A Reliable Architecture for Networks under Stress

Guy E. Weichenberg, Vincent W. S. Chan and Muriel Médard
Laboratory for Information and Decision Systems,
Massachusetts Institute of Technology

***Abstract*— In this paper, we consider Harary graphs as candidate solutions for the design of a physical network topology that achieves a high level of reliability using unreliable network elements. Our network model, which is motivated by the use of all-optical networks for high-reliability applications, is one in which nodes are invulnerable and links are subject to failure in a statistically independent fashion. Our reliability metrics are the all-terminal connectedness measure and the less commonly considered two-terminal connectedness measure. We compare in the low and high stress regimes common commercial architectures designed for all-terminal reliability in the low stress regime with the Harary graph architecture. We focus on Harary graphs as candidate topologies, as they have been shown to possess attractive reliability properties, and we derive new results for this family of graphs.**

***Index Terms*— network reliability, network design, Harary graphs**

I. Introduction and motivation

Commercial networks today are typically designed to recover from single failures at a given time, and thus provide adequate levels of reliability in the face of isolated failures with sparse connectedness. On the other hand, when very high levels of reliability are desired, or in the event of a catastrophic stress where a large portion of a network has failed, a high degree of connectedness in a network is required to ensure communication, since many links are needed to backup primary communication paths. Reliable network design must therefore be revisited for applications which demand high levels of reliability or which involve high levels of stress.

In this work, we consider highly connected Harary graphs for local-area network (LAN) design under different levels of stress. The cost of rich connectedness is a secondary issue in LANs in contrast to wide-area networks (WANs), where connectedness is hampered by the high cost of fiber runs. Owing to space constraints, we focus exclusively on the family of Harary graphs, which possess attractive reliability properties.

In this paper, we employ a model in which nodes are invulnerable and links are vulnerable which is particularly relevant to all-optical networks. In such networks, the highly-reliable passive optics in network nodes are modelled as invulnerable graph nodes, and fiber links and transmitter/receivers, which are significantly more prone to failures, are modelled as vulnerable graph edges.

The research in this paper was supported by: Defense Advanced Research Projects Agency, “Robust Architectures for Multi-Service, Multi-Level Reliability, Multi-Level Service and Multi-Priority WDM Local Area Networks”, MDA972-02-1-0021; and National Science Foundation ITR/SY, “High Speed Wavelength-Agile Optical Networks”, 008963-001.

In our reliability study, we will be solely concerned with the connectedness measures of a network. While network reliability metrics such as throughput or delay may be relevant to some network applications [1], connectedness measures are useful in situations where network performance is considered satisfactory as long as the network remains connected, or when the network's ability to provide a minimal level of service is of interest. In addition, connectedness is the relevant metric in many high-reliability applications and in LANs, where capacities of network components are over-designed, such that connectedness of nodes ensures acceptable network performance.

Most reliability studies to date have focused on the analysis and design of networks, with emphasis on all-terminal reliability, when links are very reliable. This is appropriate when modelling benign component failures due to low stress, such as normal wear of components. However, the design of networks when links are unreliable owing to high stress, which is addressed in this paper, is interesting for several reasons. In situations where the probability that a network is connected is quite small, some degree of connectedness in the network could still allow for important functions to be carried out, such as relaying emergency signals in times of distress. For example, in an aircraft application, even a small probability of connectedness could allow for adequate time for the aircraft to fail gracefully should it come under catastrophic stress.

As mentioned above, we consider both the case of low and high stress. In low stress situations, we assume that link failures occur with probability 0.2 or below, and in high stress situations, link failures occur with probability 0.8 or above. It should be emphasized that we are not assuming that networks normally operate in this latter regime of high link failure probability. Rather, high link failure probabilities are assumed given that a catastrophic stress has occurred.

Most of the necessary background is covered in the following section. Section III outlines the modelling assumptions employed in this work. In Section IV, we present bounding techniques which are valuable in the design of reliable networks. Section V justifies our pursuit of Harary graphs and specializes the techniques in Section IV to Harary graphs, and in doing so introduces new results for this family of graphs.

II. Graph theory background

We model a network as an undirected graph G with n nodes and e edges. The *incidence matrix* $\mathbf{A}$ of an undirected graph is the $n \times e$ matrix (each row corresponds to a node and each column to an edge) with the $(i,j)^{\text{th}}$ entry defined as follows:

$$a_{ij} = \begin{cases} 1, & \text{if edge } j \text{ is incident at node } i, \\ 0, & \text{otherwise.} \end{cases}$$

Two distinct nodes in an undirected graph are *connected* if there exists a path between the nodes. An undirected graph is *connected* if there exists a path between every pair of distinct nodes. A (minimal) set of edges in a graph whose removal disconnects the graph is a *(prime) edge cutset*. A (minimal) set of nodes which has the same property is a *(prime) node cutset*. The minimum cardinality of an edge cutset is the *edge connectivity* or *cohesion* $\lambda(G)$. The minimum cardinality of a node cutset is the *node connectivity* or *connectivity* $\chi(G)$. Analogous two-terminal metrics are the edge-connectivity $\lambda_{sd}(G)$ and node-connectivity $\chi_{sd}(G)$ with respect to a pair of nodes s and d. The two-terminal edge (respectively, node) connectivity of a graph is the minimum number of edges (respectively, nodes) whose removal disconnects the node pair.

