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Frequency-shift keying for Ultrawideband - How close to capacity can we get?

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Abstract

In wideband systems that decorrelate in time and frequency, capacity can be reached in the limit of infinite bandwidth by using impulsive frequency-shift keying with vanishingly small duty cycle. The richness of the codewords is then created through the large bandwidth. We show that frequency-shift keying with small duty cycle, using traditional random coding arguments, can achieve rates close to capacity in ultrawideband systems with limits on bandwidth and peak power.

1. Introduction

Wideband wireless applications have recently put transmission techniques for wideband multipath fading channel in the spotlight.

It's theoretically known that the capacity of the infinite-bandwidth general multipath-fading channel is non-zero and is equal to the capacity of the infinite-bandwidth additive white Gaussian noise (AWGN) channel with the same average received power constraint. This result has been shown by Kennedy[4], Gallager[2, §8.6] and Telatar and Tse[5].

To utilize this theoretical result in practice, one question naturally arises: given our theoretical insights, can we find actual transmission schemes with performance near the theoretical bounds? One possible way to answer this question is by directly applying a particular coding scheme and calculating its performance under general conditions.

The argument in [4] and [5] have shown that the capacity of multipath fading channel can be directly achieved by using frequency-shift keying (FSK) and non-coherent detection in a system that transmits at a low duty cycle. In contrast to wideband spread-spectrum schemes which create signals that mimic white Gaussian noise (WGN), their capacity-achieving transmission scheme is "peaky" both in frequency and time. It uses FSK with very low duty cycle. This signaling scheme yields a capacity for the Rayleigh fading channel which is the same as that for the AWGN channel in the limit of large bandwidth and large signal-to-noise ratio (SNR).

[4] and [5] only provide a capacity-achieving method for vanishingly small duty cycle and infinite bandwidth and the capacity limit for a Rayleigh fading channel. The results do not show how and how fast a system can approach this limit.

In this paper, we give a model of the system, study the performance of the FSK scheme in Rayleigh fading channels and compare it with the capacity-achieving scheme. The interplay amongst the capacity, bandwidth, SNR, and "peakiness" of the scheme is studied numerically. We show that a FSK scheme with limited bandwidth and "peakiness" can achieve performance that is of the order of that of the scheme with infinitely large bandwidth and small duty cycle.

2. The system model

The transmission scheme we examine is an impulsive FSK with small duty cycle and large bandwidth. The system is studied in Rayleigh fading channel conditions which are common in wireless communication scenarios. We send single-frequency signals which are selected from a large set of frequencies and transmitted using a low duty cycle. Because we use a large set of frequencies in a wide bandwidth, the frequency difference between two successive symbols is usually greater than the coherence frequency. Moreover, the low duty cycle means successive symbols are generally separated in time by more than a coherence time. Hence, the probability of sending two successive signals within a coherence band in the same coherence time is negligible. We can assume different symbols experience independent fading.

Assume the FSK system has M frequencies. In each symbol time, a signal may be or be not transmitted according to a probability θ $(0 < \theta \leq 1)$. θ is the duty cycle. In the interval in which some signal is transmitted, one of the M frequencies is sent. The transmitted signal can expressed as

$$x(t) = \begin{cases} \exp(2\pi i f_m t), & 0 \leq t \leq T_s; \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

where $f_m (1 \leq m \leq M)$ is the frequency of FSK signal and

T_s is the length of symbol interval. The received signal is

$$y(t) = \alpha(t)\sqrt{P}\exp\{2\pi i f_m t\} + n(t) \tag{2}$$

where $\alpha(t) = \sum_{j=1}^{K}\beta_j(t)\exp\{2\pi i f_m[-\tau_j(t)]\}$ is a circularly symmetric complex Gaussian process, K is the number of paths, $\{\beta_j(t)\}$ are the gains of paths, P is the power of the signals, $\{\tau_j(t)\}$ are delays of paths, and $n(t)$ is circularly symmetric complex Gaussian process with double-sided power density $\frac{N_0}{2}$ per dimension.

The coherence time T_c is the duration of time over which the channel remains essentially time-invariant. The delay spread T_d represents the uncertainty in the delay of the paths. In this paper, we focus on the case where the symbol time, T_s, is much less than coherence time of the channel and the delay spread is less than the symbol time. During the interval $[T_d, T_s]$, we can assume $\alpha(t) = A$ is constant, then the expression of received signal is

$$y(t) = A\sqrt{P}\exp\{2\pi i f_m t\} + n(t) \tag{3}$$

where A is circularly symmetric Gaussian variable. Without loss of generality, A can be assumed as a complex Gaussian variable with variance $\frac{1}{2}$ per dimension.

