Towards Strong Convereses for MANETs

Pierre Moulin

University of Illinois at Urbana-Champaign
Electrical and Computer Engineering

Stanford, Palo Alto, CA
March 5, 2009
Upper Bounds for MANET

- Fano’s inequality is insufficient for some simple networks
- How about this network?
A Simple Problem Where Fano Fails

- “Detect-one” problem [Moulin arxiv 09]

Error is declared if both $m_1$ and $m_2$ are incorrectly decoded.
The Gel’fand-Pinsker Problem

- Communication with side information at transmitter [GP’80]

 iid channel state sequence $s$ – e.g., known interference at encoder

- Applications to broadcast MIMO and other multiuser communication problems
Multiuser Gel’fand-Pinsker Problem

- Channel state sequence $s$ known to both transmitters

- Application to multiuser communications:

- Achievable region obtained by Somekh-Baruch and Merhav’04

- Capacity region is an open problem. **Blame Fano?**
Strong Converse (Wolfowitz’57)

- Use either $P_{e}^{\text{max}} = \max_{m} P_{e}(m)$ or $P_{e}^{\text{avg}} = 2^{-NR} \sum_{m} P_{e}(m)$
- Shannon used $P_{e}^{\text{avg}}$ only
- $\epsilon$-capacity $C(\epsilon) = \sup \{ R \text{ s.t. } P_{e} \leq \epsilon \}$
- Weak converse: $\inf_{0<\epsilon<1} C(\epsilon) = C = \max_{p_{X}} I(X;Y)$
- Strong converse: $\sup_{0<\epsilon<1} C(\epsilon) = C$
- For compound DMCs and $P_{e}^{\text{max}}$ criterion [CK 1980]

$$\inf_{0<\epsilon<1} C(\epsilon) < \sup_{0<\epsilon<1} C(\epsilon)$$
Verdú-Han’94: General Formula for Capacity

- arbitrary $p(y|x)$, not necessarily stationary or information-stable
- Fano fails
- Capacity: $C = \sup_X I(X; Y)$ where

$$I(X; Y) = p - \liminf \frac{1}{N} \log \frac{p(y|x)}{p(y)}$$

upper bound established using a variation of Wolfowitz (1957)
Strong Converse for MAC

- Error criteria:
  \[ P_e^{\text{avg}} = 2^{-N(R_1 + R_2)} \sum_{m_1, m_2} P_e(m_1, m_2) \quad \text{Ahlswede’82} \]
  \[ P_e^{\text{max}} = \max_{m_1, m_2} P_e(m_1, m_2) \quad \text{more natural & simpler} \]

- same capacity region \( \mathcal{R}^{\text{avg}} \) for strong & weak converse under \( P_e^{\text{avg}} \) criterion
- But \( \mathcal{R}^{\text{max}} \subset \mathcal{R}^{\text{avg}} \) in general [Dueck’78]
- Can enlarge \( \mathcal{R}^{\text{max}} \) using external randomness \( \Rightarrow \mathcal{R}^{\text{max}} = \mathcal{R}^{\text{avg}} \)
**Strong Converse for DMC (Wolfowitz’57)**

- Decoding sets \( \{D_m\} \), empirical pmf \( p_X \), reference pmf \( r(y) \)
- “Typical” set for each \( 1 \leq m \leq 2^{NR} \):

\[
A_\delta(m) = \{y : \frac{1}{N} \log \frac{p^N(y|x(m))}{r^N(y)} < I(X;Y) + \delta\}
\]

\[\hat{I}(x(m);y) = \text{empirical m.i.}\]

- \( P_e^{\text{max}} \leq \epsilon \) \( \Rightarrow \) lower bound on prob. that \( m \) is correctly decoded:

\[
1 - \epsilon \leq \sum_{y \in D_m} p^N(y|x(m)) = \sum_{y \in D_m \cap A_\delta(m)} + \sum_{y \notin A_\delta(m)} \leq \kappa \text{ for well-chosen } r(y)
\]

\[
1 - \epsilon - \kappa \leq 2^{N(I(X;Y)+\delta)} \sum_{y \in D_m \cap A_\delta(m)} r^N(y)
\]

