BROADCASTING BITS THROUGH QUEUES

Siva K. Gorantla and Todd P. Coleman, UIUC

{sgorant2,colemant}@uiuc.edu

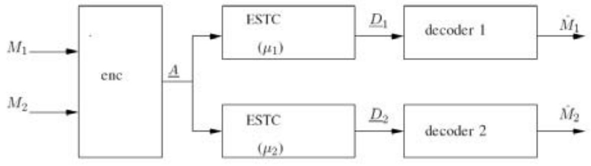
STATUS QUO

- Bits Thru Queues [1]: use *timing* as an unused degree of freedom to communicate.
- [1] Developed closed form expression for capacity of the Exponential Server Timing Channel (ESTC).
- Capacity Region of the ESTC multiple access channel (MAC) found in [2]
- What about the capacity region of a canonical Broadcast Timing Channel?

SET UP

•Encoder ***controls*** the packet timings that serve as the ***arrival process*** to the two ESTC queuing systems.

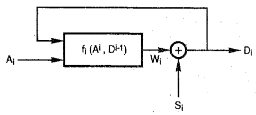
•Inter-arrival times $\{A_i\}_{i=1}^n$ must satisfy a rate constraint: $E\left[\frac{1}{n}\sum_{i=1}^{n} A_i\right] = \frac{1}{\lambda}$



$\mu_1 > \mu_2 > \lambda$

- DT model of each first-come, first-serve (FCFS) queue can be represented as[3]

$$D_{1,i} = W_i\left(\underline{D}_1^{i-1}, \underline{A}^i\right) + S_{1,i}$$
$$D_{2,i} = W_i\left(\underline{D}_2^{i-1}, \underline{A}^i\right) + S_{2,i}$$



Idling Times

$$W_i(\underline{D}, \underline{A}) \triangleq \max\left(0, \sum_{j=1}^{i} A_j - \sum_{j=1}^{i-1} D_j\right)$$

Exponential Service Times

$$\{S_{1,i}\}_{i=1}^n \sim \prod_{i=1}^n e_{\frac{1}{\mu_1}}(s_{1,i})$$
$$\{S_{2,i}\}_{i=1}^n \sim \prod_{i=1}^n e_{\frac{1}{\mu_2}}(s_{2,i})$$

•***Infinite Divisibility*** of the exponential distribution [3]: if $E_{b_k} \sim e_{b_k}$ and $b_1 < b_2$, then

- $E_{b_2} = E_{b_1} + M_{b_1 \to b_2}$ where

$$M_{b_1 \to b_2} \triangleq \begin{cases} 0 & \text{w/ Pr } \frac{b_1}{b_2} \\ \sim e_{b_2}(m) & \text{w/ Pr } 1 - \frac{b_1}{b_2} \end{cases}$$

$$E\left[M_{b_1 \to b_2}\right] = b_2 - b_1 \triangleq a_{b_1 \to b_2}$$
$$E\left[E_b\right] = b$$

$$I(M_{\mu \to \lambda}; M_{\mu \to \lambda} + S_\mu) = \log\left(1 + \frac{a_{\mu \to \lambda}}{b_\mu}\right)$$

ACHIEVABLE REGION w/ FEEDBACK

•With feedback: encoder can control idling times. Use ***rate-splitting*** analogous to the Gaussian case [4]

$$U_1 \leftrightarrow \{\underline{W}^1{}_1, \underline{W}^1{}_2\}$$
$$U_2 \leftrightarrow \{\underline{W}^2{}_1, \underline{W}^2{}_2\}$$
$$W_{1,i} = W^1{}_{1,i} + W^2{}_{1,i}$$
$$W_{2,i} = W^1{}_{2,i} + W^2{}_{2,i}$$

$$b_1 \triangleq \frac{1}{\mu_1}, \quad b_2 \triangleq \frac{1}{\mu_2}$$
$$a_1 \triangleq \frac{1}{\lambda} - b_1, \quad a_2 \triangleq \frac{1}{\lambda} - b_2$$
$$E[W^1{}_{1,i}] = \alpha a_1, \quad E[W^2{}_{1,i}] = (1-\alpha)a_1$$
$$E[W^1{}_{2,i}] = \alpha a_2, \quad E[W^2{}_{2,i}] = (1-\alpha)a_2$$

Decoder dynamics

$$\begin{aligned} D_{1,i} &= W^1{}_{1,i} + W^2{}_{1,i} + S_{1,i} \\ D_{2,i} &= W^1{}_{2,i} + W^2{}_{2,i} + S_{2,i} \\ &= \underbrace{W^2{}_{2,i}}_{M_{b_2+\alpha a_2 \to b_2+a_2}} + \underbrace{[W^1{}_{2,i} + S_{2,i}]}_{E_{b_2+\alpha a_2}} \end{aligned}$$

•Achievable Rate region using rate-splitting and onion-peeling decoding

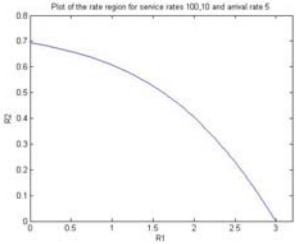
Decoder 2 treats message $W^1{}_{2,i}$ as noise:

$$\begin{aligned} R_2 &= I(W^2{}_2; D_2) \\ &= I\left(W^2{}_2; W^2{}_2 + [W^1{}_2 + S_2]\right) \\ &= \log\left(1 + \frac{(1-\alpha)a_2}{b_2 + \alpha a_2}\right) \end{aligned}$$