At the receiver, we use a bank of matched band-pass filters with central frequencies $\{f_n\}$ to detect signals. $n\,(1 \le n \le M)$ is the index of frequencies. In a certain symbol slot i, the output of the nth matched filter is

$$\widetilde{y}_n(t) = \int_{(i-1)T_s}^{t} y(\tau)e^{2\pi j f_n(t-\tau)}d\tau \quad (i-1)T_s \le t \le iT_s. \tag{4}$$

Let the frequency difference between two adjacent f_n's be $F_s = \frac{1}{T_s - T_d}$ and f_n be integer multiple of F_s. The whole bandwidth of the FSK system is $F = \frac{M-1}{T_s-T_d}$. When the mth symbol is sent, the outputs of the nth filter at time $(T_s - T_d) + iT_s$ are

$$\widehat{y}_n = \delta_{n,m}A\sqrt{P}(T_s - T_d) + \widehat{v}_n \tag{5}$$

where $\widehat{v}_n$ is a complex Gaussian variable. The $\widehat{v}_n$'s are mutually independent and $Cov[\widehat{v}_n] = (T_s - T_d)N_0$. The normalized output of the nth matched filter is

$$R_n = \frac{1}{\sqrt{N_0(T_s - T_d)}}\widehat{y}_n = S_n + Z_n \tag{6}$$

where Z_n is a complex Gaussian variable with variance $\frac{1}{2}$ per dimension. Let $\zeta = \frac{P(T_s-T_d)}{N_0}$. When $n = m$, S_n is a complex Gaussian variable with variance $\frac{\zeta}{2}$ per dimension, otherwise S_n is 0.

When the mth signal in the M frequencies is transmitted, the received $|R_m|^2$ has the probability density given by (7), otherwise, $|R_n|^2$ $(n \ne m)$ has the density given by (8).

$$P_{|R_n|^2}(r) = \frac{1}{1+\zeta}\exp\left[\frac{-r}{1+\zeta}\right] \quad (r > 0) \tag{7}$$

$$p_{|R_n|^2}(r) = \exp[-r] \quad (r > 0). \tag{8}$$

Keeping the system's average power a constant, changing the duty cycle parameter θ will affect the signal power P.

To decide which signal was transmitted, we use the maximum-a-posteriori (MAP) rule based on the observation of $|R_n|^2$ at the receiver. The probability system transmits nothing in a symbol slot is $1 - \theta$. Assume the M signals have equal probabilities to be transmitted, then the probability of transmitting the mth $(m = 1, 2, ..., M)$ signal is $\frac{\theta}{M}$.

When there is no signal being transmitted in a slot, the joint probability density of $\left(|R_1|^2, |R_2|^2, .., |R_M|^2\right)$ in the slot is

$$p_{|R_1|^2|R_2|^2,..,|R_M|^2}(r_1, r_2, ..., r_M) = \prod_{i=1}^{M}\exp(-r_i). \tag{9}$$

Otherwise, if one signal m is sent, the joint probability density in the slot should be

$$p(r_1, r_2, ..., r_M) = \frac{1}{1+\zeta}\exp\left[\frac{-r_m}{1+\zeta}\right]\prod_{i=1, i\ne m}^{M}\exp(-r_i). \tag{10}$$

The threshold used for MAP rule is

$$Z = \frac{\zeta+1}{\zeta}\ln\left((1+\zeta)\frac{(1-\theta)M}{\theta}\right). \tag{11}$$

If no $|R_n|^2$ is greater than Z, the receiver will decide that no signal was transmitted. Otherwise, the n corresponding to the largest $\{|R_n|^2\}$ is decided being transmitted.

We can determine the probability of error in this detection scheme. Let P_1 denote the probability of missing a signal, i.e. the receiver decides that nothing has been transmitted when a signal was transmitted. P_1 is

$$P_1 = \left(1 - \exp\left[\frac{-Z}{1+\zeta}\right]\right)(1 - \exp(-Z))^{M-1}. \tag{12}$$

Let P_2 denote the error probability of error detecting, i.e. the receiver decides one signal has been transmitted when actually another signal was transmitted. P_2 is given by

$$\begin{aligned} P_2 &= \int_Z^{\infty}\left(1 - (1 - \exp(-x))^{M-1}\right)\left(\frac{\exp\left[\frac{-x}{1+\zeta}\right]}{1+\zeta}\right)dx \\ &+ \left(1 - \exp\left[\frac{-Z}{1+\zeta}\right]\right)\left[1 - (1 - \exp[-Z])^{M-1}\right] \end{aligned} \tag{13}$$

Another kind of error occurs when the receiver decides a signal has been transmitted when nothing was transmitted. The probability of error is denoted by P_3:

$$P_3 = 1 - (1 - \exp(-Z))^M. \tag{14}$$

The additive noise is stationary and white. The complex gain of the channel, A, decorrelates in any two symbol slots because the symbols experience independent fading. We compute the capacity of this system using a discrete memoryless channel(DMC) model. All prior arguments justify the memoryless assumption.

