POSTERIOR MATCHING SCHEME FOR DEGRADED BROADCAST CHANNELS

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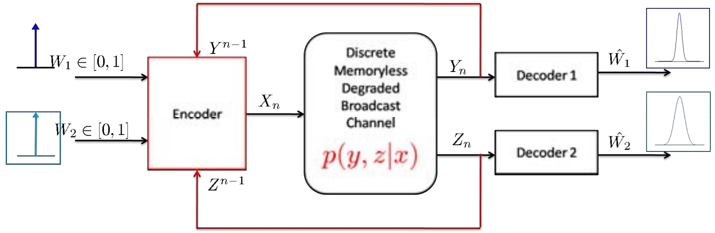
STATUS QUO

- Communication with Feedback: Reduces Encoder-Decoder complexity & Error Probability performance.
- [1] Developed a general recursive feedback scheme over memoryless channels with noiseless feedback.
- [2] Posterior Matching Scheme(PMS) shown (sometimes, using iterated function systems) to achieve capacity
- [3] Channel coding with feedback interpreted as "stochastic control of posterior". Shows PMS is an optimal solution and achieves capacity on any memoryless channel (via a simple Lyapunov function)

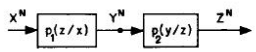
SET UP

- One Tx broadcasts 2 messages (unif on [0,1]: equivalent to an infinite sequence of 0's and 1's) to 2 Rx's



- Capacity of a Discrete Memoryless Degraded Broadcast Channel is not increased by Feedback [4].
- Discrete Memoryless nature of the channel, for any n,

$$p(Z^n|X^n) = \prod_{i=1}^n p_1(y_i|x_i)p_2(z_i|y_i)$$



- If U,X,Y,Z is a joint ensemble such that $p(u,x,y,z) = Q_1(u)Q_2(x|u)p_1(y|x)p_2(z|y)$, then

$$\mathcal{R} = \{(I(X;Y|U), I(U;Z)\} \Longleftrightarrow \mathcal{R} = \{(R_1,R_2) : R_2 + \lambda R_1 \le C(\lambda)\} \quad , C(\lambda) = \max_{Q_1(u)Q_2(x|u)} I(U;Z) + \lambda I(X;Y|U)$$

- For every $\lambda \in [0,\infty]$, there is a corresponding optimal channel input distribution. Call the optimal CDFs

F_U and $F_{X|U}$

- ***Point-Point Posterior matching scheme is an optimal policy. [2]*** $X_{n+1} = F_X^{-1}(F_n(W))$

For any posterior , next input indep. of what decoder has seen so far. $\quad I(X_n;Y_n|Y^{n-1}) = I(X_n;Y_n|F_n = f) = C$

THE SCHEME

- At every time instant, the information still missing at the receiver is extracted from the a-posteriori density function
- The signal is then reshaped to capacity-achieving (optimal) distributions.

$$\begin{aligned}
\widetilde{W}_2 &= F_{W_2|Z^{n-1}}(W_2|Z^{n-1}) \\
U_n &= F_U^{-1}(\widetilde{W}_2) \\
\widetilde{W}_1 &= F_{W_1|W_2,Y^{n-1},Z^{n-1}}(W_1|W_2,Y^{n-1},Z^{n-1}) \\
X_n &= F_{X|U}^{-1}\left(\widetilde{W}_1|U_n\right)
\end{aligned}$$

Implications of the scheme on channel

Z_n is independent of Z^{n-1}. Z_i's are i.i.d.

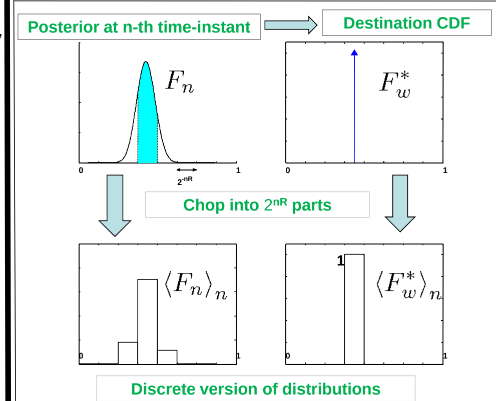
Y_n is independent of Y^{n-1} given (W_2 , Z^{n-1}).