A myriad of metrics can been defined to measure the reliability of networks. These criteria, which we now discuss, may be broadly categorized as either deterministic or probabilistic reliability metrics.

A. Deterministic metrics

The cohesion and connectivity of the graph underlying a network are prime examples of deterministic, all-terminal reliability criteria. A graph having maximum cohesion is a *max*-λ graph. Similarly, a graph having maximum connectivity is a *max*-χ graph. The following bounds relate connectivity and cohesion to the basic parameters of a graph [2]:

$$\chi \le \lambda \le \delta \le \frac{1}{n}\sum_{i=1}^{n} d_i = 2e/n \qquad (1)$$

where δ denotes the smallest node degree in the graph and d_i denotes the degree of node i. Harary has shown that the bounds in (1) can be achieved through the construction of *Harary graphs* [3].

More refined deterministic metrics for network reliability can also be defined, such as the number of edge or node cutsets of order λ or χ in a *max*-λ or *max*-χ graph, respectively. A graph is *super*-λ if it is max-λ and every edge disconnecting set of order λ isolates a point of degree λ.

An alternative measure of a graph's ability to remain connected is its number of spanning trees $t(G)$. The characterization of graphs with a maximum number of trees has been solved for sparse graphs when the number of edges is at most $n+3$, and for dense graphs when the number of edges is at most $n/2$ less than that of the complete graph K_n (the n-node graph which has all of its nodes adjacent) [4]–[6].

B. Probabilistic metrics

Sometimes deterministic reliability metrics do not provide adequate measure of the susceptibility of networks to disconnection because these metrics do not account for the reliability of network components. Probabilistic reliability criteria, on the other hand, require knowledge of deterministic network properties, in addition to the reliability of network components, and thus yield a more meaningful measure of network reliability. For this reason, this work will primarily be concerned with probabilistic reliability criteria. Probabilistic reliability metrics require the concept of a probabilistic graph. A *probabilistic graph* is an undirected graph where each node has an associated probability of being in an operational state and likewise for each edge. In probabilistic reliability analyses, networks under stress are modelled as probabilistic graphs.

Probabilistic reliability analyses have dealt mostly with the probability that a subset of nodes in a network are connected when links are very reliable. We thus define the *all-terminal reliability* of a probabilistic graph as the probability that any two nodes in the graph have an operating path connecting them. If links fail in a statistically independent fashion with probability p, then the all-terminal reliability $P_c(G,p)$ is given by:

$$P_c(G,p) = \sum_{i=n-1}^{e} A_i(1-p)^i p^{e-i} \qquad (2)$$

$$= 1 - \sum_{i=\lambda}^{e} C_i p^i (1-p)^{e-i} \qquad (3)$$

where A_i denotes the number of connected subgraphs with i edges, and C_i denotes the number of edge cutsets of cardinality i. For values of p sufficiently close to zero, $P_c(G,p)$ can be accurately approximated by $1 - C_\lambda p^\lambda (1-p)^{e-\lambda}$. In this case, an optimally reliable graph — one that achieves the maximum $P_c(G,p)$ over all graphs with the same number of nodes and edges — has a minimum number of cutsets of size $\lambda = \lfloor 2e/n \rfloor$. Therefore, in this regime of p, optimally reliable graphs are super-λ graphs. For values of p sufficiently close to unity, $P_c(G,p)$ can be accurately approximated by the first term in (2), $A_{n-1}(1-p)^{n-1}p^{e-n+1}$, where $A_{n-1} = t(G)$. Therefore, for values of p sufficiently close to unity, an optimally reliable graph has a maximum number of spanning trees.

The two-terminal reliability of a probabilistic graph is the probability that a given pair of nodes, s and d, have an operating path connecting them:

$$P_c^{sd}(G,p) = \sum_{i=w_{sd}}^{e} A_i^{sd}(1-p)^i p^{e-i} \qquad (4)$$

$$= 1 - \sum_{i=\lambda_{sd}}^{e} C_i^{sd} p^i (1-p)^{e-i} \qquad (5)$$

where w_{sd} is the shortest path length between nodes s and d, A_i^{sd} is the number of subgraphs with i edges that connect nodes s and d, λ_{sd} is the minimum number of edge failures required to disconnect nodes s and d, and C_i^{sd} is the number of cutsets with respect to nodes s and d of cardinality i. For the remainder of this work, unless otherwise stated, we redefine the *two-terminal reliability* of a probabilistic graph as $\min_{s,d}\left[P_c^{sd}(G,p)\right]$. Note that if we wish to maximize $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ when p is low, then the property of super-

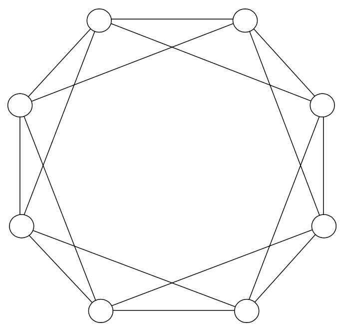

Fig. 1. The $H(8,4)$ Harary graph.

