On Networks with Side Information

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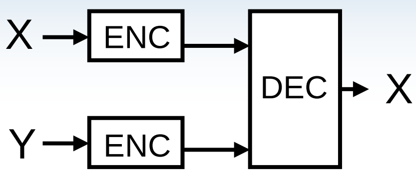
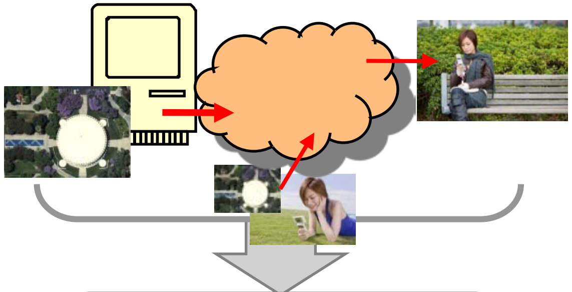
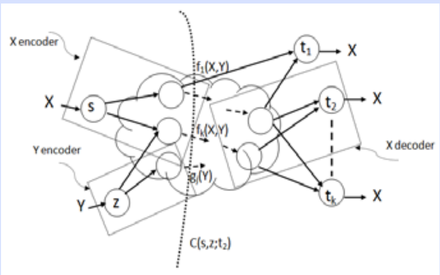
STATUS QUO



To large extent, our knowledge of networks with side information is limited to the model above. However, we are interested in more complex networks:



NEW INSIGHTS

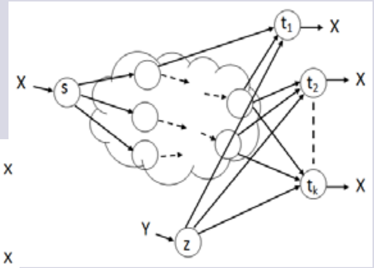


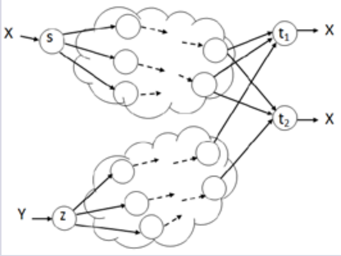
- Canonical source coding problems can be used to derive bounds for more complex networks.
- Network coding can play a key role even in non-multicast problems.

ACHIEVEMENT DESCRIPTION

MAIN ACHIEVEMENT:

- New inner and outer bounds were derived for networks with side information.
- The bounds are tight for several network topologies.



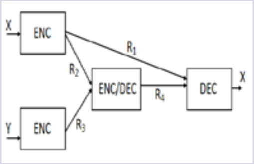


HOW IT WORKS:

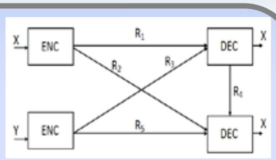
- Converse results for the canonical problem are generalized to multi-node networks.
- The achievable schemes are used at the terminals (sources and sinks), together with network coding.
- Successive refinement of both the source and side information descriptions is used when there are multiple sinks.

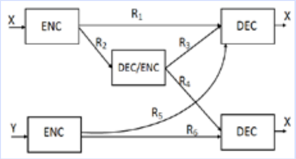
ASSUMPTIONS & LIMITATIONS:

- One source node; one helper.
- Bounds are not tight in general.



IMPACT





- Tight results for several families of networks with side information.
- A wider range of scenarios where cut-set analysis applies.
- An interesting and fruitful connection to successive refinement of information.

NEXT-PHASE GOALS

Extend this methodology to various source coding problems.

- *Derive new bounds and find network topologies for which they are tight.*
- *Different demand models(e.g. distortion)*

Strategies intended for small problems, joint with network codes, can solve complex networks

Notes

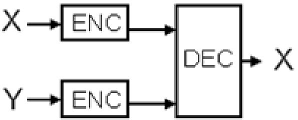
- ***Technical challenge:*** Characterize the rate region for general networks with side information.
- ***Open problems:*** The set of achievable rates and the corresponding achieving strategies for networks with more than three nodes (source, side information, destination).
- ***Current methods:*** The Ahlswede-Korner scheme for three nodes.
- ***Tools:*** A method to generalize canonical converse results to general networks. An achievable scheme which uses the solution to the three-nodes network as a base, with successive refinement of both the source and the side information descriptions and network coding at internal nodes.

