Capacity Definitions of Composite Channels with Receiver Side Information

Michelle Effros, Andrea Goldsmith and Yifan Liang

effros@caltech.edu {andrea, yfl}@wsl.stanford.edu

Shannon Capacity Definition for General Channels and Limitations

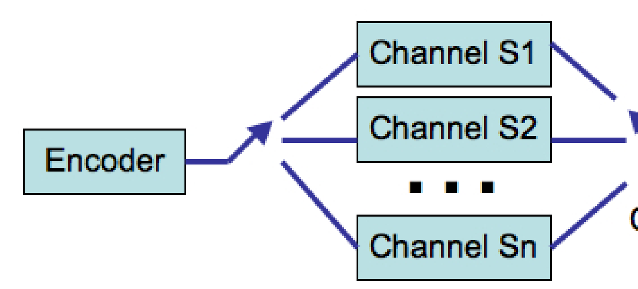
- Information Stable Channels
  - Input/output behave ergodically
  - Memoryless channels [Shannon 1948] $C = \max_X I(X;Y)$
  - With memory [Dobrushin 1963] $C = \lim_{n\to\infty} \sup_{X^n} \frac{1}{n} I(X^n;Y^n)$
- General Channels: [Verdu & Han 1994]
  - Normalized mutual information $(1/n)I(X^n;Y^n)$ may diverge
  - Distribution of information density $\frac{1}{n} i_{X^nW^n}(a^n;b^n) = \frac{1}{n}\log\frac{P_{Y^n|X^n}(b^n|a^n)}{P_{Y^n}(b^n)}$
  - Relationship $I(X^n;Y^n) = \mathbf{E}\{i_{X^nW^n}(X^n;Y^n)\}$

Coarse characterization | Refined characterization

- Shannon Capacity for General Channels: $C = \sup_X \underline{I}(X;Y)$
  - liminf in probability of normalized information density
  - Example: composite channels (information unstable)

$C = \sup_X \underline{I}(X;Y|S) = \sup_X \sup\{\alpha : \lim_{n\to\infty} \Pr[(1/n)\, i_{X^nW^n}(X^n;Y^n|S) \le \alpha] = 0\}$

  - mixture of stationary ergodic channels
  - include slowly fading wireless channels
  - Performance dominated by "worst channel"

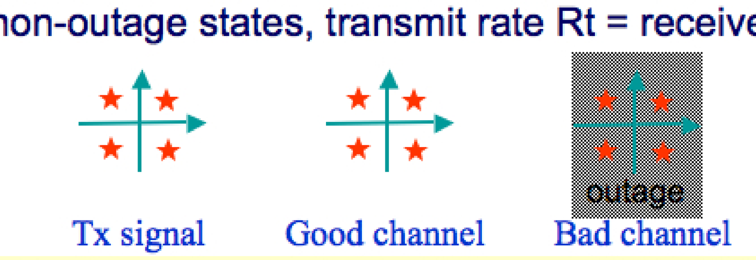


Channel Si occurs with prob pi
Fixed during transmission
Known at Rx but not Tx

- Limitations of Shannon Capacity Definition
  - Choice of codebook
    - A single encoder/decoder under Shannon definition
    - Reasonable to assume a single encoder without CSIT
    - Choice of different decoders with CSIR?
  - Rethinking: transmitted rate ?= received rate
    - Shannon: ALL Tx information decoded at ALL time
    - Relax the constraint:
      - Decode portion of the time? → Outage capacity
      - Decode partial information? → Expected capacity
      - Combined → Expected capacity with outage

Alternative Definition: Outage Capacity

- Outage Capacity
  - Decode portion (1-q) of the time
  - In non-outage states, transmit rate Rt = receive rate Rr

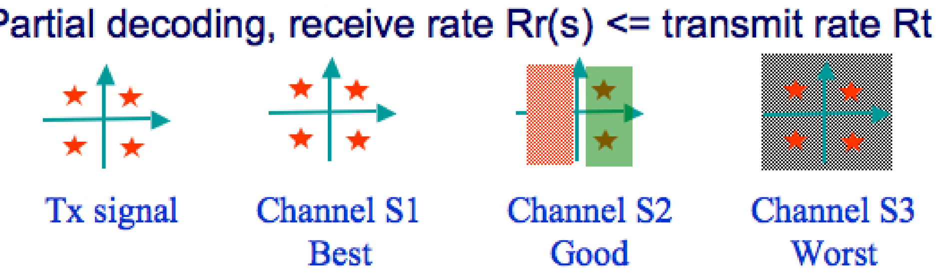


$C_q^o = (1-q)\sup_X \sup\{\alpha : \lim_{n\to\infty} \Pr[(1/n)\, i_{X^nW^n}(X^n;Y^n|S) \le \alpha] = q\}$