λ is a necessary condition since it ensures that $C^{sd}_{\lambda_{sd}}$ is upper bounded by two.

C. Harary graphs

Harary graphs, first presented in [3], achieve the bounds in (1). In a $H(n,\Delta)$ Harary graph, each node i, $0 \le i \le n-1$, is adjacent to nodes $i \pm 1, i \pm 2, \ldots, i \pm \lfloor \Delta/2 \rfloor (\text{mod } n)$; and if Δ is odd, then each node $i = 1, \ldots, \lfloor (n-1)/2 \rfloor$ is also adjacent to node $i + \lfloor n/2 \rfloor$. See Figure 1 for an example of a Harary graph. Harary graphs have the following properties [7]:

- $H(n,\Delta)$ has $e = \lceil n\Delta/2 \rceil$, $\chi = \lambda = \Delta$;
- $H(n,\Delta)$ is regular of degree Δ, unless n and Δ are both odd;
- $H(n,\Delta)$ has one node of degree $\Delta + 1$ and $n-1$ nodes of degree Δ if n and Δ are both odd.

In [8], Wang and Yang determined that even degree Harary graphs possess the fewest number of edge cutsets of cardinality i, when $\lambda \le i \le 2\Delta - 3$. Each cutset in the above range of cardinalities was shown to isolate a single node in the Harary graph.

Harary graphs, apart from the cases where n and Δ are both odd, belong to a more general family of graphs known as circulants. The *circulant* graph $C_n\langle a_1, a_2, \ldots, a_h \rangle$, where $0 < a_1 < a_2 < \ldots < a_h < (n+1)/2$, has $i \pm a_1, i \pm a_2, \ldots, i \pm a_h (\text{mod } n)$ adjacent to each node i. Owing to a theorem by Mader [9], which proves that every connected node-symmetric graph has $\lambda = \Delta$, all connected circulants are max-λ. Furthermore, the only circulants which are not super-λ are the cycles and the graphs $C_{2m}\langle 2, 4, \ldots, m-1, m \rangle$ with $m \ge 3$, and m an odd integer [10]. In [11], Wang and Yang derive a useful result for the number of spanning trees for the family of circulant graphs.

III. Network model

As mentioned in the previous section, networks will be modelled as probabilistic graphs. In addition, we assume the following about the the graphs underlying the networks considered in this paper:

- Nodes are invulnerable;
- Edges fail in a statistically independent fashion with probability p;
- Edge capacities are assumed to be sufficiently large to carry any possible network flow;
- Once an edge fails it cannot be repaired.

IV. Bounds on probabilistic reliability metrics

In this section[1], we introduce new and simple techniques to bound the probability of connection of a network and the probability of connection of a node pair in a network. The quality of these bounds are illustrated for the ten node, degree three Harary graph in Figures 5, 6, and 7.

A. All-terminal reliability when p is low

In this subsection, we derive upper and lower bounds for the probability that graph G is connected $P_c(G,p)$. The general approach we follow is based on enumeration of prime failure events. We define a *prime failure event* as an event in which a subset of nodes becomes disconnected from the rest of the graph through the failure of the minimal set of edges. Clearly, prime failure events constitute only a subset of all possible graph disconnection events, since graph disconnection can also occur when more than a minimal set of edges fail. Therefore, we may obtain an upper bound for $P_c(G,p)$ by subtracting from unity the probabilities of the mutually exclusive prime failure events:

$$P_c(G,p) \le 1 - \sum_{i=\lambda}^{e} B_i p^i (1-p)^{e-i} \tag{6}$$

where B_i is the number of distinct prime failure events of cardinality i. To obtain a lower bound for $P_c(G,p)$, we note that any failure scenario requires that at least one of the prime failure events occur. Therefore, we obtain a lower bound for $P_c(G,p)$ by subtracting from unity the union bound of the prime failure events:

$$P_c(G,p) \ge 1 - \sum_{i=\lambda}^{e} B_i p^i. \tag{7}$$

It remains to determine the coefficients B_i. If the graph under consideration is either trivially small, or simple and symmetric as is the case with Harary networks, then closed form, analytic solutions or bounds are obtainable; otherwise, one must resort to more general techniques.

We now introduce a technique to determine the coefficients B_i for general graphs. It is known that a vector representation of the prime failure events of a graph can be expressed in two ways as the modulo two sum of a subset of rows of a graph's incidence matrix [12]. Specifically, a prime failure event partitions a network into two subsets of nodes. Therefore, we can obtain a prime failure event by adding modulo two the rows that correspond to each of the nodes in one of the partitions. Hence, we can find all prime failure events of a graph by summing modulo two the rows of the $2^{n-1} - 1$ subsets of the rows the incidence matrix which yield distinct

[1] In the discussion that follows, we assume that all graphs are Δ regular and have maximum connectivity.

partitions of the network[2]. The B_i coefficients are determined by simply counting the number of distinct prime failure events obtained which have cardinality i.

