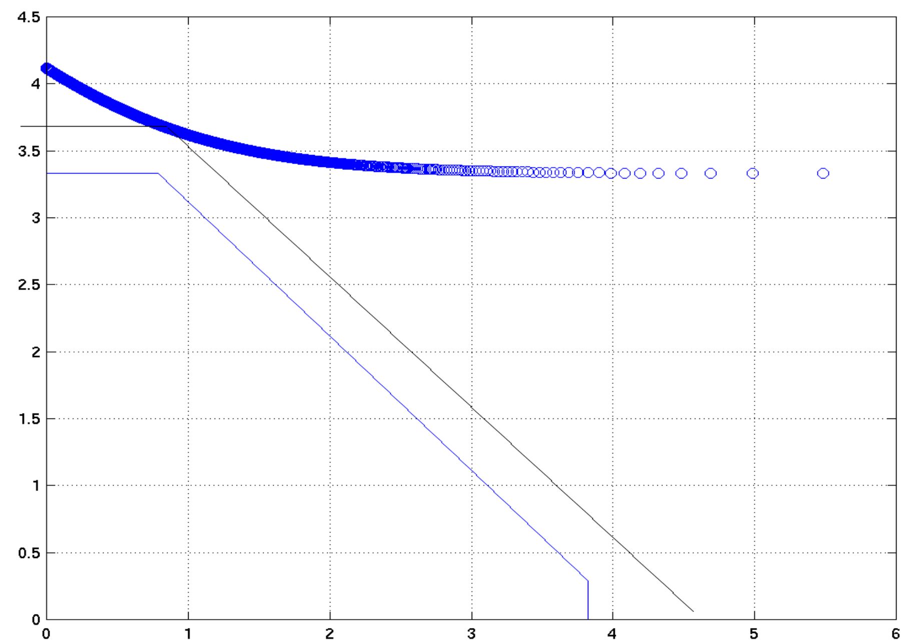
MAC channel:

