The Vámos Network

Randall Dougherty, Chris Freiling, and Kenneth Zeger

***Abstract*— The well-studied Vámos matroid has provided a wealth of interesting theoretical results in matroid theory. We use the Vámos matroid to construct a new network, which we call the Vámos network. We then exploit the Vámos network to answer in the negative the open question as to whether Shannon-type information inequalities are in general sufficient for computing network coding capacities. To accomplish this, we first determine the smallest coding capacity upper bound that can be obtained for the Vámos network using only Shannon-type information inequalities. Then, we prove that a smaller capacity upper bound for the Vámos network can be obtained by using a non-Shannon-type information inequality discovered in 1998 by Zhang and Yeung. This is the first published application of a non-Shannon-type inequality to network coding. Finally, we demonstrate that one can compute the exact routing capacity and linear coding capacity of the Vámos network.**

I. INTRODUCTION

In this paper, unless stated otherwise, a *network* is a directed acyclic multigraph, some of whose nodes are information sources or receivers (e.g. see [18]). Associated with the sources are *messages* that they generate, which are assumed to be vectors of k arbitrary elements of a fixed finite alphabet of size at least 2. At any node in the network, each out-edge carries a vector of n alphabet symbols which is a function (called an *edge function*) of the vectors of symbols carried on the in-edges to the node, and/or a function of the node's message vectors if it is a source. Associated with each receiver are *demands*, which are a subset of all the messages of all the sources. Each receiver has *decoding functions* which map the receiver's inputs to vectors of symbols in an attempt to produce the messages demanded at the receiver. The goal is for each receiver to deduce its demanded messages from its in-edges and sources by having information propagate from the sources through the network. Each edge is allowed to be used at most once (i.e. at most n symbols can travel across each edge). Special cases of interest include linear codes, where the edge functions and decoding functions are linear, and routing codes, where the edge functions and decoding functions simply copy input components to output components.

This work was supported by the Institute for Defense Analyses, the National Science Foundation, and the UCSD Center for Wireless Communications.

R. Dougherty is with the Center for Communications Research, 4320 Westerra Court, San Diego, CA 92121-1969 (rdough@ccrwest.org).

C. Freiling is with the Department of Mathematics, California State University, San Bernardino, 5500 University Parkway, San Bernardino, CA 92407-2397 (cfreilin@csusb.edu).

K. Zeger is with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093-0407 (zeger@ucsd.edu).

A (k, n) *fractional code* is a collection of edge functions, one for each edge in the network, and decoding functions, one for each demand of each node in the network. For a (k, n) fractional code, the ratio k/n is called a *coding rate*. A (k, n) *fractional solution* is a (k, n) fractional code which results in every receiver being able to compute its demands via its demand functions. If a network has a (k, n) fractional solution over some alphabet, then we say the coding rate k/n is *achievable* for the network. A network is said to be *solvable* if it has a (k, n) fractional solution for the case $k = n = 1$.

The *coding capacity of a network with respect to an alphabet $\mathcal{A}$ and a class $\mathcal{C}$ of network codes* (a related definition appears in [18, p. 339]) is

$$\sup\{\frac{k}{n} : \exists\ (k, n) \text{ fractional coding solution in } \mathcal{C} \text{ over } \mathcal{A}\}.$$

If $\mathcal{C}$ consists of all network codes, then we simply refer to the above quantity as the *coding capacity* of the network with respect to $\mathcal{A}$. If the class $\mathcal{C}$ of network codes consists of all routing codes or all linear codes, then the coding capacity is referred to as the *routing capacity* or *linear coding capacity*, respectively. The coding capacity of a given network is said to be *achievable* if there is some fractional (k, n) solution for the network for which k/n equals the capacity.

Ahlswede, Cai, Li, and Yeung [1] showed that for a general network the linear coding capacity can be larger than the routing capacity. Li, Yeung, and Cai [11] showed in the special case of a multicast network (i.e. a network with a single source and each receiver demanding all messages), the coding capacity and the linear coding capacity are equal. It was shown in [3] that for all networks the coding capacity is independent of the alphabet size. Clearly the routing capacity is also independent of the alphabet size. However, it was shown in [4] that the linear coding capacity of a network can depend on the alphabet size and the largest linear coding capacity of a network over any finite field alphabet can be smaller than the network's coding capacity. It was also shown in [3] that the routing capacity is always rational, achievable, and computable by an algorithm.