B. Two-terminal reliability when p is low

If instead of the probability that graph $G = (N, E)$ is connected $P_c(G, p)$, we desire the probability that nodes $s, d \in N$ are connected $P_c^{sd}(G, p)$, we can use an approach similar to that of Section IV-A to obtain the following bounds:

$$1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i \leq P_c^{sd}(G,p) \leq 1 - \sum_{i=\lambda_{sd}}^{e} B_i^{sd} p^i (1-p)^{e-i} \tag{8}$$

where B_i^{sd} is the number of distinct prime failure events with respect to nodes s and d of cardinality i, and λ_{sd} is the minimum number of edge failures required to disconnect nodes s and d.

In order to determine the coefficients B_i^{sd}, we use an approach similar to that of Section IV-A. Since we are only interested in prime failure events of G which disconnect nodes s and d, we add modulo two to the row corresponding to s all possible subsets of the remaining rows of the incidence matrix, except for the row corresponding to d. Clearly, there are 2^{n-2} such possible subsets. This will provide us with a binary vector representation of all possible prime failure events which disconnect s and d.

C. All-terminal reliability when p is high

We approach the task of bounding $P_c(G, p)$ in the regime of high p in an analogous fashion to Section IV-A. The events of interest here, however, are the existence of spanning trees rather than prime failure events. A lower bound for $P_c(G, p)$ is obtained by summing the events that correspond to a spanning tree existing *and* the remaining links in the network being inoperative:

$$P_c(G,p) \geq t(G)(1-p)^{n-1} p^{e-n+1}. \tag{9}$$

An upper bound for $P_c(G, p)$ can be obtained be invoking the union bound on the spanning tree events:

$$P_c(G,p) \leq t(G)(1-p)^{n-1}. \tag{10}$$

It now remains to determine $t(G)$. Fortunately, this is a well studied problem, and $t(G)$ is known to be the determinant of an $(n-1) \times (n-1)$ matrix $\mathbf{T}(\mathbf{G})$ whose $(i, j)^{\text{th}}$ entry is defined as follows [13]:

$$t_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -1, & \text{if } i \text{ and } j \text{ are adjacent}, \\ 0, & \text{otherwise.} \end{cases}$$

[2]Note that if we sum modulo two the rows of all 2^n possible subsets, then we are counting every partitioning scenario twice, including the null and complete partitions.

D. Two-terminal reliability when p is high

When p is high, most of the links in a network have failed and the underlying graph has relatively few edges. In such sparsely connected graphs, the disconnection of nodes s and d is nearly equivalent to a set of edge-disjoint paths between s and d all having failed. To be precise, the disconnection of nodes s and d actually implies the failure of a set of Δ edge-disjoint paths between s and d, but the converse is not necessarily true. This is because each of the edge-disjoint paths can fail but there may still exist a path between s and d through the use of segments of the failed disjoint paths. Hence, we can lower bound $P_c^{sd}(G, p)$ as follows:

$$\begin{aligned} P_c^{sd}(G,p) &\geq 1 - \text{Pr}(\Delta \text{ edge-disjoint paths fail}) \\ &= 1 - \prod_{i=1}^{\Delta} \text{Pr(path } i \text{ fails)} \\ &= 1 - \prod_{i=1}^{\Delta} \left[1 - (1-p)^{l_i}\right] \end{aligned} \tag{11}$$

where l_i is the length of the ith edge-disjoint path, and the second and third lines follow from the independence of edge failures.

The value of $\min_{s,d} \left[P_c^{sd}(G,p)\right]$ when p is sufficiently high corresponds to a node pair with shortest path length equal to the graph diameter $k(G)$. A simple lower bound for $\min_{s,d} \left[P_c^{sd}(G,p)\right]$ is $(1-p)^{k(G)}$, which is just the probability that the shortest path between the most distant node pair is available:

$$(1-p)^{k(G)} \leq \min_{s,d} \left[P_c^{sd}(G,p)\right]. \tag{12}$$

A tighter lower bound for $\min_{s,d} \left[P_c^{sd}(G,p)\right]$ can be derived using (11) if the lengths or an upper bound on the lengths of the edge-disjoint paths joining the most distant node pair is available.

V. Analysis of Harary graphs

In this section, we specialize the results of the previous section to the family of Harary graphs. We focus on Harary graphs because they possess good reliability properties, particularly in the low p regime.

We showed in Section II-B that when p is sufficiently low, a necessary condition for $P_c(G, p)$ and $\min_{s,d} \left[P_c^{sd}(G,p)\right]$ to be maximized is that G must be super-λ. Among super-λ graphs, even degree Harary graphs are especially good when p is low, since they achieve the fewest number of cutsets of cardinality i, when $\lambda \leq i \leq 2\Delta - 3$. Furthrmore, as we show in the next subsection, Harary graphs possess good reliability properties relative to commercial architectures, especially in the low p regime.