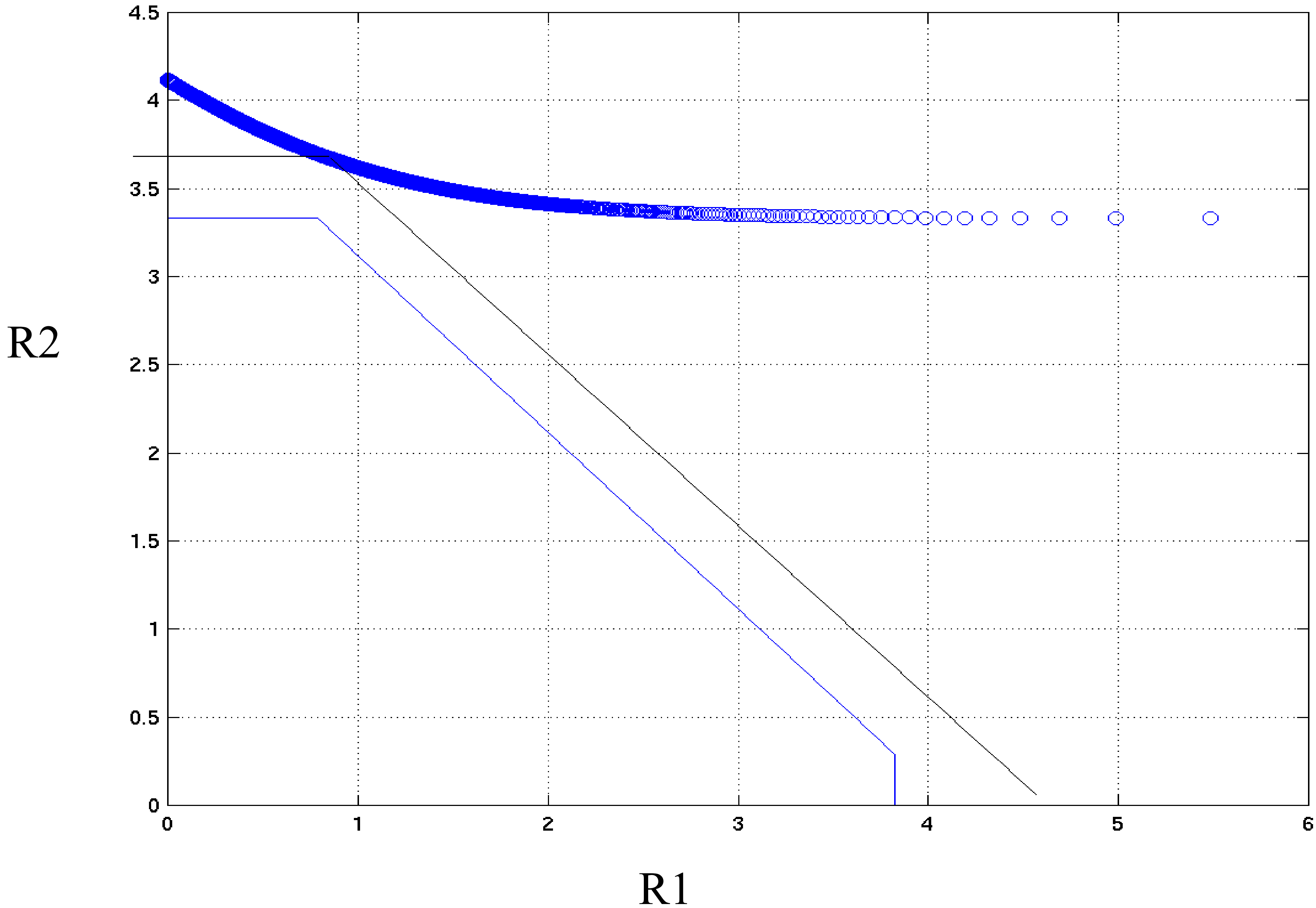


May be represented as:



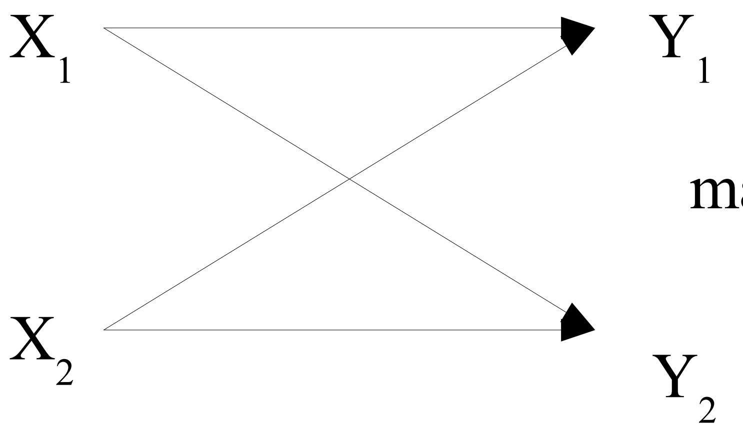
$R_1 > I(X_1;U)$, $\quad R2 > I(X_2;Y|U)$



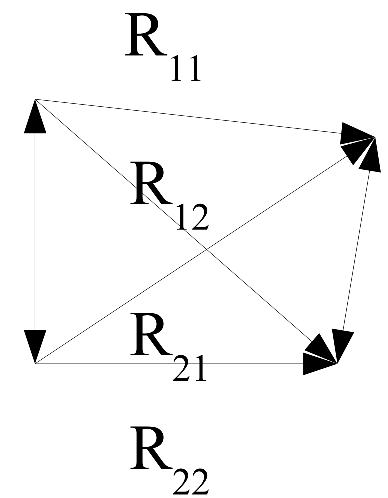


R2
R1
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4
5
6

Interference channel:



may be represented as:



R11> I(X1;U11)
R12> I(X1;U12|U11)
R21> I(X1,X2;Y1|U11)
R22> I(X1,X2;Y2|U11,U12,Y1)

Summary:

In order to develop a capacity tool of networks we need to find a good approximation for the overall transition probabilities.