For an undirected network (i.e. using undirected edges for information flow), the coding capacity can be larger than the routing capacity. However, for undirected networks where each message is demanded by exactly one receiver (called "multiple unicast"), it is presently unknown whether the coding capacity can be larger than the routing capacity. Li and Li [10] and Harvey, Kleinberg, and Rasala Lehman [6] have conjectured that the two capacities must always be equal for multiple unicast undirected networks. This conjecture remains an open question, although it was shown to be true for certain special undirected bipartite networks by Rasala Lehman [9].

Although the routing capacity of an arbitrary network is always computable, there is no known computationally efficient algorithm for such a task. Unfortunately, it is not even presently known whether or not there exist algorithms that can compute the coding capacity or the linear coding capacity of an arbitrary network. In fact, computing the exact coding capacity or linear coding capacity of even relatively simple networks can be a seemingly non-trivial task. At present, very few exact coding capacities have been rigorously derived in the literature. It is also known that a arbitrary network need not even be able to achieve its coding capacity [5].

As an alternative to determining exact coding capacities, it can be useful to determine bounds on the coding capacity and linear coding capacity of a network. One approach to obtaining capacity bounds (and possibly exact capacities) is to use information theoretic entropy arguments. The basic idea is to assume a network's source messages are i.i.d. uniform random variables on some finite alphabet and then to use standard information theory identities and inequalities to derive bounds on the largest possible ratio of the message dimension to the edge capacity.

However, it has been an open question whether standard Shannon-type information theoretic identities and inequalities are sufficient for computing the exact coding capacity or linear coding capacity of an arbitrary network, or whether they are sufficient for obtaining the best possible capacity bounds. We answer these open questions in the negative.

Specifically, we construct a network, which we call the Vámos network, and demonstrate that no collection of Shannon-type information inequalities can produce an upper bound on the coding capacity which is as small as an upper bound obtainable using a certain non-Shannon-type information inequality. To prove this result, we first show that Shannon-type information inequalities can only produce a coding capacity upper bound as low as 1 (Theorem VII.1), and then show that a non-Shannon-type information inequality argument can produce a coding capacity upper bound of $10/11$ (Theorem VII.2). Additionally, for the Vámos network, we compute the exact routing capacity (Theorem VII.4) and the the exact linear coding capacity over every finite field (Theorem VII.5). To establish these results we demonstrate a close relationship between the Vámos network and the well-known Vámos matroid. Most proofs are omitted here due to space limitations.

II. Network Fundamentals

We will assume throughout that there is at least one directed path from each source node to every receiver that demands a message generated by the source node.

Suppose a network $\mathcal{N}$ has message set μ, node set ν, and edge set ϵ. For each network node x, let $Z_{in}(x)$ denote the union of the set of source messages generated by x and the set of symbols carried on the in-edges of x, and let $Z_{out}(x)$ denote the union of the set of source messages demanded at x and the set of symbols carried on the out-edges of x. $Z_{in}(x)$ and $Z_{out}(x)$ are, respectively, called the *in-symbols* and *out-symbols* at node x. If we are considering solutions over alphabet $\mathcal{A}$, then let entropy H be defined in terms of logarithms base $|\mathcal{A}|$.

Lemma II.1. *If a network has a (k,n) fractional coding solution over alphabet $\mathcal{A}$ and the message components are independent random variables uniformly distributed over $\mathcal{A}$, then*

(N1) $H(\mu) = k|\mu|$ *and* $H(x) = k \quad \forall x \in \mu$
(N2) $H(x) \le n \quad \forall x \in \epsilon$
(N3) $H(y|Z_{in}(x)) = 0 \quad \forall x \in \nu,\ \forall y \in Z_{out}(x)$.

We call (N1)-(N3) the *network entropy conditions*.

III. Matroid Fundamentals

For a detailed background in matroid theory, the reader is referred to reference [15].

A *matroid* M is an ordered pair $(E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a set of subsets of E satisfying the following three conditions:

(I1) $\emptyset \in \mathcal{I}$.
(I2) If $I \in \mathcal{I}$ and $J \subseteq I$,then $J \in \mathcal{I}$.
(I3) If $I, J \in \mathcal{I}$ and $|J| < |I|$, then $\exists e \in I - J$ such that $J \cup \{e\} \in \mathcal{I}$.