Notes (cont.)

- ***Intermediate achievement:*** New outer and inner bounds on the set of achievable rates. The bounds are tight for several families of networks. The results extend the network scenarios for which cut-set analysis is known to yield tight results.
- ***Long-term objectives/ alignment with the project roadmap:*** Obtain fundamental limits for wider sets of network topologies, as well as understand more general source and side information models and more diverse demands.
- ***Thrust 1: New paradigms for upper bounds:*** "means and methods to evaluate the achievable performance of different strategies".

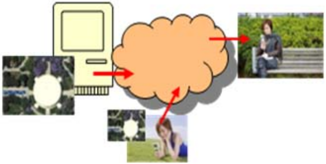
Problem Statement and Motivation

Side information in networks can significantly reduce the rate required from the main server: we want a reduced rate from X.



The model is interesting in cases where:

1. Side information is "close" to the destination.
2. Collaborative networks (sensor, organizational, army). The helper is aware of what we are doing.



We are interested in the following questions:

1. Which X descriptions should we create?
2. Which Y descriptions should we create?
3. What is required from the network to support these rates?

Problem Formulation

- We denote a side information network by $(\mathcal{V}, \mathcal{E}, s, z, T)$.
  - $(\mathcal{V}, \mathcal{E})$ is a directed graph, where $\mathcal{V}$ is the set of vertices (nodes) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges (links).
  - s and z are the source and side information nodes. T is a set of sinks.
  - $o(e)$ and $d(e)$ denote the edge's origin and destination.
- Let $\{(X_i, Y_i)\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed pairs of discrete random variables with alphabet $\mathcal{X} \times \mathcal{Y}$. X is the source and Y is the side information.
- For any vector of rates $(R_e)_{e \in \mathcal{E}}$, a $((2^{nR_e})_{e \in \mathcal{E}}, n)$ network code comprises the following mappings

$$
\begin{array}{ll}
g_n^e : \mathcal{X}^n \mapsto \{1, \ldots, 2^{nR_e}\} & e \in \mathcal{E}, o(e) = s \\
g_n^e : \mathcal{Y}^n \mapsto \{1, \ldots, 2^{nR_e}\} & e \in \mathcal{E}, o(e) = z \\
g_n^e : \prod_{e' : d(e') = o(e)} \{1, \ldots, 2^{nR_{e'}}\} & \\
\qquad \mapsto \{1, \ldots, 2^{nR_e}\} & e \in \mathcal{E}, o(e) \notin \{s, z\} \\
h_n^t : \prod_{e : d(e) = t} \{1, \ldots, 2^{nR_e}\} \mapsto \mathcal{X}^n & t \in T.
\end{array}
$$

Objective

For each $t \in T$, we use $\hat{X}_t^n$ to denote the reproduction of X^n found by decoder h_n^t. Denote by $c(e)$ the capacity of an edge $e \in \mathcal{E}$.

Definition

A set of values $(c(e))_{e \in \mathcal{E}}$ is achievable if for any $\epsilon > 0$ there exists a sufficiently large n and a $((2^{nR_e})_{e \in \mathcal{E}}, n)$ code with $R_e \leq c(e)$ for all $e \in \mathcal{E}$, such that $\Pr(\hat{X}_t^n = X^n) \geq 1 - \epsilon$ for all sinks $t \in T$.

We call the closure of this set of rate vectors the set of achievable rates, which we denote by $\mathcal{R}(\mathcal{V}, \mathcal{E}, s, z, T)$.