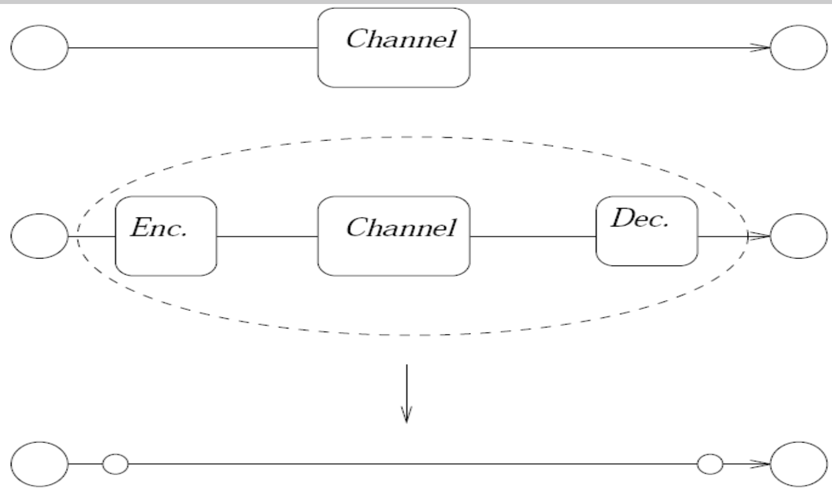
This amounts to an efficient factorization of P(Y|X)

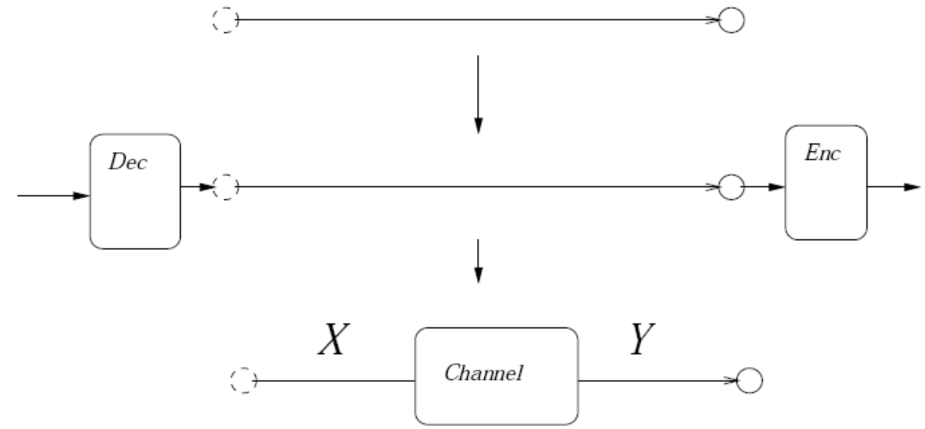
As factors we find small components: links, broadcasts, MAC, interference channel

The goal is to develop a library of „factors“ as well as a factorization approach.

Thanks you!

Proof hint: information theoretic elements (unfortunately not so simple)





A channel can operate as arbitrarily reliable bit-pipe at the channels capacity

a

A bit-pipe can be used to emulate the statistical relationship between X and Y if the bit pipe rate is at least as large as I(X;Y)

C. H. Bennett and P. W. Shor, Entanglement-Assisted Capacity of a Quantum Channel and the Reverse Shannon Theorem, IEEE IT, 2002

P.Cuff, Communication Requirements for Generating Correlated Random Variables, ISIT 2008

Some of the issues:

The input to the same channel may be differently distributed at different instances in time