In each symbol slot, we choose to transmit nothing or one of M signals. Hence, the discrete model has an input alphabet with size $M+1$, denoted as a_0, a_1, a_2, ..., a_M. a_0 means no signal is transmitted, and a_j $(1 \le j \le M)$ means the jth FSK signal is transmitted. The output is decided by the vector $\{|R_1|^2, |R_2|^2, |R_3|^2, ..., |R_M|^2\}$.

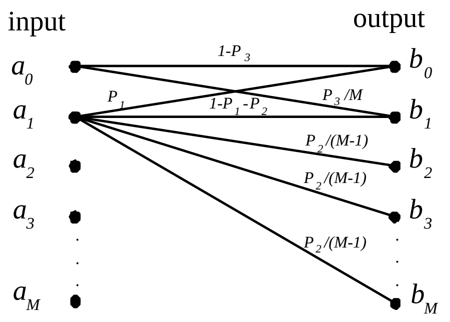

Figure 1: The discrete model

The transmission probabilities will determine the capacity of a DMC channel. We have already obtained these probabilities and can map them into the DMC model. It's known that the mutual information of a DMC channel is a concave function of the probability of input. The inputs a_1, a_2, ..., a_M are symmetric. Our prior assumption gives $P(a_0) = 1 - \theta$, $P(a_1) = P(a_2) = ... = P(a_M) = \frac{\theta}{M}$, which makes the mutual information achieve capacity. The capacity of the DMC channel is

$$C = H\left(\overline{b}\right) - H\left(\overline{b}|\overline{a}\right). \tag{15}$$

Then the capacity of the FSK system is $\frac{C}{T_s}$. In following section, we use numerical methods to calculate this capacity.

3. Bound the capacity

Before we discuss the capacity of the system, we will first look into the bound on capacity. From [5], we know that the capacity of the Rayleigh fading channel is the same as that of the AWGN channel in the limit of large bandwidth. So a tight bound of the capacity is

$$C = \frac{T_s - T_d}{T_s} F \ln(1 + \frac{P}{2N_0 F}) \tag{16}$$

where F is the bandwidth of the system, P is the average signal power, N_0 is single-sided power density per dimension for additive noise. The factor $\frac{T_s - T_d}{T_s}$ is introduced in because the effective time of transmission is $[T_d, T_s]$. When F goes to infinity, the bound approaches $(1 - \frac{T_d}{T_s})\frac{P}{2N_0}$. For the purpose of convenience, we will call this bound *infinite bandwidth bound*.

Another bound on capacity is deduced from the discrete model of the system. Owing to the limitation for the capacity of a discrete channel, we get

$$C \le \frac{\ln(M+1)}{T_s}. \tag{17}$$

This bound will be effective when SNR is very large as we will see in following discussion. We denote this bound as *limited bandwidth bound*.

4. Numerical results

We assume $T_d = 1\mu s$, T_s is between $10\mu s$ and $10ms$, F is between $1MHz$ and $10THz$.

A high peak signal power makes transmission reliable. When the bandwidth is very large, which means there is a large number of transmission symbols, we need to improve the peak power of the signal to get reliable transmission. However, when the average received signal power is constant, we should lower the signal duty cycle in order to improve the peak signal power, which will put a limitation on the data rate. Hence, we need to adjust the duty cycle parameter θ to optimize the system capacity. In our simulation, all results are optimized with respect to θ.

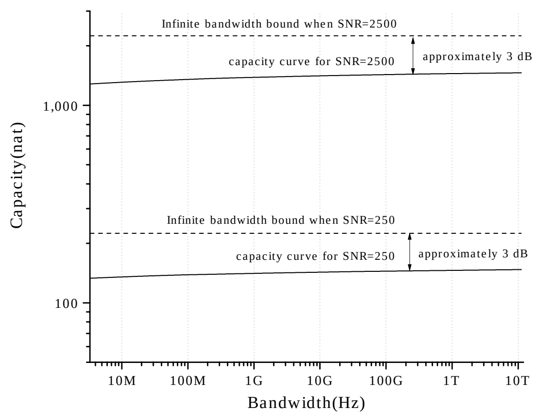

Figure 2: Capacity vs System Bandwidth

In Figure 2, we fix the SNR and let the symbol time T_s be $10\mu s$, and find as expected that the capacity increases with the bandwidth of system. However, it grows very slowly, and roughly has a gap of 3dB with the infinite bandwidth bound when the system bandwidth is between $1MHz$ and $10THz$. Note that with moderate bandwidth we can achieve a capacity very close to that with very large bandwidth.

