Towards Strong Converses for MANETs

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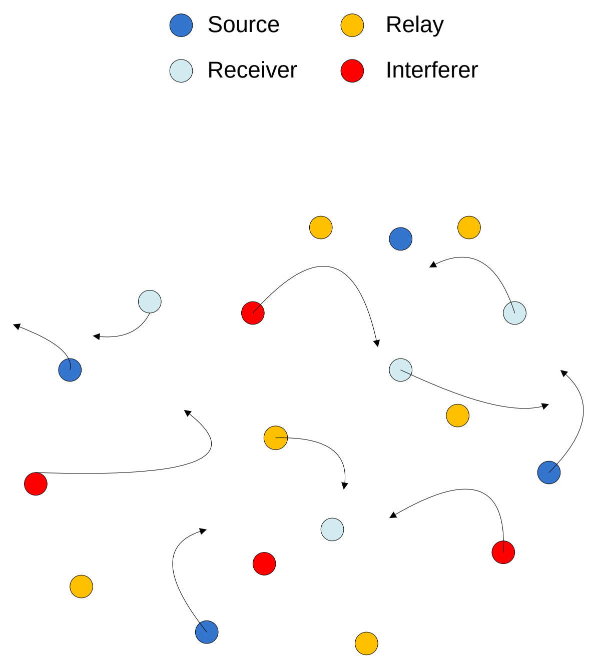
March 5, 2009

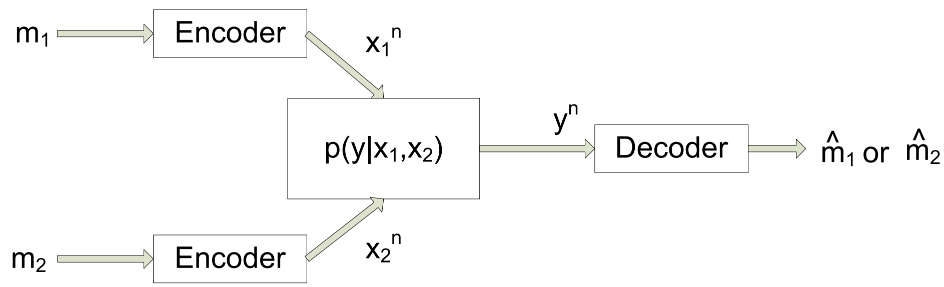
Upper Bounds for MANET

- Fano's inequality is insufficient for some simple networks
- How about this network?



A Simple Problem Where Fano Fails

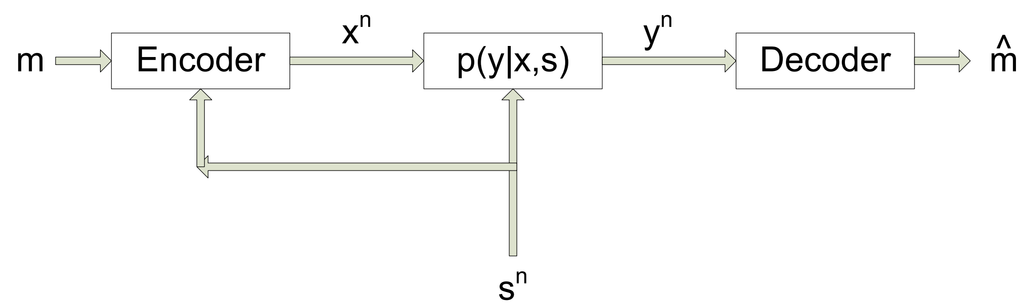
- "Detect-one" problem [Moulin arxiv 09]