The set E is called the *ground set* and the matroid $M = (E, \mathcal{I})$ is called a matroid *on* E. The members of $\mathcal{I}$ are called *independent sets* and any subset of E not in $\mathcal{I}$ is called a *dependent set*.

A maximal independent set of a matroid is called a *base* of the matroid. It can be shown that all bases are equinumerous.

There are many equivalent definitions of a matroid. One such alternate definition, which is particularly useful for us, uses the notion of a rank function.

Let M be the matroid $(E, \mathcal{I})$, let $X \subseteq E$, let $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}\}$, and let $M|X = (X, \mathcal{I}|X)$. Then $M|X$ is a matroid and the *rank* of X, denoted $r(X)$, is the size of a base of $M|X$.

Lemma III.1. *[15, pp. 22-23] If r is the rank function of a matroid with ground set E, then the following three conditions hold:*

(R1) *If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.*
(R2) *If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.*
(R3) *If $X, Y \subseteq E$, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.*

Lemma III.2. *[15, p. 23] Let E be a set and let $r : 2^E \to \{0, 1, 2, \ldots\}$ be a mapping satisfying $(R1)-(R3)$. Let $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. Then $(E, \mathcal{I})$ is a matroid having rank function r.*

Two matroids $(E, \mathcal{I})$ and $(E', \mathcal{I}')$ are said to be *isomorphic* if there exists a bijection $f : E \to E'$ such that $I \in \mathcal{I}$ if and only if $f(I) \in \mathcal{I}'$.

IV. Networks from Matroids

Suppose a network $\mathcal{N}$ has message set μ, node set ν, and edge set ϵ, and assume the messages in μ are independent random variables uniformly distributed over $\mathcal{A}$. Let be $M = (E, \mathcal{I})$ be a matroid with rank function r. We say that $\mathcal{N}$ is a *matroidal network* (or an M-*network*) if there exists a function $f : \mu \cup \epsilon \to E$ such that:

- (i) f is one-to-one on μ
- (ii) $f(\mu) \in \mathcal{I}$
- (iii) $r(f(Z_{in}(x))) = r(f(Z_{in}(x) \cup Z_{out}(x)))$, for every $x \in \nu$.

We have devised a method (omitted here to conserve space, but to be presented at the workshop) that can be useful for constructing an M-network from a matroid M. Such constructions allow us to transfer various interesting properties of matroids to networks. As matroid theory is a field rich in important results, the goal in constructing matroidal networks is to obtain some analogues for networks.

IV-A. The Vámos Matroid and Network

The *Vámos matroid* is an 8-element rank-4 matroid $(E, \mathcal{I})$ with $E = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and whose dependent sets are the 4-element sets which are coplanar in the three-dimensional drawing in Figure 1 (i.e. precisely $\{1, 2, 3, 4\}$, $\{2, 3, 5, 6\}$, $\{1, 4, 5, 6\}$, $\{2, 3, 7, 8\}$, and $\{1, 4, 7, 8\}$) and all subsets of E of cardinality at least 5.

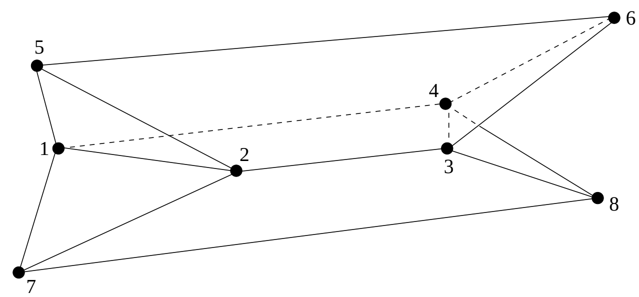

Fig. 1. A geometric description of the Vámos matroid.

We call the network shown in Figure 2 the *Vámos network*, since it is a matroidal network constructed from the Vámos matroid. The network has 13 nodes and 4 message variables. Nodes $n_9, \ldots, n_{13}$ are receiver nodes, each demanding one source message, except for n_{11}, which demands two source messages. As depicted in Figure 2, various sources generate the same messages (e.g. message c is generated by nodes n_1, n_5, n_7, n_{10}, and n_{12}). One could equivalently add 4 new nodes to the network, each generating a unique message, and then connect each source node to all of the nodes in Figure 2 which are shown to have the corresponding source message.

