On Linear Representation for Network Solutions

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Status Quo

Network coding capacities are well-understood only for some demands.

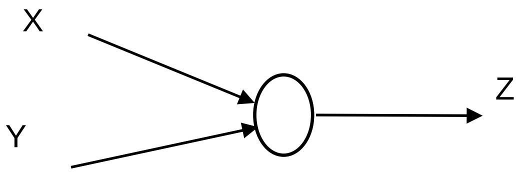
- Multicast: linear coding suffices.
- General demands: negative results, scaling laws and suboptimal bounds.

Open questions:

- When will linear codes suffice?
- If we relax the “zero-error requirement”, can we find simpler solutions?
- When can we guarantee the existence of a “simple” solution?

New Insight

- Dependence constraints within the network limit the space of entropy vectors substantially (hence, limits the space of possible solutions), thus allowing simpler representations, such as binary linear.



H(Z|X,Y)=0

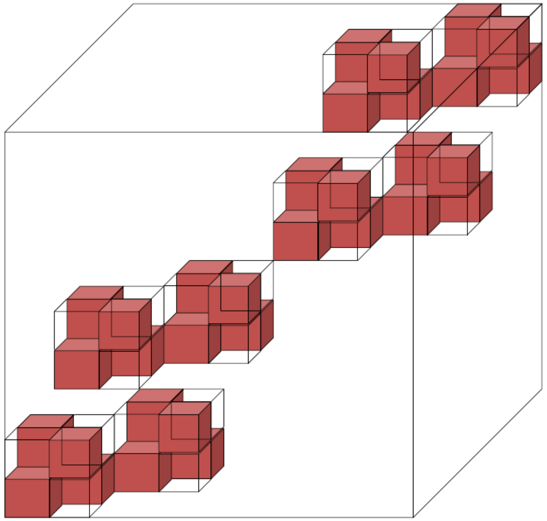
- Bits with integer entropy are well-structured in terms of operations (in fact, this is a *matroidal representation*).
- By finding the conditions for this linear representation, we might answer the question:

  When do linear codes suffice?

  Technical challenge: Discover the limitations of linear coding and its relation to the space of possible network solutions.

Main Results

- A simple model for random variables (RVs) whose bit representation has integer entropies.
- Identifying sufficient conditions for up to 5 random variables (but using a scalable method).
- Understanding the limitations this structure imposes: Non-overlapping typical sets
- **Conjecture:** This model implies linear operations!



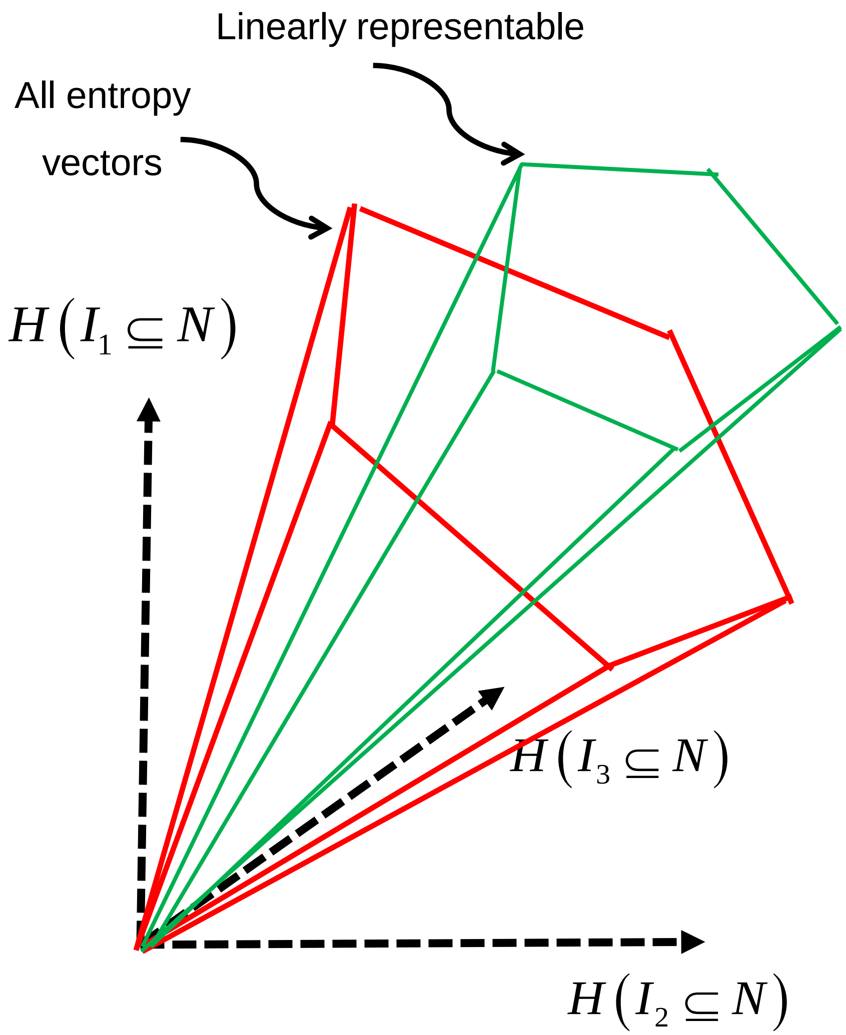
Tools: For a given network topology, identify the solution-space and try to match a linear representation.