Main Objective

Derive general inner and outer bounds on this set and investigate the scenarios in which these bounds are tight.

Outline of Results

- We derive inner and outer bounds on the set of achievable rates.
- The bounds are tight for some specific network scenarios, and are sometimes tighter than known results for general networks.
- We extend the range of network scenarios for which min-cut analysis is known to yield tight results.
- We show how solutions to small, canonical problems can be used as building blocks in analyzing large networks.
- This method may generalize to other source coding problems, not necessarily the coded side information we use here.
- The work reveals an interesting connection between coding for networks with side information and successive refinement.

Outer Bound

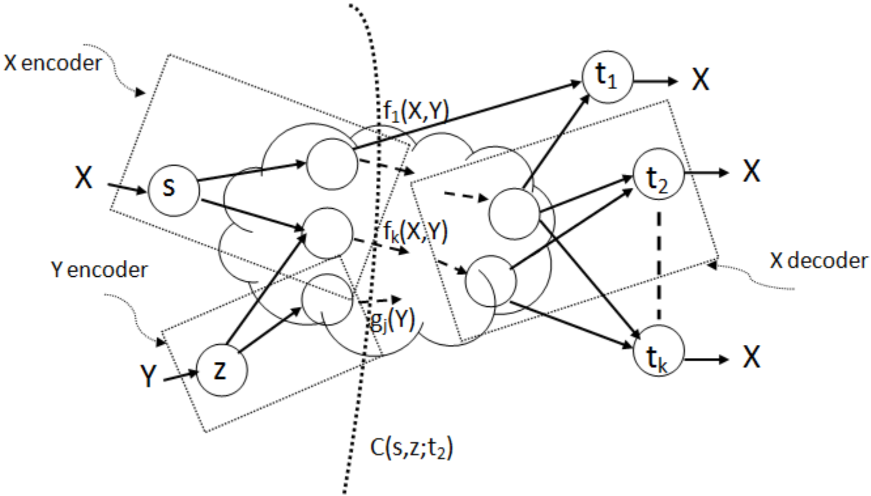
Let $\mathcal{E}_{XY} \subseteq \mathcal{E}$ denote the set of edges for which there is a directed path from s to $o(e)$ with a strictly positive capacity, and denote by $\mathcal{E}_Y$ the set $\mathcal{E} \setminus \mathcal{E}_{XY}$. Given any non-intersecting sets $A, B \subset \mathcal{V}$, we use $\mathcal{V}_{A;B}$ to denote a cut with $A \subseteq \mathcal{V}_{A;B}$ and $B \cap \mathcal{V}_{A;B} = \emptyset$. Let $\mathcal{C}(\mathcal{V}_{A;B})$ be the set of edges $e \in \mathcal{E}$ for which $o(e) \in \mathcal{V}_{A;B}$ and $d(e) \notin \mathcal{V}_{A;B}$.

Theorem

Given a side information network $(\mathcal{V}, \mathcal{E}, s, z, T)$, if $(c(e) : e \in \mathcal{E}) \in \mathcal{R}(\mathcal{V}, \mathcal{E}, s, z, T)$, then for each $t \in T$ and each cut $\mathcal{V}_{s,z;t}$ there exists a random variable $U \in \mathcal{U}$ such that $U \leftrightarrow Y \leftrightarrow X$, $|\mathcal{U}| \leq |\mathcal{Y}|$, and

$$\begin{aligned} \sum_{e \in \mathcal{E}_{XY} \cap \mathcal{C}(\mathcal{V}_{s,z;t})} c(e) &\geq H(X|U) \\ \sum_{e \in \mathcal{E}_{Y} \cap \mathcal{C}(\mathcal{V}_{s,z;t})} c(e) &\geq I(Y;U). \end{aligned}$$

Proof Sketch



Inner Bound

Let $\mathcal{V}^*_{A;B}$ denote the cut for which $\sum_{e\in\mathcal{C}(\mathcal{V}_{A;B})} c(e)$ is minimal among all cuts $\mathcal{V}_{A;B}$.