The specific mapping f used to define the network from the matroid in Figure 2 is determined by:

$$\begin{array}{cc} b \to 1 & c \to 2 \\ y \to 3 & x \to 4 \\ a \to 5 & w \to 6 \\ d \to 7 & z \to 8 \end{array}$$

V. Information Inequalities

Denote the joint entropy of any set of discrete random variables $X_1, \ldots, X_s$ over alphabet $\mathcal{A}$ with joint probability mass function p by

$$\begin{aligned} &H(X_1, \ldots, X_s) \\ &= - \sum_{u_1, \ldots, u_s} p(u_1, \ldots, u_s) \log_{|\mathcal{A}|} p(u_1, \ldots, u_s), \end{aligned}$$

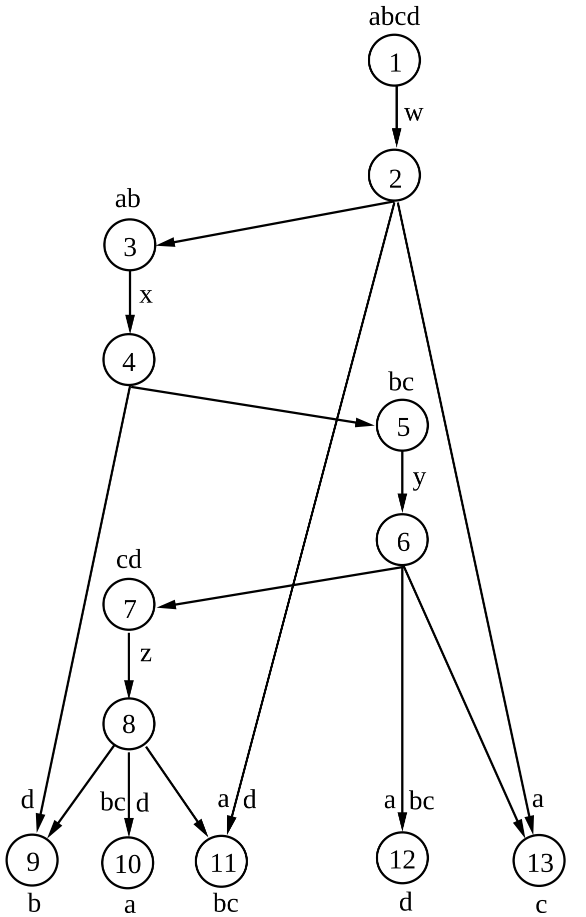

Fig. 2. The Vámos network. Message variables a, b, c, d are labeled above the sources generating them. Demand variables are labeled below the nodes demanding them.

the conditional entropy of $X_1, \ldots, X_s$ given $Y_1, \ldots, Y_t$ by

$$H(X_1, \ldots, X_s | Y_1, \ldots, Y_t) = H(X_1, \ldots, X_s, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t), \quad (1)$$

the mutual information between random variables $X_1, \ldots, X_s$ and $Y_1, \ldots, Y_t$ by

$$I(X_1, \ldots, X_s; Y_1, \ldots, Y_t) = H(X_1, \ldots, X_s) - H(X_1, \ldots, X_s | Y_1, \ldots, Y_t)$$

and the conditional mutual information between random variables $X_1, \ldots, X_s$ and $Y_1, \ldots, Y_t$, given $Z_1, \ldots, Z_w$ by

$$I(X_1, \ldots, X_s; Y_1, \ldots, Y_t | Z_1, \ldots, Z_w) = H(X_1, \ldots, X_s | Z_1, \ldots, Z_w) - H(X_1, \ldots, X_s | Y_1, \ldots, Y_t, Z_1, \ldots, Z_w).$$

It then easily follows that for any three collections p, q, and r of jointly related random variables, one can write:

$$I(p; q) = H(p) + H(q) - H(pq)$$
$$I(p; q|r) = H(p, r) + H(q, r) - H(r) - H(p, q, r).$$