A. Comparison of Harary graphs and commercial networks

We now conduct a comparison among Harary graphs — our candidate topology — and some topologies employed in commercial networks — dual-homed switch graphs, rings, and multi-rings.

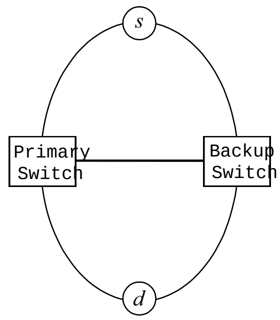

Fig. 2. Dual-homed switch topology (Ethernet).

The dual-homed switch architecture is illustrated in Figure 2. In this topology, each node is connected to a primary and a secondary switch through a dedicated link. In addition, the two switches are bridged. Communication between a node pair can be carried out via any available path. Switched Ethernet is a very common example of the dual-homed switch architecture [14], and we will therefore refer to the dual-homed switch architecture simply as Ethernet. In an m multi-ring graph, there are m undirected edges between nodes that would otherwise have one undirected edge in a ring graph. Multi-ring graphs are fairly accurate representations of two-fiber unidirectional path-switched rings (UPSRs) when $m = 1$, and four-fiber bidirectional path-switched rings (BLSRs) when $m = 1, 2$ [15].

In our comparison, each graph supports 14 nodes and the degree of the multi-ring and the Harary graph is four. We further assume that nodes, including the two switches in the Ethernet topology, are invulnerable and that the Ethernet bridge reliability is identical to that of the other links in the network.

Figure 3 depicts the performance of the topologies when $p \leq 1/2$. The Harary graph $H(14, 4)$ is seen to achieve the best all- and two-terminal reliability performance of all the topologies considered. $H(14, 4)$ outperforms the double ring because the reliability of Harary graphs scales more weakly than that of multi-rings with the number of nodes; and it outperforms Ethernet and the ring largely due to its higher node degree. With respect to all-terminal reliability, $H(14, 4)$, since it is super-λ, possesses $n = 14$ cutsets of order four, whereas the double ring, the other degree four topology, possesses $\binom{n}{2} = 91$ cutsets of order four. Ethernet's all-terminal reliability performance is largely governed by its $n = 14$ cutsets of order two, and the ring by its $\binom{n}{2} = 91$ cutsets of order two. For two-terminal reliability, the number of cutsets of order four is two in $H(14, 4)$, whereas it is $n^2/4 = 49$ for the double ring. The number of cutsets of order two is two in Ethernet, whereas it is $n^2/4 = 49$ in the ring.

In Figure 4, the performance of the topologies is plotted when $p \geq 1/2$. Again, with respect to all-terminal reliability, $H(14, 4)$ outperforms the other topologies considered, owing to its larger number of spanning trees. $H(14, 4)$ has 1.9898×10^6 spanning trees, whereas the double ring has

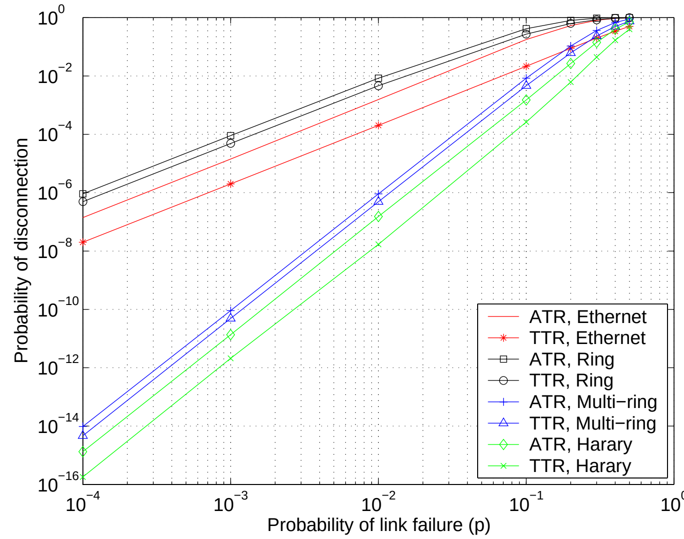

Fig. 3. Probability of disconnection versus p for the 14 node Ethernet, ring, double-ring and $H(14, 4)$ graphs when $p \leq 1/2$.

$n2^{n-1} = 1.1469 \times 10^5$ spanning trees and the ring only $n = 14$ spanning tress. With respect to two-terminal reliability, Ethernet achieves the best performance of all, owing to its diameter of two. $H(14, 4)$ has the next best two-terminal performance with a diameter of four, whereas the ring and the double ring possess diameters of $\lfloor n/2 \rfloor = 7$.