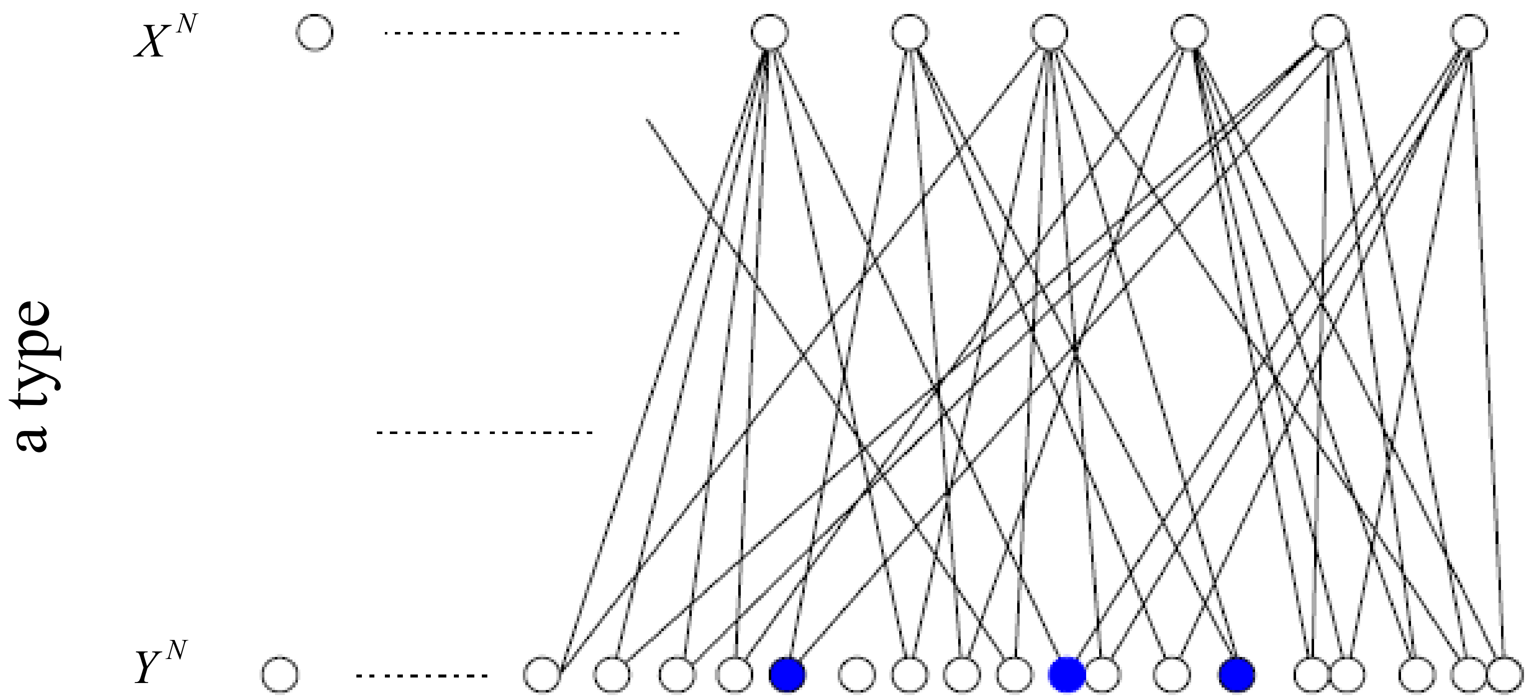
Continuity of the rate region as a function of $P(Y|X)$ is unclear.

NOTHING is known (to us) about the capacity achieving way to operate the network)

Setting up error analysis in this situation is non-trivial without having an algorithm to analyze.

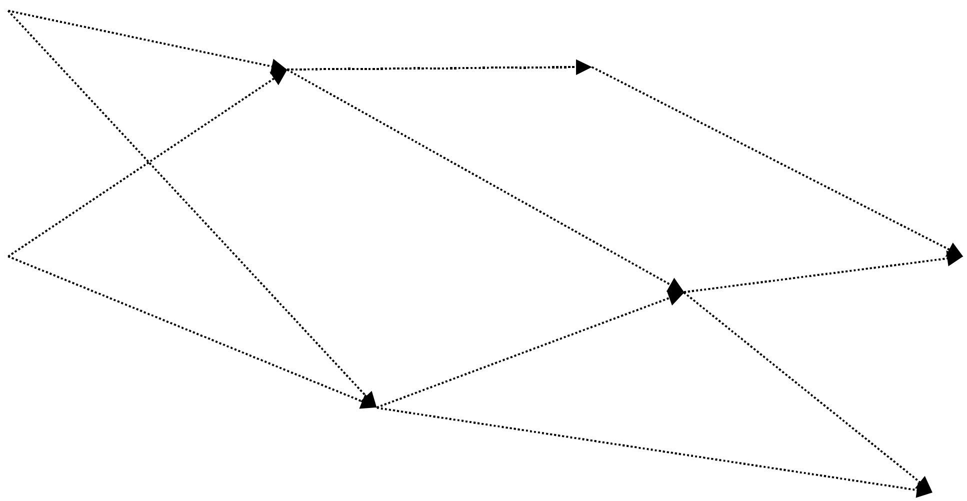
The proof proceeds by showing the the existence of a solution at a rate point in some network implies the existence of a solution in another network.