We will make use of the following basic information theoretic facts [18] (where p, q, and r are each collections of jointly related random variables):

$$H(p) \leq H(p, q) \quad (2)$$
$$H(p, q) \leq H(p) + H(q) \quad (3)$$
$$H(p|q) \geq 0 \quad (4)$$
$$H(p) \geq 0 \quad (5)$$
$$I(p; q) \geq 0. \quad (6)$$

An *information inequality* is any inequality of the form

$$\sum_i C_i H(X_{g(i,1)}, X_{g(i,2)}, \ldots) \geq 0$$

(where s is a positive integer, $1 \leq g(i, j) \leq s$ for all i and j, and each C_i is a real number) that holds for all joint distributions of $X_1, \ldots X_s$. Since all conditional entropies and all conditional mutual informations can be written in terms of only joint entropies, any linear inequality involving conditional entropies and conditional mutual informations is an information inequality.

The information inequalities in (2)-(6) were originally given in 1948 by Shannon [17] (they follow from the convexity of the logarithm function) and can all be obtained as special cases of the inequality [18]

$$I(p; q|r) \geq 0. \quad (7)$$

A *Shannon-type information inequality* is any information inequality that is (or can be rearranged to be) of the form (where each c_i is a nonnegative real number):

$$\sum_i c_i I(p_i; q_i | r_i) \geq 0.$$

Virtually every known result in information theory that makes use of an information inequality only makes use of Shannon-type information inequalities.

Information inequalities provide a powerful tool for analyzing the capacity and solvability of networks. In particular, one can derive capacity bounds for a network by assuming its sources are random variables and using the fact that network nodes obey the entropy conditions given in Lemma II.1.

Suppose a network $\mathcal{N}$ has message set μ (with messages of dimension k), and edge set ϵ (with edges of capacity n). We will say that a number B is *a Shannon-type capacity upper bound* for the network if the network entropy conditions in Lemma II.1, together with the

Shannon-type information inequality (7) applied to all $p, q, r \subseteq \mu \cup \epsilon$, imply that $k \le Bn$. In this case, the coding capacity of $\mathcal{N}$ must be at most B, because if we take any (k, n) fractional solution to the network with alphabet $\mathcal{A}$, and let the elements of μ be independent random variables uniformly distributed over $\mathcal{A}$, then the resulting joint entropies will satisfy the network entropy conditions and the Shannon-type inequalities, and must therefore satisfy $k/n \le B$.

Thus, a Shannon-type capacity bound must not require non-Shannon-type information inequalities for its derivation. In particular, one can demonstrate that a number B is not a Shannon-type capacity upper bound by finding positive integers k and n with $k/n > B$, and finding a mapping $\hat{H} : 2^{\mu\cup\epsilon} \to \mathbf{R}$ such that (N1)-(N3) and (7) (for all $p, q, r \subseteq \mu \cup \epsilon$) are satisfied when replacing H by $\hat{H}$.

Lemma V.1. *Suppose a network has a message z which is demanded by a node y and is produced by exactly one upstream source node x of y. If there is a unique directed path from x to y, then the coding capacity of the network is at most* 1.

We note that Lemma V.1 can readily be generalized to the case where all paths in a network from some particular message go through one specific (“bottleneck”) edge.

Any information inequality that cannot be derived from (7) will be called a *non-Shannon-type information inequality*. It is known [18, p. 308] that all information inequalities containing three or fewer random variables are Shannon-type inequalities. The first known non-Shannon-type information inequality was published in 1998 by Zhang and Yeung and is stated in the following lemma. To date, it is the only published unconstrained non-Shannon-type information inequality for four random variables.

Lemma V.2. *[19] For any jointly related discrete random variables X_1, X_2, X_3, X_4 the following information inequality holds:*

$$\begin{aligned} 2I(X_3; X_4) \le\ & I(X_1; X_2) + I(X_1; X_3X_4) \\ & +3I(X_3; X_4|X_1) + I(X_3; X_4|X_2). \end{aligned}$$

Furthermore, this inequality cannot be derived purely from Shannon-type information inequalities.

In what follows, we will exploit Lemma V.2 to obtain an upper bound on the coding capacity of the Vámos network and we will then show that this bound is strictly tighter than any such bound obtainable using only Shannon-type information inequalities. Thus, we demonstrate that Shannon-type information inequalities are generally insufficient for computing the coding capacity of a network.