The comparison conducted in this subsection substantiates our claim that Harary graphs possess reliability advantages relative to topologies employed in present-day commercial networks. In fact, Harary graphs outperformed the other topologies except when two-terminal reliability in the high p regime was considered. Of course, the price paid for the better reliability of Ethernet in this respect is the cost of the switches. We conclude that there is a significant reliability advantage in strategic positioning of link capacity, as in a Harary graph, rather than adding redundant backup links. It should be noted, however, that when p is high it is possible to find circulant graphs with the same number of nodes and edges which possess more spanning trees and smaller diameters, and hence better reliability performance when p is high, than the corresponding Harary graphs [16].

B. A useful Harary graph result

Before beginning our analysis of Harary graphs, we prove an intuitive and useful theorem regarding this family of graphs.

Theorem 1 *Consider a Harary graph $H(n, \Delta)$, where Δ is even. Partition the n nodes into a subset of j nodes S_j and a subset of $n - j$ nodes S_{n-j}, where $j \leq n - j$. Then, the minimum number of edges joining S_j to S_{n-j} occurs when the j nodes in S_j (and hence, the $n - j$ nodes in S_{n-j}) are consecutively numbered (modulo n).*

To prove the theorem, we need the following lemma:

Lemma 1 *Partition the n nodes of the $H(n, \Delta)$ Harary graph into a subset of $j \leq n - j$ nodes S_j, and a subset of $n - j$*

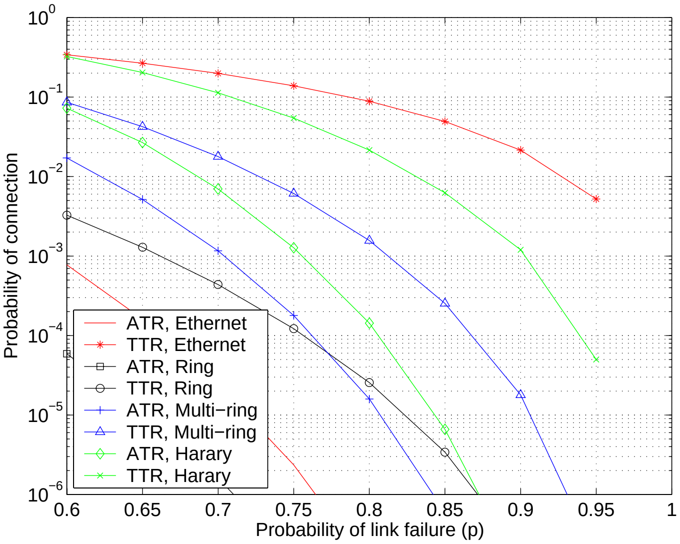

Fig. 4. Probability of connection versus p for the 14 node Ethernet, ring, double-ring and $H(14,4)$ graphs when $p \geq 1/2$.

nodes S_{n-j}, such that the nodes in S_j (and hence the nodes in S_{n-j}) are consecutively numbered (modulo n). Then, the number of edges joining S_j to S_{n-j} is:

$$\begin{cases} \Delta, & \text{if } j = 1, \\ j\Delta - 2\binom{j}{2}, & \text{if } 2 \leq j \leq \lfloor \Delta/2 \rfloor + 1, \\ \lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil, & \text{otherwise.} \end{cases} \quad (13)$$

Proof. The case of $j = 1$ is trivial. When $2 \leq j \leq \lfloor \Delta/2 \rfloor + 1$, a consecutive partition of j nodes allows the nodes in S_j to be fully connected. In this case, the number of edges joining S_j to S_{n-j} follows from the fact that the total number of edge endpoints incident at S_j's nodes is $j\Delta$ and that the total number of edge endpoints in a fully connected subgraph of j nodes is $2\binom{j}{2}$. For the remaining case, when the nodes are consecutively arranged, the nodes at either end of the S_j partition possess $\lceil \Delta/2 \rceil$ connections to S_{n-j}, the nodes which are second from either end of the partition possess $\lceil \Delta/2 \rceil - 1$ connections to S_{n-j}, and so on. Hence, the total number of edges joining S_j to S_{n-j} is the constant $2\sum_{i=1}^{\lceil \Delta/2 \rceil} i = \left(\lceil \Delta/2 \rceil^2 + \lceil \Delta/2 \rceil\right)$, as required. □

We are now ready to prove Theorem 1:

Proof of Theorem 1. The case of $j = 1$ is trivial. Consider now the case of $2 \leq j \leq \Delta/2+1$. Note that minimizing the number of edges joining S_j to S_{n-j} is equivalent to maximizing the number of internal edges shared by the nodes of one of the partitions. When $2 \leq j \leq \Delta/2 + 1$, a consecutive partition of j nodes allows the nodes in S_j to be fully connected, yielding the maximum number of internal connections, and hence the minimum number of external edges.

For the remaining case where $\Delta/2 + 2 \leq j \leq n/2$, we carry out the proof by induction. We may use our result for $j = \Delta/2 + 1$ as our base case. Now, assume that a consecutive arrangement of j nodes achieves the minimum number of external edges. Let us now proceed by contradiction by assuming the existence of a partition S'_{j+1} of $j + 1$ nodes which achieves a smaller number of external edges than the number achieved by a consecutive arrangement of $j+1$ nodes in Lemma 1.