USER 2	USER 1
$\begin{aligned} I(W_2;Z^n) &= \sum_{i=1}^N H(Z_n\|Z^{n-1}) - H(Z_n\|W_2,Z^{n-1}) \\ &= \sum_{i=1}^N H(Z_n) - H(Z_n\|W_2,Z^{n-1},Y^{n-1}) \\ &= \sum_{i=1}^N I(Z_n;\tilde{U}_n) \end{aligned}$	$\begin{aligned} I(W_1;Y^n\|W_2) &= I(W_1;Y^n,Z^n\|W_2) \\ &= \sum_{i=1}^N I(W_1;Y_n,Z_n\|\tilde{U}_n) \\ &= \sum_{i=1}^N I(X_n;Y_n\|\tilde{U}_n) \end{aligned}$

$$I(W_2;Z) + \lambda I(W_1;Y|W_2) = \sum_{i=1}^N I(Z_n;\tilde{U}_n) + \lambda I(X_n;Y_n|\tilde{U}_n) = NC(\lambda) \qquad \tilde{U}_n = (W_2,Y^{n-1},Z^{n-1})$$

PERFORMANCE OF THE SCHEME

- "Error Probability at the Decoders goes to 0" – is shown by using Lyapunov Theory on KL-Distance.

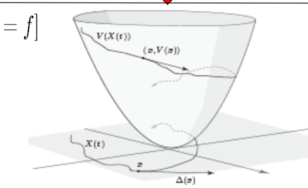


$$\mathbb{P}\left(|W - \hat{W}_n| > 2^{-nR}\right) \to 0$$

$$V_n(F_n) \to 0 \Longleftrightarrow D\left(\langle F_w^* \rangle_n \,\|\, \langle F_n \rangle_n\right) \to 0$$

$$\mathbb{E}\left[V_\infty(F_{n+1}) - V_\infty(F_n)|F_n = f\right] = -C + C1_{\{V_\infty(f) \le 0\}}$$

V_n(f) is a LYAPUNOV FUNCTION



$$V_n(f) \triangleq \int_{w=0}^1 D\left(\langle F_w^* \rangle_n \,\|\, \langle f \rangle_n\right) f(w) dw$$

Whenever $I(W;Y_n|Y^{n-1}) = I(W;Y_n|F_n = f) = C$ **holds, Drift of Lyapunov function is -C**

- Z_n is independent of Z^{n-1}. $\rightarrow$ $I(W_2;Z_n|Z^{n-1}) = I(W_2;Z_n|F_{2,n} = f) = I(U;Z) = C_2$
- Y_n is independent of Y^{n-1} given (W_2 , Z^{n-1}) $\rightarrow$ $I(W_1;Y_n|Y^{n-1},W_2) = I(W_1;Y_n|W_2,F_{1,n} = f) = I(X;Y|W_2) = C_1$

$$\begin{aligned} &\mathbb{E}\left[V_{i,n+1}(F_{i,n+1}) - V_{i,n}(F_{i,n})|F_{i,n} = f\right] \\ &\le -(C_i - R_i - \epsilon) + (C_i - R_i)1_{\{V_{i,n}(f) \le \epsilon\}} \forall i = 1,2 \end{aligned} \Rightarrow \begin{aligned} V_{1,n}(F_{1,n}) &\to 0 \\ V_{2,n}(F_{2,n}) &\to 0 \end{aligned} \Rightarrow \begin{aligned} &\mathbb{P}\left(|W_1 - \hat{W}_{1,n}| > 2^{-nR_1}\right) \to 0 \\ &\mathbb{P}\left(|W_2 - \hat{W}_{2,n}| > 2^{-nR_2}\right) \to 0 \end{aligned}$$

SUMMARY – HOW IT WORKS

A. Proposed a Scheme to simplify encoder-decoder design for a Degraded Broadcast Channel with Feedback.

How? → Interpret feedback communication encoder design as stochastic control of posterior towards certainty

B. Showed that the scheme is optimal and achieves Capacity, by using Lyapunov Theory.

How? → An optimal policy implies the existence of a Lyapunov function, which is in essence a KL divergence

IMPLICATIONS, EXTENSIONS, AND FUTURE WORK

- Posterior Matching Scheme for a Degraded Broadcast Channel opens up several directions for simplifying encoder-decoder schemes for many Multi-terminal problems with feedback.
- Can develop fundamental limits of error exponents for feedback w/ fixed block length using Martingale condition implied by Lyapunov function.

REFERENCES

[1] O. Shayevitz and M. Feder, "Communication with feedback via posterior matching," in *IEEE International Symposium on Information Theory,* Nice, France, June 2007.

[2] ——, "The posterior matching feedback scheme: Capacity achieving and error analysis," in *IEEE International Symposium on Information Theory, Toronto, Canada, July 2008.*

[3] T. P. Coleman "A Stochastic Control Approach to 'Posterior Matching'-style Feedback Communication Systems', submitted to *IEEE International Symposium on Information Theory, Seoul, Korea, July 2009.*

[4] A. El Gamal "The feedback Capacity of Degraded Broadcast Channels", *IEEE Transactions on Information Theory, vol. 24, pp. 379–381.*

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