Lemma

Let $(\mathcal{V},\mathcal{E},s,z,\{t\})$ be a side information network for which graph $(\mathcal{V},\mathcal{E})$ has no cycles. If there exists a vertex $v\in\mathcal{V}$ and a random variable $U\in\mathcal{U}$ such that $U\leftrightarrow Y\leftrightarrow X$, $|\mathcal{U}|\le|\mathcal{Y}|$, and

$$\sum_{e\in\mathcal{C}(\mathcal{V}^*_{s;v})} c(e)\ge H(X|U);\qquad \sum_{e\in\mathcal{C}(\mathcal{V}^*_{z;v})} c(e)\ge I(Y;U) \tag{1}$$

$$\sum_{e\in\mathcal{C}(\mathcal{V}^*_{s,z;v})} c(e) \ge H(X|U)+I(Y;U) \tag{2}$$

$$\sum_{e\in\mathcal{C}(\mathcal{V}^*_{v;t})} c(e) \ge H(X), \tag{3}$$

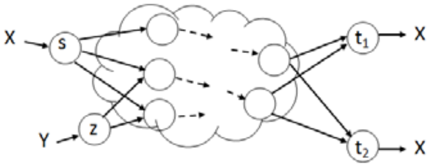
then $(c(e))_{e\in\mathcal{E}}\in\mathcal{R}(\mathcal{V},\mathcal{E},s,z,\{t\})$.

The main idea:

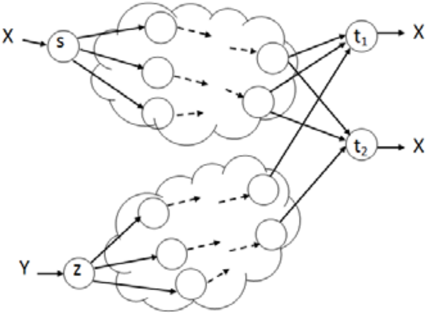
X might be reconstructed at internal node/nodes.

Two Sinks

We wish to characterize the rate region for networks of the following form:



We will completely characterize the rate region for networks of the form below, from which achievable rates for the general case result.

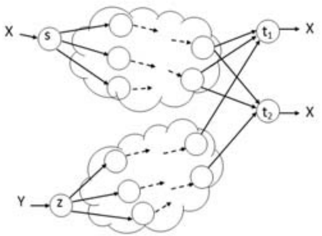


Two Sinks - Arbitrary Network from Y

Theorem

Assume both X and Y are binary symmetric. The outer bound is tight for the network to the right, and $(c(e) : e \in \mathcal{E}) \in \mathcal{R}(\mathcal{V}, \mathcal{E}, s, z, \{t_1, t_2\})$ iff $\exists\ U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ such that $U_1 \leftrightarrow Y \leftrightarrow X$, $U_2 \leftrightarrow Y \leftrightarrow X$, $|\mathcal{U}_1| \leq |\mathcal{Y}|$, $|\mathcal{U}_2| \leq |\mathcal{Y}|$, and

$$\begin{aligned}
\sum_{e \in \mathcal{C}(\mathcal{V}^*_{s;t_1})} c(e) &\geq H(X|U_1) \\
\sum_{e \in \mathcal{C}(\mathcal{V}^*_{z;t_1})} c(e) &\geq I(Y;U_1) \\
\sum_{e \in \mathcal{C}(\mathcal{V}^*_{s;t_2})} c(e) &\geq H(X|U_2) \\
\sum_{e \in \mathcal{C}(\mathcal{V}^*_{z;t_2})} c(e) &\geq I(Y;U_2)
\end{aligned}$$



Cut-sets

The rate region is fully characterized using cut-set analysis.

Proof Sketch - cont.

Descriptions:

From the source s: two descriptions of rates $H(X|U_1)$ and $H(X|U_2)$.
From the side information z: two descriptions of rates $I(Y;U_1)$ and $I(Y;U_2)$.