If we can find a node in S'_{j+1} which contains at least $\Delta/2$ edges to S'_{n-j-1}, then we move this node to S'_{n-j-1}. This creates a partitioning of the graph into j and $n - j$ nodes which achieves fewer edges joining the two partitions than a consecutive arrangement. This would contradict our induction hypothesis, implying that a consecutive arrangement of nodes is optimal.

Now, let us consider the case where there does not exist a node in S'_{j+1} which contains at least $\Delta/2$ edges to S'_{n-j-1}. We proceed by finding a pair of consecutive nodes in the graph such that one of the nodes u belongs to S'_{j+1} and the other node v belongs to S'_{n-j-1}. Examining the window of $\Delta + 1$ consecutive nodes centered at u, our assumption that there does not exist a node in S'_{j+1} which has at least $\Delta/2$ edges to S'_{n-j-1} requires that at least $\Delta/2 + 2$ nodes in this window belong to S'_{j+1}. We now consider the window of $\Delta+1$ consecutive nodes centered at v. Since the window formed by the union of u and v's windows of length $\Delta+1$ has size $\Delta+2$ nodes, there can be at most $\Delta/2$ nodes in this larger window that belong to S'_{n-j-1}. By moving v to S'_{j+1}, we create a partitioning of the graph into $j+2$ and $n-j-2$ nodes which achieves fewer edges joining the two partitions than that of the S'_{j+1} and S'_{n-j-1} partitioning, and hence, fewer than that of a consecutive arrangement of j and $n-j$ nodes. Note that by moving v to S'_{j+1}, we have not created a node in S'_{j+1} which possesses at least $\Delta/2$ edges to the other partition. This is because the $j + 1$ nodes initially in S'_{j+1} only gain internal edges by moving v to S'_{j+1}, and v now possesses fewer than $\Delta/2$ edges to the other partition. Thus, we can continue in this way – finding a pair of consecutive nodes in different partitions and moving one node to the other partition, always decreasing the number of edges connecting the partitions, until we have increased the size of our initial partition of j nodes to $n-j$ nodes. At this point, we have created a partitioning of the graph into j and $n-j$ nodes which achieves fewer edges joining the partitions than the partitioning of the graph in our induction hypothesis, which was assumed to be optimal. This is a contradiction, implying that a consecutive arrangement of nodes is optimal. □

C. All-terminal reliability when p is low

Every graph disconnection scenario can be viewed as a partitioning of the graph into two subsets of nodes which are disconnected. Now, since a partition of j consecutive nodes minimizes the number of edges joining S_j to S_{n-j} in an even degree Harary graph, the probability that a partition of j nodes becomes disconnected from a partition of S_{n-j} nodes is maximized when the partition of j nodes are consecutive. We can therefore form an upper bound for the probability of graph disconnection (and hence, a lower bound for the probability of graph connection) by upper bounding the probability of S_j and S_{n-j} becoming disconnected by the consecutive case, and then employing a union bound on these events. Furthermore, since

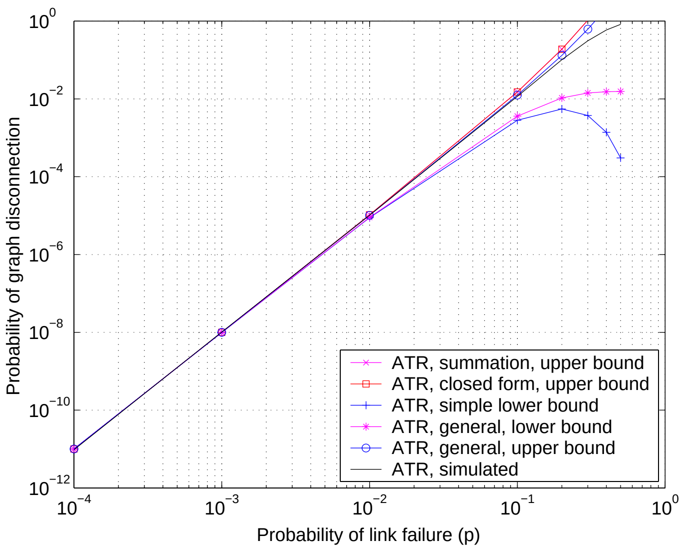

Fig. 5. Probability of graph disconnection versus p for $H(10,3)$. "ATR, simple lower bound" refers to $np^{\Delta}(1-p)^{e-\Delta}$, "ATR, general, lower bound" refers to (6), "ATR, general, upper bound" refers to (7), "ATR, summation, upper bound" refers to (14), and "ATR, closed form, upper bound" refers to (15).