VI. Matroidal Network Capacity Bound from Shannon-type Information Inequalities

Lemma VI.1. *[18, p.297] Let E be a finite collection of jointly related discrete random variables. Then the following “polymatroidal axioms” hold:*

(H1) $H(\emptyset) = 0$

(H2) *If $X \subseteq Y \subseteq E$, then $H(X) \le H(Y)$*

(H3) *If $X, Y \subseteq E$, then $H(X \cup Y) + H(X \cap Y) \le H(X) + H(Y)$.*

The polymatroid axioms (H1)-(H3) are known [18, p.297] to be equivalent to the Shannon-type inequalities.

Lemma VI.2. *For any matroidal network, the least upper bound on the coding capacity obtainable from Shannon-type information inequalities is at least* 1.

Proof. Let M be a matroid and let $\mathcal{N}$ be an M-network; we will show that any Shannon-type capacity upper bound for $\mathcal{N}$ must be at least 1. Let μ and ϵ be the message set and edge set of $\mathcal{N}$. Let f be the given mapping from $\mu \cup \epsilon$ to the ground set E of M. The rank function r of M satisfies conditions $(R1) - (R3)$ in Lemma III.1. Hence, if we assign a value $H(Y)$ to each subset Y of $\mu\cup\epsilon$ by letting $H(Y) = r(f(Y))$, then H will satisfy the entropy conditions $(H1) - (H3)$ and hence all of the Shannon-type information inequalities. Also, $r(f(x)) = 1$ for each $X \in \mu$ and $r(f(\mu)) = |\mu|$, so network condition (N1) is satisfied with $k = 1$. Similarly, $r(f(x)) \le 1$ for each $x \in \epsilon$, so network condition (N1) is satisfied with $n = 1$. Furthermore, for any $y \in \mu \cup \epsilon$ and $X \subseteq \mu\cup\epsilon$, if y is an out-edge or demanded message of a node whose in-edges and source messages are the set X, then part (iii) of the definition of ‘matroidal’ gives $r(f(X \cup \{y\})) = r(f(X))$, which implies network condition (N3). Thus, the network conditions in (N1)-(N3) are satisfied with $k = n = 1$. So the Shannon inequalities and the network conditions do not imply $k \le Bn$ for any $B < 1$, and hence no $B < 1$ is a Shannon-type capacity upper bound for $\mathcal{N}$. ■

VII. Bounds on Coding Capacity of the Vámos Network

In general, the routing capacity of an arbitrary network can in principle be determined using a linear programming approach [3], although the computational complexity can be prohibitive for even relatively small networks. It is thus generally non-trivial to efficiently determine the routing capacity. In addition, there are presently no known techniques for computing the coding capacity or the linear coding capacity of an arbitrary network.[1] In fact, the linear coding capacity of a network depends, in general, on the finite field alphabet used [4], whereas the routing capacity and coding capacity do not depend on the alphabet size [3]. However, somewhat surprisingly, the exact routing capacity and linear coding capacity of the Vámos network can be computed, and the linear coding capacity of the Vámos network turns out to be independent of the finite field alphabet.

In what follows, we first determine the best possible upper bound on the coding capacity of the Vámos network, under the restriction that we only use Shannon-type information inequalities. Then we show that the upper bound on the coding capacity of the Vámos network can be improved if we allow the use of non-Shannon-type information inequalities. Specifically, we exploit the Zhang-Yeung non-Shannon-type information inequality given in Lemma V.2 and obtain a smaller upper bound on the coding capacity of the Vámos network than is obtainable using Shannon-type information inequalities. To the best of our knowledge, this is the first published application of a non-Shannon-type information inequality to network coding. Finally, we compute the exact routing capacity and the exact linear coding capacity of the Vámos network.

Theorem VII.1. *The least upper bound on the coding capacity of the Vámos network that can be derived purely from Shannon-type information inequalities is* 1.

Proof. By Lemma VI.2 the least upper bound is greater than or equal to 1. Since there is a unique path in the Vámos network from the source node n_1 to the node n_{12} which demands message d, the capacity can no be greater than 1 (by Lemma V.1). ■

The following theorem demonstrates that non-Shannon-type information inequalities can give tighter upper bounds on a network's capacity than can only Shannon-type information inequalities. In particular, Shannon-type information inequalities do not by themselves guarantee that the Vámos network is unsolvable, whereas adding one non-Shannon inequality indeed confirms the unsolvability of the network (since the coding capacity is strictly smaller than 1).