- Direct generalization of Shannon capacity $C = C_0^o$
- q can be chosen to max C_q^o
- When outage occurs
  - ARQ for retransmission
  - Approximate unreliable data from surroundings

Alternative Definitions: Expected Capacity and Expected Capacity with Outage

- Expected Capacity
  - Decode all the time
  - Partial decoding, receive rate Rr(s) <= transmit rate Rt



  - Definition: supremum of $\mathbf{E}_S R_S$ with $\mathbf{E}_S P_e^{(n,S)} \to 0$
  - Capacity Bounds

$\sup_q C_q^o \le C^e \le \sup_X \limsup_{n\to\infty} \mathbf{E}_{X^nY^n|S}\{(1/n)\, i_{X^nW^n}(X^n;Y^n|S)\}$

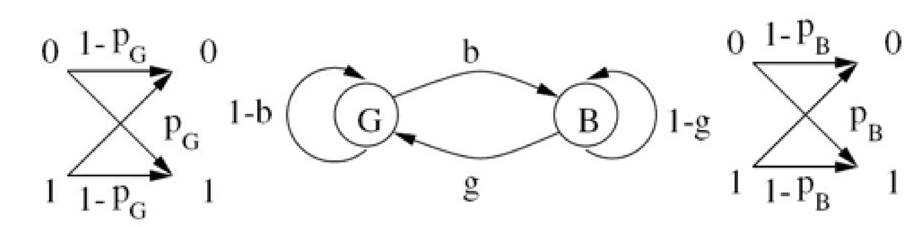
Achievable with outage-q code | Assuming Tx Side Information

- Broadcast Channel Codes for Expected Capacity
  - Channel states ↔ virtual multiple receivers
  - BC strategy for slowly fading channels [Shamai 2003]
  - Gilbert-Elliott channel, BSC with random crossover prob
- Expected Capacity with Outage
  - $C_q^e \ge \sup_{\varepsilon\ge 0} C_{q+\varepsilon}^o$ achievable with outage $(q+\varepsilon)$ code

$C_q^e \le (1-q)\sup_X \sup_\alpha \limsup_{n\to\infty} \mathbf{E}\left\{\frac{1}{n} i(X^n;Y^n|S) \,\middle|\, \frac{1}{n} i(X^n;Y^n|S) > \alpha\right\}$

Subject to outage constraint | Assuming CSIT for non-outage states

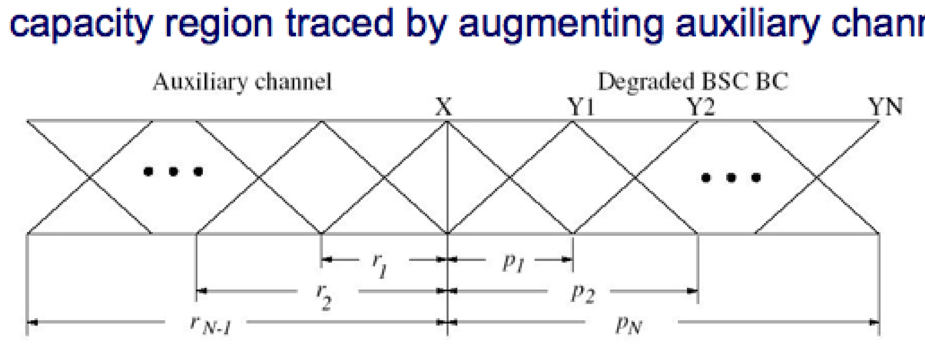
Example 1: Gilbert-Elliott Channel with Different Configurations



- Two-state Markov Chain
- Initial state distribution π_G π_B
- Stationarity and ergodicity varies with channel parameters
- Case 1: Stationary and Ergodic $\pi_G = g/(g+b)$, $\pi_B = b/(g+b)$
  - Shannon capacity = Expected Capacity = $\pi_G C_G + \pi_B C_B$
  - Outage C (expected C with outage) = $(1-q)(\pi_G C_G + \pi_B C_B)$
- Case 2: Nonstationary and Ergodic $\pi_G \ne g/(g+b)$, $b \ne 0, g \ne 0$
  - Effect of initial state distribution: transient
  - All four capacity definitions same with Case 1
- Case 3: Stationary and Nonergodic $b = g = 0$
  - Initial state chosen according to $\{\pi_G, \pi_B\}$ then fixed
  - Shannon Capacity $C = C_B = 1 - h(p_B)$ h(p) binary entropy
  - Outage capacity $C_q^o = (1-q)C_B$ $q < \pi_B$ otherwise $C_q^o = (1-q)C_G$
  - Expected capacity $\max_{0\le r\le 1/2} 1 - h(r*p_B) + \pi_G[h(r*p_G) - h(p_G)]$
    where a*b = a(1-b) + b(1-a), binary convolution
    - strictly less than $\pi_G C_G + \pi_B C_B$
      loss incurred from lack of CSIT
    - BC code achieves 1-h(r*p_B) for bad channel, and additional h(r*p_G)-h(p_G) for good channel