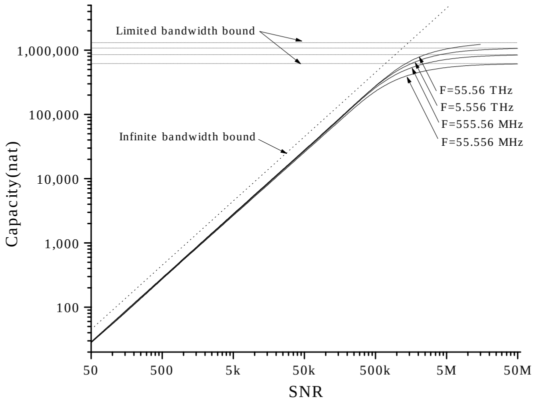

Figure 3: Capacity vs Signal-to-Noise Ratio

Figure 3 shows the capacity vs SNR performance under different bandwidth constraints and explores how the bounds effect. Only for very large SNR, the limited bandwidth bound will obviously effect on the capacity.

The symbol time T_s is also an important parameter for the FSK system. On the one hand, the greater T_s, the greater ζ, which will reduce the error probability, and thus improve capacity. On the other hand, increasing T_s will lower the symbol rate. If the average power and bandwidth are fixed, we can adjust T_s to maximize the capacity. With two different bandwidths, we show how symbol time T_s affects on the capacity in Figure 4, where SNR is 2500.

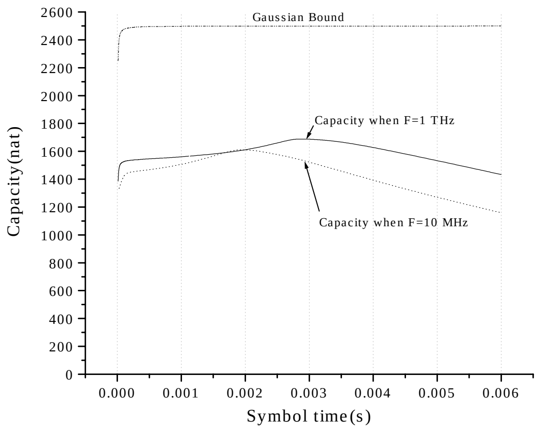

Figure 4: Capacity vs Symbol Time

The simulation shows that when SNR is reduced, the peaks of the curves move to the right, which means that a greater symbol time improves system capacity when average signal power is lower. Another observation is that the duty cycle for optimizing capacity is a non-decreasing function of SNR.

For high SNR, the limited bandwidth bound begins to take effect, high θ and small T_s are needed to achieve the maximum capacity. For lower SNR, θ decreases and T_s increases. When the SNR is very small, we need vanishingly a low duty cycle and very long T_s to reach the maximum.

5. Conclusions

When the received signal power is very large, the limited bandwidth bound will mainly constrain the capacity of the FSK system in a Rayleigh Fading Channel. Otherwise, the capacity grows slowly with bandwidth, and is nearly 3 dB lower than the infinite bandwidth bound for bandwidths commensurate with general mobile communication conditions. Using a moderate bandwidth, we can approach the capacity achieved by using very large bandwidth. To achieve the best performance in such a FSK system, we need to select optimal symbol time and duty cycle. Large θ and small T_s are needed for high SNR, small θ and long T_s are needed for low SNR.

References

[1] R.G. Gallager. A simple derivation of the coding theorem and some applications. *IEEE Trans. Inform. Theory*, pages 3–18, January 1965.

[2] R.G. Gallager. *Information Theory and Reliable Communication*. John Wiley & Sons, New York, NY, 1968.

[3] Linda W. Hughes. A simple upper bound on the error probability for orthogonal signals in white noise. *IEEE Trans. Commun.*, 40(4):670, April 1992.

[4] R.S. Kennedy. *Fading Dispersive Communication Channels*. Wiley Interscience, New York, NY, 1969.

[5] I.Telatar and D.N.C. Tse. Capacity and mutual information of wideband multipath fading channels. *IEEE Trans. Inform. Theory*, 46(4):1384–1400, July 2000.