Error is declared if **both** m_1 **and** m_2 are incorrectly decoded

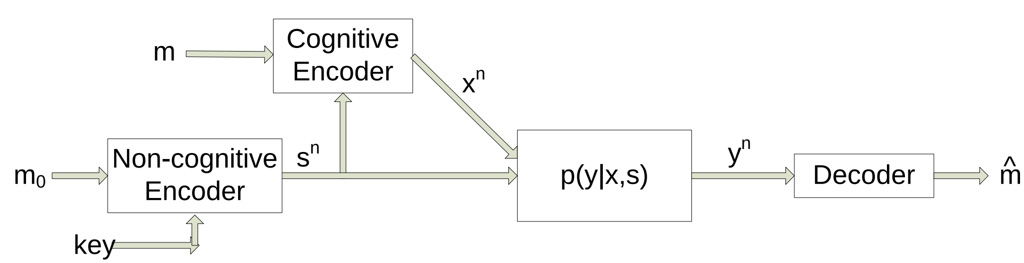
The Gel'fand-Pinsker Problem

- Communication with side information at transmitter [GP'80]



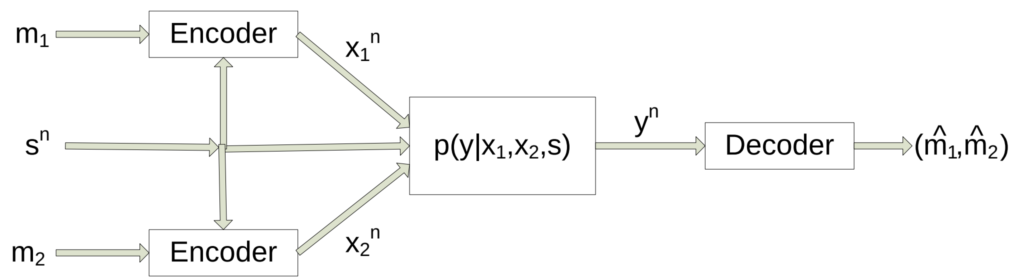
iid channel state sequence **s** – e.g., known interference at encoder

- Applications to broadcast MIMO and other multiuser communication problems

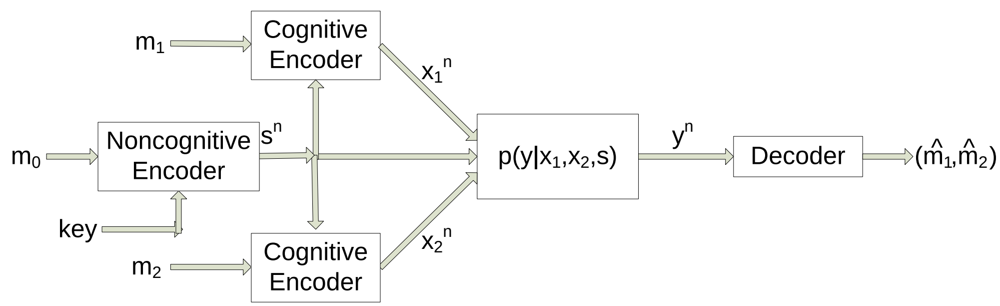


Multiuser Gel'fand-Pinsker Problem

- Channel state sequence **s** known to both transmitters

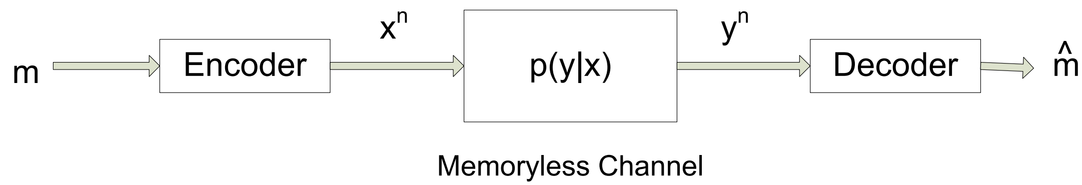


- Application to multiuser communications:



- Achievable region obtained by Somekh-Baruch and Merhav'04
- Capacity region is an open problem. **Blame Fano?**