Min-cuts:

From the source s: $\sum_{e \in \mathcal{C}(\mathcal{V}^*_{s;t_1})} c(e) \geq H(X|U_1)$ and $\sum_{e \in \mathcal{C}(\mathcal{V}^*_{s;t_2})} c(e) \geq H(X|U_2)$.
From the side information z: $\sum_{e \in \mathcal{C}(\mathcal{V}^*_{z;t_1})} c(e) \geq I(Y;U_1)$ and $\sum_{e \in \mathcal{C}(\mathcal{V}^*_{z;t_2})} c(e) \geq I(Y;U_2)$.

The rates are too high to send the descriptions separately

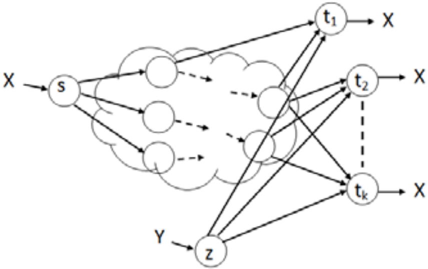
To achieve these generalized cut-set bounds, some kind of an incremental description is required.

K Sinks with Direct Links from Y - Result

Theorem

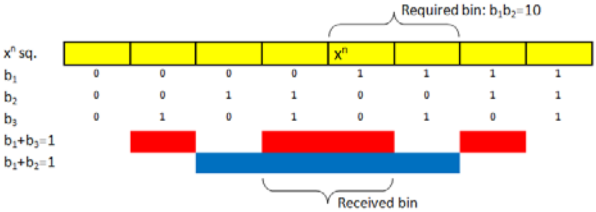
$(c(e) : e \in \mathcal{E}) \in \mathcal{R}(\mathcal{V}, \mathcal{E}, s, z, \{t_i\}_{i=1}^{K})$ *iff for any* $1 \leq i \leq K$ *there exist* $U_i \in \mathcal{U}_i$ *such that* $U_i \leftrightarrow Y \leftrightarrow X$, $|\mathcal{U}_i| \leq |\mathcal{Y}|$ *and*

$$\begin{aligned} \sum_{e \in \mathcal{C}(\mathcal{V}^*_{s;t_i})} c(e) &\geq H(X|U_i) \\ c((z, t_i)) &\geq I(Y; U_i) \end{aligned}$$



Proof Sketch - random binning and independent equations

The figure below represents binning to 8 bins and its binary representation:



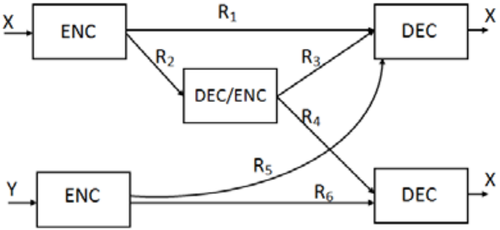
The bin we received is not the one we "intended" to send, but:

1. It has the correct size.
2. It contains the true x^n.
3. It is just as random.

The power of random binning

We can refine the random binning without having a true incremental multicast network code.

Example



$(R_1, \ldots, R_6) \in \mathcal{R}(\mathcal{V}, \mathcal{E}, s, z, \{t_1, t_2\})$ if and only if there exist random variables $U_1 \in \mathcal{U}_1$ and $U_2 \in \mathcal{U}_2$ such that $U_1 \leftrightarrow Y \leftrightarrow X$, $U_2 \leftrightarrow Y \leftrightarrow X$, $|\mathcal{U}_1| \leq |\mathcal{Y}|$, $|\mathcal{U}_2| \leq |\mathcal{Y}|$, and

$$\begin{aligned} R_1 + \min(R_2, R_3) &\geq H(X|U_1); \quad R_5 \geq I(Y; U_1) \\ \min(R_2, R_4) &\geq H(X|U_2); \quad R_6 \geq I(Y; U_2). \end{aligned}$$