A general covering theorem [Johnson, Stein, Lovacs]



Let a bipartite, biregular graph be given with vertex classes V_1; V_2 of degree d_1; d_2. There exists of subset U of k vertices of V_2 such that every vertex in V_1 is adjacent to U and $k \le \frac{V_2}{d_1}(1+\log(d_2))$
(a weak form of the Johnson-Stein-Lovasz Theorem)

The network related side of the proof

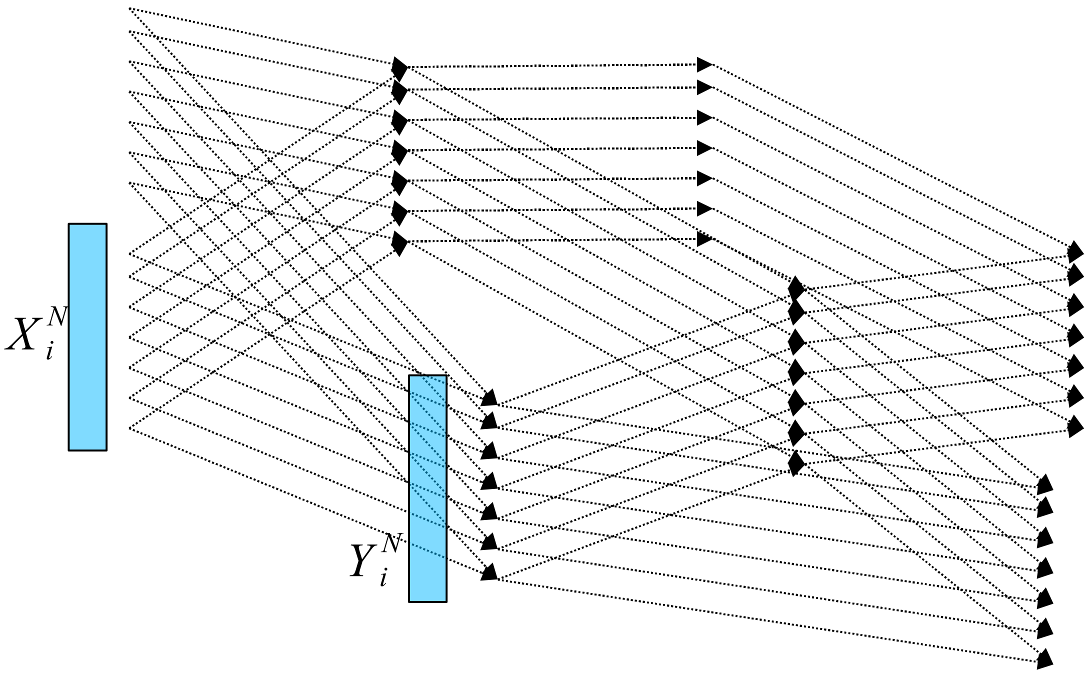


Assume there exists a sequence
of transmissions in the network of
channels that achieves a certain rate point

->

there exists a sequences of trasnmissions
in the bit-pipe netwokr that also achieves
this rate point.

The network related side of the proof



Assume there exists a sequence
of transmissions in the network of
DMCs that achieves a certain rate point

->

there exists a sequences of trasnmissions
in the bit-pipe netwokr that also achieves
this rate point.

X_i^N, Y_i^N will be typical w.h.p. => we can emulate
the behavior of the trasnmission with a bit
pipe of width N I(X;Y) p