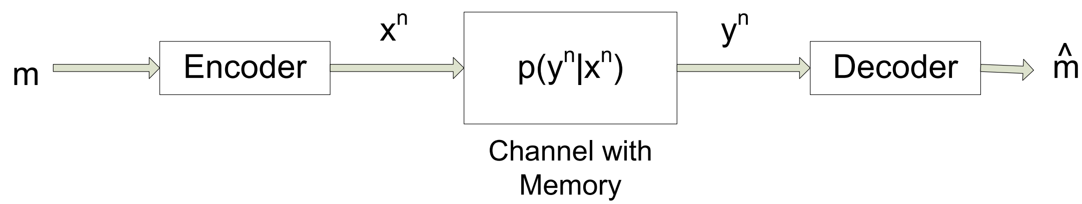
Strong Converse (Wolfowitz'57)



- Use either $P_e^{\max} = \max_m P_e(m)$ or $P_e^{\text{avg}} = 2^{-NR} \sum_m P_e(m)$
- Shannon used P_e^{avg} only
- ϵ-capacity $C(\epsilon) = \sup\{R \text{ s.t. } P_e \leq \epsilon\}$,
- Weak converse: $\inf_{0<\epsilon<1} C(\epsilon) = C = \max_{p_X} I(X;Y)$
- Strong converse: $\sup_{0<\epsilon<1} C(\epsilon) = C$
- For compound DMCs and $P_e^{\max}$ criterion [CK 1980]

$$\inf_{0<\epsilon<1} C(\epsilon) < \sup_{0<\epsilon<1} C(\epsilon)$$

Verdú-Han'94: General Formula for Capacity

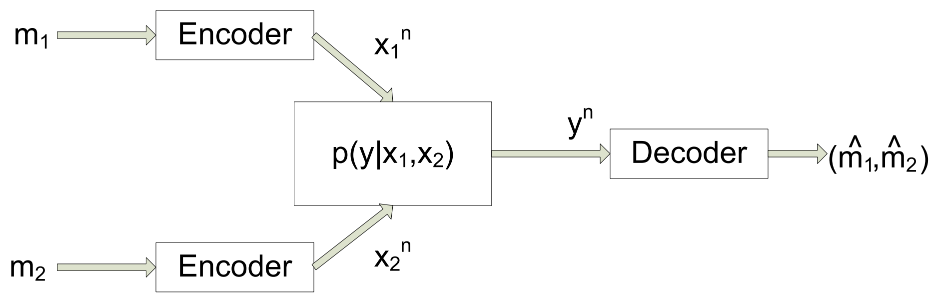


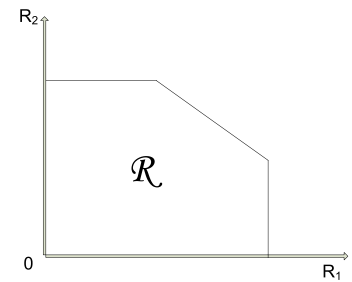
- arbitrary $p(\mathbf{y}|\mathbf{x})$, not necessarily stationary or information-stable
- Fano fails
- Capacity: $C = \sup_{\mathbf{X}} \underline{I}(\mathbf{X};\mathbf{Y})$ where

$$\underline{I}(\mathbf{X};\mathbf{Y}) = p - \liminf \frac{1}{N} \log \frac{p(\mathbf{y}|\mathbf{x})}{p(\mathbf{y})}$$

upper bound established using a variation of Wolfowitz (1957)

Strong Converse for MAC





- Error criteria:

$$P_e^{\text{avg}} = 2^{-N(R_1+R_2)} \sum_{m_1,m_2} P_e(m_1,m_2) \quad \text{Ahlswede}'82$$

$$P_e^{\max} = \max_{m_1,m_2} P_e(m_1,m_2) \quad \text{more natural \& simpler}$$

- same capacity region $\mathscr{R}^{\text{avg}}$ for strong & weak converse under P_e^{avg} criterion
- But $\mathscr{R}^{\max} \subset \mathscr{R}^{\text{avg}}$ in general [Dueck'78]
- Can enlarge $\mathscr{R}^{\max}$ using external randomness $\Rightarrow \mathscr{R}^{\max} = \mathscr{R}^{\text{avg}}$

Strong Converse for DMC (Wolfowitz'57)