Example 2: Binary Symmetric Channel with Random Crossover Probabilities

- Random crossover probability p with distribution f(p), $0 \le p \le (1/2)$ corresponding cdf F(p)
- Shannon capacity C = 1 – h(p*), where p* = inf{p: F(p) = 1}
- Outage capacity, $C_q^o = (1-q)[1-h(p_q)]$ with $p_q = \inf\{p : F(p) \ge 1-q\}$
- Expected capacity ← characterization of BC rate region
  - stochastically degraded BC → cascade of BSC
  - capacity region traced by augmenting auxiliary channels



In continuous case: define increasing function r(p) as overall crossover prob of auxiliary channels up to that indexed by p

Incremental rate for layer p $|dR(p)| = h(p*r_p) - h(p*r_{p-dp})$

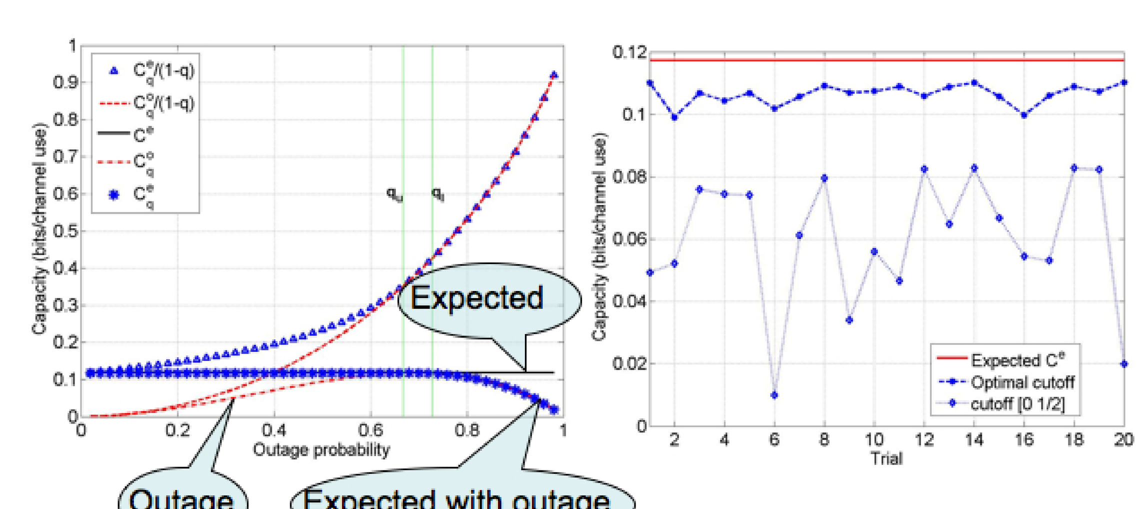
Expected rate $C^e = \max_{r(p)} \int_0^{1/2} f(p)R(p)dp = -\max_{r(p)} \int_0^{1/2} F(p)dR(p)$

optimization solved by functional analysis and Euler eqn

- Expected Capacity with outage:
  Same as C^e of a modified composite channel with pdf

$$\tilde{f}(p) = \begin{cases} f(p)/(1-q), & p \le p_q \\ 0, & p > p_q \end{cases}$$

Numerical example: uniformly distributed f(p) on [0,1/2]



- Shannon capacity = 0
- Nonzero incremental rate dR(p) only for $0.136 = p_l \le p \le p_u = 1/6$
- cutoff range [p_l, p_u] captures most benefit from optimization

Future Work: Joint Source-Channel Coding

- New end-to-end performance metric: expected distortion
  - Multi-resolution source code (SC) combined with BC channel code (CC): rates in BC region to min expected distortion
  - Distortion with multi-resolution SC: Gaussian source successively refinable, bounds exist for general sources
  - Transmit over two parallel non-ergodic links [Effros et al 04]: compare with multi-description SC and time-sharing CC
- Non-ergodic sources
  - Ergodic source/channel separation: simple interface H<C
  - Source/Channel codes separable without loss for some notion of capacity and some end-to-end performance metric?
  - In case separation fails, choice of interface? Performance vs. complexity tradeoff1