Error Exponents for Channel Coding and Signal Constellation Design[1]

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Abstract — **We consider the optimization of the random coding error exponent of a class of memoryless channels where the input is subject to a peak and average-power constraints. The capacity-achieving input is shown to be typically *discrete* and an algorithm is developed for computing these optimal distributions. As one application, we propose a new signal constellation design which out-performs QAM and PSK for certain channels.**

I. Introduction

In [1] it is showed that under a very common communication channel setting, the capacity-achieving input distribution has a discrete set of amplitudes and can be approximated by simple discrete distributions even if the optimal one is continuous. A novel capacity-computation algorithm is proposed to generate a sequence of discrete distributions which converge to the optimal distribution.

In this paper, these results are extended to the random coding exponent. Consider a stationary, memoryless channel with input alphabet X, output alphabet Y, and transition density defined by

$$\mathsf{P}(Y \in dy \mid X = x) = p(y|x)\, dy\,, \qquad x \in \mathsf{X},\ y \in \mathsf{Y},$$

where Y is equal to either $\mathbb{R}$ or $\mathbb{C}$, and X is a closed subset of $\mathbb{R}$. The *random coding exponent* $E_r(R)$ may be expressed,

$$\begin{aligned} E_r(R) &:= \max_{0\le\rho\le 1}[-\rho R + \max_\mu(-\log(F(\rho,\mu)))], \\ F(\rho,\mu) &:= \int \left[\int \mu(dx) p(y|x)^{1/(1+\rho)}\right]^{1+\rho} dy, \end{aligned}$$

where $R \ge 0, \rho \ge 0$, and μ belongs to the set of probability distributions on X, subject to peak and average-power constraints

$$\mathcal{M}(\sigma_P^2, M, \mathsf{X}) := \Big\{\mu \in \mathcal{M} : \langle \mu, x^2\rangle \le \sigma_P^2,\ \mu\{[-M,M]\} = 1\Big\}.$$

II. Optimization Theory & Algorithms

Optimization of the random coding exponent is expressed as the convex problem,

$$\textbf{min}\ F(\rho,\mu)\,, \qquad \textbf{subject to}\ \ \mu \in \mathcal{M}(\sigma_P^2, M, \mathsf{X}). \tag{1}$$

Under some general assumptions on the channel, the following results summarize the structure of optimal input distributions.

Proposition. (i) *$F(\rho,\mu) = \langle \mu, g_\mu^\rho\rangle,\ \rho \ge 0,$ and*

$$g_\mu^\rho(x) := \int \left[\int \mu(dz) p(y|z)^{1/(1+\rho)}\right]^\rho p(y|x)^{1/(1+\rho)} dy.$$

(ii) *Fix $\rho \ge 0$, $\mu^\circ \in \mathcal{M}$, the first order sensitivity is*

$$\frac{d}{d\theta} F(\rho, \mu_\theta)\Big|_{\theta=0} = (1+\rho)\langle \mu - \mu^\circ, g_{\mu^\circ}^\rho\rangle.$$

(iii) *For each $\rho \ge 0$, there exists $\mu_\rho^* \in \mathcal{M}(\sigma_P^2, M, \mathsf{X})$ that is optimal, and hence achieves $E_r(R(\rho))$.*

(iv) *A given distribution $\mu \in \mathcal{M}(\sigma_P^2, M, \mathsf{X})$ is optimal if and only if there exists a real number λ_1^* and a negative real number λ_2^* such that*

$$\begin{aligned} g_\mu^\rho(x) &\ge \lambda_1^* + \lambda_2^* x^2, \quad x \in \mathsf{X}, \\ g_\mu^\rho(x) &= \lambda_1^* + \lambda_2^* x^2, \quad a.e.\ [\mu]. \end{aligned}$$

(v) *Suppose that $\sigma_P^2 = \infty$, and $M < \infty$. Then, for each fixed $\rho \ge 0$, there exists an optimal input distribution μ^* which is discrete, with a finite number of mass points.*

III. Cutting plane algorithm

The algorithm is initialized with an arbitrary distribution $\mu_0 \in \mathcal{M}(\sigma_P^2, M, \mathsf{X})$, and inductively constructs a sequence of distributions as follows. At the nth stage of the algorithm, we are given n distributions $\{\mu_0, \mu_1, \ldots, \mu_{n-1}\} \subset \mathcal{M}(\sigma_P^2, M, \mathsf{X})$ and

(i) The piecewise linear approximation $F_n(\rho,\mu) := \max_{0\le i\le n-1}\{(1+\rho)\langle \mu, g_{\mu_i}^\rho\rangle - \rho\langle \mu_i, g_{\mu_i}^\rho\rangle\}$.

(ii) The next distribution $\mu_n = \arg\min\{F_n(\rho,\mu) : \mu \in \mathcal{M}(\sigma_P^2, M, \mathsf{X})\}$.

It is shown under general conditions that the algorithm is convergent, in the sense that $F_n(\rho, \mu_n) \to F(\rho, \mu^*)$, and that the sequence of distributions $\{\mu_n\}$ converges weakly to an optimal distribution.

IV. Signal Constellation Design

For a symmetric, complex channel the optimal input distribution is circularly symmetric on $\mathbb{C}$, and we may apply the cutting plane algorithm by considering the distribution of the *magnitude* of X. We propose a approach to signal constellation design and coding in which the signal alphabet and associated probabilities are chosen to provide a random code that approximates the random code obtained through the nonlinear program (1). Numerical results on complex AWGN and Rayleigh fading channels show how a better designed code can greatly out-perform QAM and PSK for $R < C$.

References

[1] J.Huang and S. Meyn, "Characterization and Computation of Optimal Distributions for Channel Coding", to appear, *IEEE Trans. Inform. Theory.*

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