- Sum over \( 1 \leq m \leq 2^{NR} \) \( \Rightarrow \)

\[
(1 - \epsilon - \kappa)2^{NR} \leq 2^{N(I(X;Y)+\delta)} \rightarrow R < \max_{p_X} I(X;Y) + \delta + o(1)
\]
Strong Converse for DMC (Wolfowitz’61)

- Useful for deriving more precise asymptotics (limited $N$)
- There exists no $(N, \mathcal{M}_N, \epsilon)$ code such that
  \[
  |\mathcal{M}_N| \geq 2^{NC + K\sqrt{N}} \quad \text{and} \quad P_e \leq \epsilon
  \]
- There exists a $(N, \mathcal{M}_N, \epsilon)$ code such that
  \[
  |\mathcal{M}_N| \geq 2^{NC - K'\sqrt{N}} \quad \text{and} \quad P_e \leq \epsilon
  \]
- Relates to recent work by Verdú (2008) and Hayashi (2008)
Strong Converse for Gelfand-Pinsker Channel

(ISIT 2009 submission)

- Think of $s$ as interference known to encoder but not decoder
- GP’80: $C = \max_{p_{X|US}} [I(U; Y) - I(U; S)]$, achieved by random binning
- Strong converse yields coding interpretation for $U$
- Define alphabet $\mathcal{U}$, function $f : \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{X}$, and virtual DMC $p(y|u, s) = \sum_x p(y|x, s) \mathbb{1}\{x = f(u, s)\}$
- Wlog define codewords as $\{u(m, s)\}$ with $x_i = f(u_i, s)$ for $1 \leq i \leq N$
- Indeed can always adopt trivial choice $\mathcal{U} = \mathcal{X}$ and $x = f(x, s)$
Think of $s$ as interference known to both encoders but not decoder.

Random binning achieves the following rate region [Somekh’04]

For a pmf $P$ of the form $p_s p_t p_{X_1V_1|ST} p_{X_2V_2|ST} p_Y|X_1X_2S$, let $\mathcal{R}^{\text{in}}(L, P)$ be the region of rate pairs $(R_1, R_2)$ that satisfy

$$R_1 < I(V_1; Y|V_2, T) - I(V_1; S|V_2, T)$$

$$R_2 < I(V_2; Y|V_1, T) - I(V_2; S|V_1, T)$$

$$R_1 + R_2 < I(V_1, V_2; Y|T) - I(V_1, V_2; S|T)$$

where the alphabets for $V_1$ and $V_2$ have arbitrarily large cardinality $L$. 

Multiuser Gel’fand-Pinsker Channel
Weak Converse for Multiuser GP Channel

- For a pmf $P$ of the form $p_S p_T p_{V_1, V_2 | ST} p_{X_1 | V_1, ST} p_{X_2 | V_2, ST} p_{Y | X_1, X_2, S}$, let $\mathcal{R}^{\text{out}}(L, P)$ be the region of rate pairs $(R_1, R_2)$ that satisfy

$$
R_1 < I(V_1; Y | V_2, T) - I(V_1; S | V_2, T)
$$

$$
R_2 < I(V_2; Y | V_1, T) - I(V_2; S | V_1, T)
$$

$$
R_1 + R_2 < I(V_1, V_2; Y | T) - I(V_1, V_2; S | T)
$$

where the alphabets for $V_1$ and $V_2$ have cardinality $L$.

- No rate pair outside $\bigcup_{L \geq 1} \mathcal{R}^{\text{out}}(L, P)$ is achievable

- Apparently $\mathcal{R}^{\text{in}} \subset \mathcal{R}^{\text{out}}$
Strong Converse for Multiuser GP Channel

- $\mathcal{R}^\text{in}$ is the capacity region of the multiuser GP channel.

- Furthermore, in the definition of $\mathcal{R}^\text{in}$ it suffices to consider $p_{X_iV_i|ST}$ of the form $p_{V_i|ST} 1\{X_i = f_i(V_i, S)\}$ for $i = 1, 2$

- Two basic ideas of the proof:
  - Extend methods from single-user case
  - Use of the $P_e^\text{max}$ criterion eliminates the need of Dueck and Ahlswede’s wringing techniques
Conclusion

• Strong converses are a powerful alternative to Fano-based weak converses

• Use of $P^\text{max}_e$ criterion is simpler and arguably more natural for multiuser communications

• Method should be applied to a suite of problems with increasing difficulty, including degraded broadcast channel, etc.