- Decoding sets $\{\mathcal{D}_m\}$, empirical pmf p_X, reference pmf $r(y)$
- "Typical" set for each $1 \le m \le 2^{NR}$:

$$\mathcal{A}_\delta(m) = \{\mathbf{y} \ : \ \underbrace{\frac{1}{N}\log\frac{p^N(\mathbf{y}|\mathbf{x}(m))}{r^N(\mathbf{y})}}_{\hat{I}(\mathbf{x}(m);\mathbf{y}) \ = \ \text{empirical m.i.}} < I(X;Y)+\delta\}$$

- $P_e^{\max} \le \epsilon \ \Rightarrow$ lower bound on prob. that m is correctly decoded:

$$\begin{aligned} 1-\epsilon &\le \sum_{\mathbf{y}\in\mathcal{D}_m} p^N(\mathbf{y}|\mathbf{x}(m)) = \sum_{\mathbf{y}\in\mathcal{D}_m\cap\mathcal{A}_\delta(m)} + \underbrace{\sum_{\mathbf{y}\notin\mathcal{A}_\delta(m)}}_{\le\kappa \text{ for well-chosen } r(y)} \\ 1-\epsilon-\kappa &\le 2^{N(I(X;Y)+\delta)} \sum_{\mathbf{y}\in\mathcal{D}_m\cap\mathcal{A}_\delta(m)} r^N(\mathbf{y}) \end{aligned}$$

- Sum over $1 \le m \le 2^{NR} \Rightarrow \ (1-\epsilon-\kappa)2^{NR} \le 2^{N(I(X;Y)+\delta)}$
  $\Rightarrow \ R < \max_{p_X} I(X;Y) + \delta + o(1)$

Strong Converse for DMC (Wolfowitz'61)

- Useful for deriving more precise asymptotics (limited N)
- There exists no $(N, \mathcal{M}_N, \epsilon)$ code such that

$$|\mathcal{M}_N| \geq 2^{NC+K\sqrt{N}} \quad \text{and} \quad P_e \leq \epsilon$$

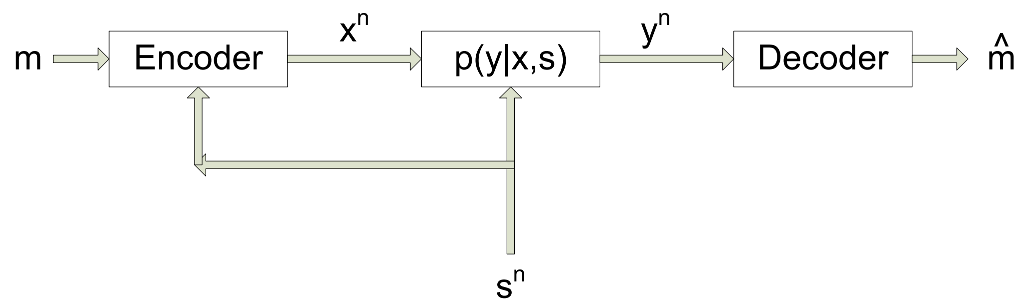
- There exists a $(N, \mathcal{M}_N, \epsilon)$ code such that

$$|\mathcal{M}_N| \geq 2^{NC-K'\sqrt{N}} \quad \text{and} \quad P_e \leq \epsilon$$

- Relates to recent work by Verdú (2008) and Hayashi (2008)

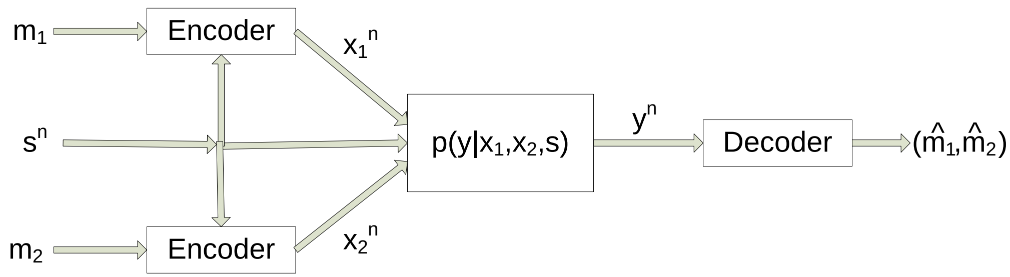
Strong Converse for Gelfand-Pinsker Channel