Matroids, Entropy Vectors and Linear Representation

- For a random vector X^n over a finite alphabet, the entropy function h maps each subset $\alpha \subseteq N = \{1, 2, \ldots, n\}$ to $\mathbb{R}^+$.
- Thus, each random vector is mapped to a point in $\mathbb{R}^{P(N)}$, called an entropy vector, where $P(N)$ is the power set of N.
- All these points satisfy the polymatroid axioms:
  1. $h(\phi) = 0$,
  2. $h(I) \le h(J)$ for $I \subseteq J \subseteq N$,
  3. $h(I) + h(J) \ge h(I \cup J) + h(I \cap J)$ for any $I, J \subseteq N$.
- Integer valued polymatroids with a rank function bounded by the cardinality $h(I) \le |I|$ are called matroids.
- Matroids whose rank function is proportional to the dimensions of these subsets of some linear space are linearly representable.

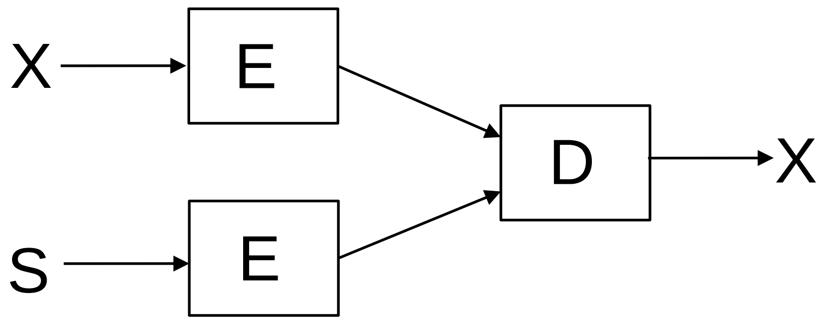
How it Works

- RVs represent the inputs, link messages and outputs.
- For a given topology, identify the dependencies and the resulting constraints.
- Check: is the resulting space of possible solutions within the cone of linearly representable RVs?

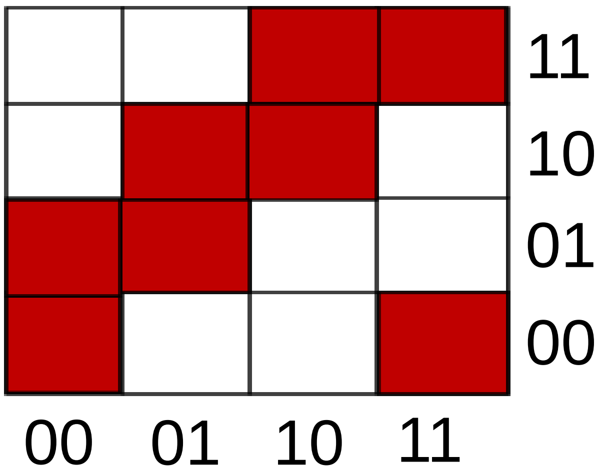


Assumptions and Limitations

Limitations on the dependence structure of the random variables: less important for internal links, might be restrictive for inputs (independent inputs are OK).



X and S are dependent

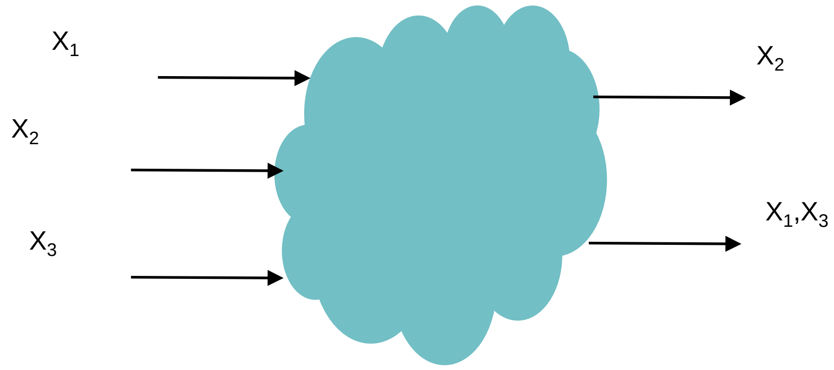


Currently use a computer program to retrieve the vertices of the dependence-cone: might be prohibitively complex for even medium size networks.

Impact

- A better understanding of the space of linearly representable random variables.
- Might give an answer to when CAN we use linear codes (more general sufficient conditions).
- Might find applications in other areas of information theory.

X_1, X_2, X_3 are independent



Goals

Long-term objectives/ alignment with the project roadmap: Obtain fundamental limits of network coding under complexity constraints, e.g., codes with a binary linear representation. Identify networks and inputs for which this representation suffices.

- Identify an analytical method to extend the result to any number of random variables.
- Identify specific networks we can solve.
- Identify new problems in information theory which can benefit from this representation and understanding.

New paradigms for upper bounds: "means and methods to evaluate the achievable performance of different strategies".