Decoder 1 first decodes U_2 then onion-peels $W^2{}_1$ out:

$$\begin{aligned} R_1 &= I(W^1{}_1; D_1 | W^2{}_1) \\ &= I(W^1{}_1; W^1{}_1 + W^2{}_1 + S_1 | W^2{}_1) \\ &= I(W^1{}_1; W^1{}_1 + S_1) \\ &= \log\left(1 + \frac{\alpha a_1}{b_1}\right) \end{aligned}$$

$$\mathcal{R} = \{R_1(\alpha), R_2(\alpha) : \alpha \in [0,1]\}$$



CONVERSE IN LOW SNR REGIME

- The structure of the converse is in the spirit of the Bergman's converse[4] for the Gaussian Broadcast channel

Proof by contradiction: Assume (R_1, R_2) in outer exterior of $\mathcal{R}$.

USER 1

(i) $h\left(\underline{D}_1 | U_2\right) \geq n\left[1 + \log(b_1 + \alpha a_1)\right] - n\epsilon_1$

USER 2

(ii) $\Rightarrow h(\underline{D}_2) \geq n\log(b_2 + a_2) - n\log(b_2 + \alpha a_2) + n\delta + h(\underline{D}_2 | U_2) - n\epsilon_2$

$\theta = \frac{1}{\mu}, \quad \theta_0 = \frac{1}{\lambda}$ Consider $\{\text{ESTC}(\theta)\}$ for a fixed $P(\underline{A})$.

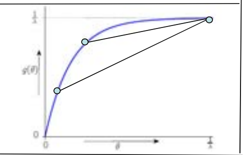
- Queuing Stability Corollary (*): $f_{D^n}(d^n; \theta_0) = \prod_{i=1}^n e_\lambda(d_i)$
- $\mathcal{H}_D(\theta) \triangleq \lim_{n\to\infty} \frac{1}{n} h\left(f_{D^n}(d^n; \theta)\right)$
- Timing Entropy Power: $g(\theta) \triangleq \exp\left(\mathcal{H}_D(\theta) - 1\right)$
- Maximum-Entropy (rate of $\underline{D}$ is λ): $g(\theta) \leq g(\frac{1}{\lambda}) = \frac{1}{\lambda}$

Converse follows directly from the *concavity* of g:

$$\frac{g(\theta_2) - g(\theta_1)}{\theta_2 - \theta_1} \geq \frac{g(\theta_1) - g(\frac{1}{\lambda})}{\theta_1 - \frac{1}{\lambda}}$$

$$g(\theta_2 | U_2) \geq (1-\alpha)(\theta_2 - \theta_1) + g(\theta_1 | U_2) \underbrace{\geq}_{(i)} \theta_2 + \alpha_2$$

$$g(\theta_2) \underbrace{>}_{(ii)} b_2 + a_2 = \frac{1}{\lambda}$$



Concavity of g in the low-SNR regime:

$$g'(\theta) = g(\theta)\mathcal{H}'_D(\theta)$$
$$g''(\theta) = g(\theta)\left[\mathcal{H}''_D(\theta) + \left(\mathcal{H}'_D(\theta)\right)^2\right] \quad (**)$$

$$\begin{aligned} \mathcal{D}(\theta) &\triangleq \lim_{n\to\infty} \frac{1}{n} D\left(f_{D^n}(d^n;\theta) \| f_{D^n}(d^n;\theta_0)\right) \\ \Rightarrow \mathcal{H}_D(\theta) &\underbrace{=}_{(*)} -\mathcal{D}(\theta) + C \\ \Rightarrow \mathcal{H}'_D(\theta) &= -\mathcal{D}'(\theta) \\ \Rightarrow \mathcal{H}''_D(\theta) &= -\mathcal{D}''(\theta) \end{aligned}$$

Relation between KL divergence and Fisher Information [5]:

$$\begin{aligned} \mathcal{H}'_D(\theta_0) &= -\mathcal{D}'(\theta_0) = 0 \\ \mathcal{H}''_D(\theta_0) &= -\mathcal{D}''(\theta_0) = -F\left(\frac{1}{\lambda}\right) \leq 0 \\ \Rightarrow g''(\theta) &\underbrace{\leq}_{(1)} 0 \end{aligned}$$

DISCUSSION

•The structure of the broadcasting bits through ESTC queues problem is shown to be intimately related to the AWGN broadcast problem [4].

•We have developed an appropriate ***entropy power for timing channels*** and derived its concavity in the low-SNR regime

IMPLICATIONS, EXTENSIONS, AND FUTURE WORK

•Explore the concavity question beyond just the low SNR regime using information geometry.

•Explore achievable schemes without feedback that achieve the same rates

REFERENCES

[1] V. Anantharam and S. Verdu, "Bits through queues," *IEEE Transactions on Information Theory*, vol. 42, no. 1, pp. 4–18, 1996.

[2] R. Sundarasen and S. Verdu, "Capacity of queues via point process channels", *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2697–2709, June 2006.

[3] S. Verdu, "The exponential distribution in information theory", *Problems of Information Transmission*, vol 32, no. 1, pp 86-95, Jan-Mar 1996.

[4] P. Bergmans "A simple converse for broadcast channels with additive white Gaussian noise (Corresp.), IEEE Trans Info. Theory, vol. IT- 20, pp. 279 – 280, March 1974.

[5] S. Kullback. Information Theory and Statistics. Wiley, New York, 1959.