Summary: “Channel Coding Rate in the Finite Blocklength Regime” - Polyanskiy et al.

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This paper investigates the performance of finite blocklength codes over general classes of channels. In particular, results on the maximum achievable rate at a particular blocklength and a target error probability are presented. A new converse result is also presented, and it is shown that in a number of cases, the bounds are quite tight. In brief, the results in the paper may be divided into two regimes, asymptotic and non-asymptotic; a brief review of the results along with the essence of the proof techniques follows.

I. Non-asymptotic regime

The results presented for this regime involve fixed blocklengths and code size (i.e., rate).

A. *Results and proof techniques*

1) Achievability using random coding: the first result is an exact analysis of the average error probability of the maximum-likelihood (ML) decoder averaged over all random codes (with fixed length and rate) drawn according to a particular distribution. Although exact, the resulting expression is in terms of higher order moments of random variables that are functions of the information density, and is not easily computeable.
2) Random Coding Union (RCU) bound: This is a simpler upper-bound on the average error probability of a random coding ensemble. The proof technique involves upper-bounding the error probability of the ML decoder using a union bound. It is shown that this bound may be weakened to earlier recover achievability results of Gallager and Shannon.
3) Dependence Testing (DT) bound: Once again, this result is an upper bound on the average error probability. The bound itself is easier to compute than most previous bounds, since it does not involve optimizations over auxiliary random variables. The proof technique makes use of Shannon’s random coding with Feinstein’s suboptimal decoder, and makes connections with a Bayesian hypothesis testing problem. The bound is also extended to the case of a channel with input constraints (for example, an AWGN channel).
4) Maximal error probability bound: A general upper bound on the *maximal* error probability of a random code ensemble is presented. The achievability strategy is based on a particular sequential random coding technique, and a suboptimal decoder based on information density. Versions of the bound that are easier to compute

in the special case of the BEC and BSC are presented, and extensions to the case of channels with input constraints is also presented. It is shown that this bound is stronger than the conventional input-constrained version of Feinstein's bound.

5) $\kappa\beta$ bound: Another maximal error probability achievability bound, based on the Neyman-Pearson lemma and uses an auxiliary output distribution. Also works for cost constrained channels.
6) General converse: Two general converse results are presented, for average and maximal error probabilities. The result is derived using connections with hypothesis testing and the Neyman Pearson lemma. The bounds are in terms of minimax problems involving input and output densities (difficult to solve in general), but weaker results that are easier to compute are also presented. It is shown that previous converse results, such as Fano's inequality,Wolfowitz's strong converse and the Shannon-Gallager-Berlekamp converse may be obtained as special cases of this result.

B. Numerical results

Numerical results of the proposed bounds are presented for the BSC, BEC and the AWGN. The RCU bound is shown to be the tightest for the BSC, this improves quite a bit on previous achievability results such as Gallager and Feinstein. For the BEC, the DT bound is the best among the ones presented in this paper, but a bound that was previously proposed by Ashikhmin is almost as good as this one (in fact, Ashikhmin's bound is fractionally better for certain blocklengths). For the AWGN channel, the achievability bounds proposed in this paper are weaker than Shannon's earlier achievability result: the advantage of the bounds presented are that they are easier to compute than Shannon's, and are not too away from it for blocklengths that are not too small (although, one might argue that since computation of the bounds is an "offline" task, one should not worry too much about computational complexity, as long as the complexity is within reasonable limits).

II. Asymptotic regime

In this section, the main result is a normal approximation refinement of the coding theorem. It is shown for both discrete memoryless channels (DMC) and Gaussian channels,

$$\log M^*(n,\epsilon) = nC - \sqrt{nV}Q^{-1}(\epsilon) + O(\log n),$$

where $M^*(n,\epsilon)$ is the cardinality of a code of blocklength n and error probability ϵ, $Q^{-1}(\cdot)$ is the inverse Q-function, and V is a parameter referred to as the channel dispersion, which measures the stochastic variability of the channel relative to a deterministic channel with the same capacity. For DMCs, the result above is a refinement of Strassen's result in [1], with some minor improvements:

1) The present paper shows that the $\log n$ term cannot be negative (proved using the DT bound)
2) Strassen only proved the result for the case of $\epsilon < 0.5$, and claimed that it would extend to $\epsilon > 0.5$, whereas the current paper demonstrates that this extension to $\epsilon > 0.5$ is only true for certain DMCs termed as "nonexotic channels".

3) The present paper proves the above result for both average and maximal error probabilities, while [1] considers only maximal error probabilities.

Further, Strassen's result is not valid for channels with input constraints (in particular, the AWGN channel), while the present paper proves the result for the AWGN channel as well. For the special case of the BSC and BEC, improvements to the $\log n$ term are given: these improved estimates are shown to work much better than the naive estimates by numerically plotting the achievable rate against the converse. For blocklengths more than 200 for the BSC and BEC (800 for the AWGN), the normal approximation is seen to be very close to numerical evaluations of the finite-length achievable rates.

III. Overall merit

Overall, there are several positive aspects about the paper. Firstly, the introduction presents a good review of previous works, and presents the relevant achievability and converse results in a unified framework with common notation, which is very useful to someone new to the area. There is a significant amount of results in the paper (53 pages in double column), and a few proof techniques are quite novel. The results are quite relevant to a coding theorist: some numerical results presented with state-of-the-art codes show that about one half of the gap to capacity is due to the fundamental backoff due to finite blocklength; the other half of the gap is bridgeable with future advances in coding theory. In terms of actual numerical improvement of achievability results in comparison to prior works, the situation is not entirely rosy: the bounds for the BSC show reasonable improvement, but those for the BEC and AWGN are as good as or worse than prior results.

References

[1] V. Strassen, "Asymptotische abschätzungen in Shannons informationstheorie," in *Trans. 3rd Prague Conf. Inf. Theory*, Prague, 1962, pp. 689723.