Theorem VII.2. *The coding capacity of the Vámos network is at most* $10/11$.

Corollary VII.3. *Shannon-type information inequalities are in general insufficient for determining the coding capacity of a network.*

[1] An exception is for multicast networks, where it is known that the coding capacity equals the linear coding capacity and is computable [11].

Theorem VII.4. *The routing capacity of the Vámos network is* $2/5$.

Theorem VII.5. *The linear coding capacity of the Vámos network over every finite field is* $5/6$.

REFERENCES

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, "Network information flow", *IEEE Transactions on Information Theory*, vol. IT-46, no. 4, pp. 1204–1216, July 2000.
[2] P.B. Bhattacharya, S.K. Jain, and S.R. Nagpaul, *Basic Abstract Algebra*, Cambridge University Press, New York, 1986.
[3] J. Cannons, R. Dougherty, C. Freiling, and K. Zeger, "Network routing capacity", *IEEE Transactions on Information Theory*, (to appear 2006).
[4] R. Dougherty, C. Freiling, and K. Zeger, "Insufficiency of linear coding in network information flow", *IEEE Transactions on Information Theory*, vol. 51, no. 8, pp. 2745-2759, August 2005.
[5] R. Dougherty, C. Freiling, and K. Zeger, "Unachievability of network coding capacity", joint issue of *IEEE Transactions on Information Theory & IEEE/ACM Transactions on Networking* (to appear June 2006).
[6] N. J. A. Harvey, R. D. Kleinberg, and A. Rasala Lehman, "Comparing network coding with multicommodity flow for the k-pairs communication problem", *Technical Report 964*, M.I.T. LCS, September 2004.
[7] N. J. A. Harvey, R. D. Kleinberg, and A. Rasala Lehman, "On the Capacity of Information Networks", *preprint*, 2005.
[8] A. Rasala Lehman and E. Lehman, "Complexity classification of network information flow problems", *41st Annual Allerton Conference on Communication Control and Computing*, Monticello, Illinois, October 2003.
[9] A. Rasala Lehman, "*Network Coding*", Ph.D. thesis, Massachusetts Institute of Technology, Department of Electrical Engineering and Computer Science, February 2005. [available on-line at: http://theory.lcs.mit.edu/ arasala/thesis.pdf]
[10] Z. Li and B, Li, "Network coding in undirected networks", *38th Annual Conference on Information Sciences and Systems (CISS)*, 2004.
[11] S.-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding", *IEEE Transactions on Information Theory*, vol. IT-49, no. 2, pp. 371-381, February 2003.
[12] R. Lnenicka, "On the tightness of the Zhang-Yeung inequality for Gaussian vectors", *Communications in Information and Systems*, vol. 3, no. 1, pp. 041-046, June 2003.
[13] K. Makarychev, Y. Makarychev, A. Romashchenko, and N. Vereshchagin, "A new class of non-Shannon-type inequalities for entropies", *Communications in Information and Systems*, vol. 2, no. 2, pp. 147-166, December 2002.
[14] D. Hammer, A.E. Romashchenko, A. Shen, and N.K. Vereshchagin, "Inequalities for Shannon entropies and Kolmogorov Complexities", *12th IEEE Annual Conference on Computational Complexity (CCC'97)*, p. 13.
[15] J. G. Oxley, Matroid Theory Oxford University Press, New York, 1992.
[16] I. Satake, *Linear Algebra*, Marcel Dekker, Inc., New York, 1975.
[17] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379-423 and 623-656, July and October, 1948.
[18] R. W. Yeung, *A First Course in Information Theory*, Kluwer, 2002.
[19] Z. Zhang and R. W. Yeung, "On characterization of entropy function via information inequalities", *IEEE Transactions on Information Theory*, vol. IT-44, no. 4, pp. 1440-1452, July 1998.
[20] Z. Zhang, "On a new non-Shannon type information inequality", *Communications in Information and Systems*, vol. 3, no. 1, pp. 047-060, June 2003.