(ISIT 2009 submission)



- Think of **s** as **interference** known to encoder but not decoder
- GP'80: $C = \max_{p_{X|US}} [I(U;Y) - I(U;S)]$, achieved by random binning
- Strong converse yields **coding interpretation** for U
- Define alphabet $\mathcal{U}$, function $f \ : \ \mathcal{U} \times \mathcal{S} \to \mathcal{X}$,
  and virtual DMC $p(y|u,s) = \sum_x p(y|x,s) \mathbb{1}\{x = f(u,s)\}$
- Wlog define codewords as $\{\mathbf{u}(m,\mathbf{s})\}$ with $x_i = f(u_i, s)$ for $1 \le i \le N$
- Indeed can always adopt trivial choice $\mathcal{U} = \mathcal{X}$ and $x = f(x,s)$

Multiuser Gel'fand-Pinsker Channel



- Think of **s** as **interference** known to both encoders but not decoder
- **Random binnning** achieves the following rate region [Somekh'04]
  For a pmf P of the form $p_S\, p_T\, p_{X_1 V_1|ST}\, p_{X_2 V_2|ST}\, p_{Y|X_1 X_2 S}$, let $\mathscr{R}^{\text{in}}(L,P)$ be the region of rate pairs (R_1,R_2) that satisfy

$$\begin{aligned} R_1 &< I(V_1;Y|V_2,T) - I(V_1;S|V_2,T) \\ R_2 &< I(V_2;Y|V_1,T) - I(V_2;S|V_1,T) \\ R_1+R_2 &< I(V_1,V_2;Y|T) - I(V_1,V_2;S|T) \end{aligned}$$

  where the alphabets for V_1 and V_2 have arbitrarily large cardinality L

Weak Converse for Multiuser GP Channel

- For a pmf P of the form $p_S\, p_T\, p_{V_1 V_2|ST}\, p_{X_1|V_1 ST}\, p_{X_2|V_2 ST}\, p_{Y|X_1 X_2 S}$, let $\mathscr{R}^{\text{out}}(L, P)$ be the region of rate pairs (R_1, R_2) that satisfy

$$\begin{aligned} R_1 &< I(V_1; Y|V_2, T) - I(V_1; S|V_2, T) \\ R_2 &< I(V_2; Y|V_1, T) - I(V_2; S|V_1, T) \\ R_1 + R_2 &< I(V_1, V_2; Y|T) - I(V_1, V_2; S|T) \end{aligned}$$

  where the alphabets for V_1 and V_2 have cardinality L.

- No rate pair outside $\cup_{L \geq 1} \mathscr{R}^{\text{out}}(L, P)$ is achievable

- Apparently $\mathscr{R}^{\text{in}} \subset \mathscr{R}^{\text{out}}$

Strong Converse for Multiuser GP Channel

- $\mathscr{R}^{\text{in}}$ **is the capacity region** of the multiuser GP channel.
- Furthermore, in the definition of $\mathscr{R}^{\text{in}}$ it suffices to consider $p_{X_i V_i|ST}$ of the form $p_{V_i|ST}\,\mathbb{1}\{X_i = f_i(V_i, S)\}$ for $i = 1, 2$
- Two basic ideas of the proof:
  - Extend methods from single-user case
  - Use of the $P_e^{\max}$ criterion eliminates the need of Dueck and Ahlswede's wringing techniques

Conclusion

- Strong converses are a powerful alternative to Fano-based weak converses
- Use of $P_e^{\max}$ criterion is simpler and arguably more natural for multiuser communications
- Method should be applied to a suite of problems with increasing difficulty, including degraded broadcast channel, etc.