A Game Theoretic Approach to Network Coding

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Approach provides guarantees independent of network structure. Guarantees existence of an equilibrium that achieves a system cost of at most 50% higher than the optimal. This offers an improvement over opportunistic coding.

**Global Objective:** Efficiently use network using network coding

**Approach:** Centralized solutions. (e.g., opportunistic coding) Fix paths, use coding opportunities if available

**NEW INSIGHTS**

What about **distributed** solutions?

What if flows were allowed to select path in response to local "cost"?

**Goal:** Let users create coding opportunities to improve efficiency

**MAIN ACHIEVEMENT:**

Introduced game theory as a distributed tractable mechanism to obtain good network performance

- **game theory for social sciences:** "descriptive agenda"
- **game theory for engineering:** "prescriptive agenda"

**HOW IT WORKS:**

- Model interactions as a non-cooperative game
  - players (unicast flows)
  - actions (available paths)
- Assign each player a "cost" function
- Analyze efficiency of equilibrium behavior

**ASSUMPTIONS AND LIMITATIONS:**

- Limited form of network coding (reverse carpool)
- Players have knowledge of available paths
- Players equilibrate faster than network changes

**ACHIEVEMENT DESCRIPTION**

**MAIN ACHIEVEMENT:**

- Introduced game theory as a distributed tractable mechanism to obtain good network performance

**IMPACT**

Approach provides guarantees independent of network structure. Guarantees existence of an equilibrium that achieves a system cost of at most 50% higher than the optimal. This offers an improvement over opportunistic coding.

**NEXT-PHASE GOALS**

- Understand the potential of game theory in network coding problems
  - Establish desirable distributed learning algorithms with good convergence rates
  - Extend game theoretic approach to more general network coding problems

Game theory is an applicable tool for distributed optimization in network coding
Example: Network coding

**Features:**
- large common network
- large # of users
- different network demands

**Global objective:** Allocate users efficiently over network (utilizing network coding)
- maximize throughput
- minimize # of transmissions

**Challenges:**
- centralized optimization is not feasible
- network coding capacity is unsolved
Multiple unicast flows in shared network environment

possible transmissions highlighted by edges on graph

Cost of allocation = number of transmission
Limited form of network coding: “Reverse carpooling”

opportunity for network coding arises when two unicasts traverse the same node in opposite directions

without network coding - 4 transmissions required

with network coding - 3 transmissions required
**Approach:** Model interactions as a non-cooperative game

- **Players (unicast flows):** \( \{1, 2, \ldots, n\} \quad (s_i, t_i) \)
- **Actions (available paths):** \( a_i \in \mathcal{A}_i \)
  \[ \mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \]
- **System cost:** \( C(a) = \sum_e \max\{\#_e^0(a), \#_e^1(a)\} \)

**Goal:** Design local players’ cost functions so that equilibrium behavior is desirable

**Cost functions:** \( J_i(a) = J_i(a_i, a_{-i}) \)

**Equilibrium behavior:** Pure Nash equilibrium

\[ J_i(a_i^*, a_{-i}^*) \leq J_i(a_i, a_{-i}^*) \]
Efficiency

Let

$$E(G) := \{a \in A : a \text{ is a Nash equilibrium of game } G\}$$

$$a^{\text{opt}} \in \arg \min_{a \in A} C(a)$$

**Price of Anarchy**

$$POA = \sup_G \max_{a \in E(G)} \frac{C(a)}{C(a^{\text{opt}})}$$

worst case performance of any NE

**Price of Stability**

$$POS = \sup_G \min_{a \in E(G)} \frac{C(a)}{C(a^{\text{opt}})}$$

worst case performance of best NE

*(independent of network structure or demands)*
Cost design: Wonderful life design

Decision Makers

\[ J_i(a_i, a_{-i}) \]

Global Behavior

\[ C(a) = \sum_e \max\{\#_0 e(a), \#_1 e(a)\} \]

Wonderful Life:

\[ J_i(a) = C(a) - C(a^0_i, a_{-i}) \]

Positives

- local

NE exists (minimizer C)

POS = 1

Negatives

- POA unbounded

s_1 \quad t_2 \quad s_2 \quad t_1
Select any global cost \( \phi : \mathcal{A} \rightarrow \mathbb{R} \)

\[
J_i(a) = \phi(a) - \phi(a_i^0, a_{-i})
\]

**Positives**

NE exists (minimizer \( \phi \))

local

\[ \phi(a) = C(a) \]

**Negatives**

POS = 1

POA unbounded

Can we choose a cost that gives us better equilibrium efficiency?
Consider

\[ \phi(a) = (\alpha - 1)C(a) + \sum_{i} |a_i| \quad \alpha \geq 0 \]

New design:

\[ J_i(a) = \phi(a) - \phi(a_i^0, a_{-i}) \]
\[ = \alpha N_i^{(>)}(a) + N_i^{(\leq)}(a) \]

Old design:

\[ J_i(a) = C(a) - C(a_i^0, a_{-i}) \]
\[ = N_i^{(>)}(a) \]
What is the cost of this assignment for player 1?

\[ J_i(a) = \phi(a) - \phi(a_i^0, a_{-i}) \]

\[ = \alpha N_i^{(>)}(a) + N_i^{(\leq)}(a) = 2\alpha + 3 \]
Main result

Price of Anarchy and Price of Stability in Reverse Carpooling Game

\[ J_i(a) = \alpha N_i^{(\geq)}(a) + N_i^{(\leq)}(a) \]

Is this good? Yes and No!

POS = 3/2  POA = 2

Can we do better? No!
Conclusions

Recap:
- Derived “optimal” cost functions for a simple network coding problem (WL)
- The results on efficiency have implications beyond simple setup

Goals:
- Illustrate potential of game theory as mechanism for distributed optimization
- Results stem for recent research connecting “potential games” and cooperative control problems


Left unsaid:
- How do players reach equilibrium in a distributed fashion?
- Existing literature on learning in games provides some algorithms