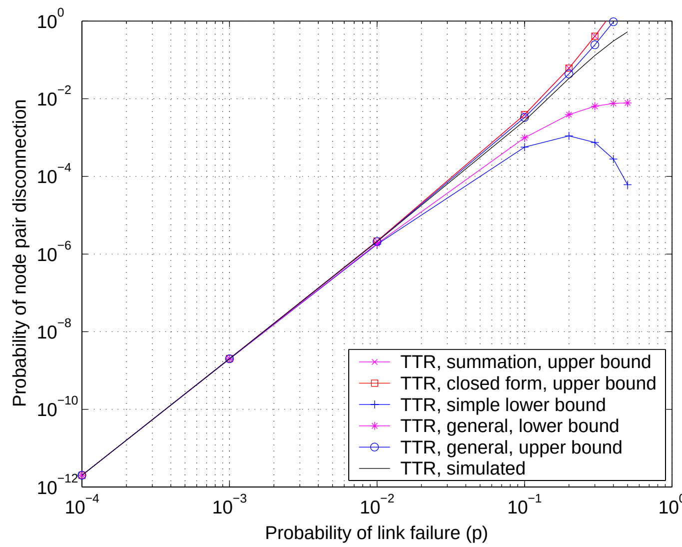

Fig. 6. Worst-case probability of node pair disconnection versus p for $H(10,3)$. "TTR, simple lower bound" refers to $2p^{\Delta}(1-p)^{e-\Delta}$, "TTR, general, lower bound" refers to the right inequality of (8), "TTR, general, upper bound" refers to the left inequality of (8), "TTR, summation, upper bound" refers to (16), and "TTR, closed form, upper bound" refers to (17).

the $H(n, 2\lfloor\frac{\Delta}{2}\rfloor)$ Harary graph is a subgraph of the $H(n,\Delta)$ Harary graph, the all-terminal reliability of an odd degree Harary graphs is lower bounded by the all-terminal reliability of the Harary graph with degree one less. Thus, a lower bound for $P_c(G,p)$ for a Harary graph $H(n,\Delta)$ is:

$$P_c(G,p) \geq 1 - \left(np^{\Delta} + \sum_{i=2}^{\lfloor\Delta/2\rfloor+1} \binom{n}{i} p^{i\Delta-2\binom{i}{2}} + \sum_{i=\lfloor\Delta/2\rfloor+2}^{\lfloor n/2\rfloor} \binom{n}{i} p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil} \right). \quad (14)$$

Because prime failure events were used to derive (14), the bound is tight for low p. We can derive a slightly looser lower bound for $P_c(G,p)$ by bounding some of the terms in (14) [16]:

$$P_c(G,p) \geq 1 - \left(np^{\Delta} + \left\lfloor\frac{\Delta}{2}\right\rfloor \binom{n}{\lfloor\frac{\Delta}{2}\rfloor+1} \left(p^{2\Delta-2} - p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right) + p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil} \left[2^{n-1} + \frac{1}{2}\binom{n}{\lfloor n/2\rfloor} - n - 1\right]\right). \quad (15)$$

The quality of these bounds is illustrated in Figure 5 for the ten node, degree three Harary graph. The more useful upper bounds on the probability of disconnection are tighter than the lower bounds. Furthermore, these upper bounds are quite tight for values of p less than approximately 0.2.

D. Two-terminal reliability when p is low

The derivation of a lower bound for the node pair connection probability $P_c^{sd}(G,p)$ is virtually identical to that of $P_c(G,p)$ for low p in Section V-C. The difference is that we are only interested in partitions of the network nodes that result in nodes s and d residing in different partitions. Hence, we modify (14) to obtain:

$$P_c^{sd}(G,p) \geq 1 - \left(2p^{\Delta} + 2\sum_{i=2}^{\Delta/2+1} \binom{n-2}{i-1} p^{i\Delta-2\binom{i}{2}} + 2\sum_{i=\Delta/2+2}^{\lfloor n/2\rfloor} \binom{n-2}{i-1} p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil} \right). \quad (16)$$

In a manner similar to Section V-C, we can derive a slightly looser upper bound for $P_c^{sd}(G,p)$ [16]:

$$P_c^{sd}(G,p) \geq 1 - \left(2p^{\Delta} + 2\left\lfloor\frac{\Delta}{2}\right\rfloor \binom{n-2}{\lfloor\frac{\Delta}{2}\rfloor} \left(p^{2\Delta-2} - p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil}\right) + p^{\lceil\Delta/2\rceil^2+\lceil\Delta/2\rceil} \left[2^{n-2} + \binom{n-2}{\lfloor\frac{n-2}{2}\rfloor} - 2\right]\right). \quad (17)$$

The quality of these bounds is illustrated in Figure 6 for the ten node, degree three Harary graph. As in the all-terminal case, the two-terminal upper bounds are quite tight for values of p less than approximately 0.2.

E. All-terminal reliability when p is high

For high p, we bound $P_c(G,p)$ using the approach outlined in Section IV-C, which requires knowledge of the number of spanning trees in a graph. We specialize Wang and Yang's result [11] for the number of spanning trees in circulant graphs to Harary graphs:

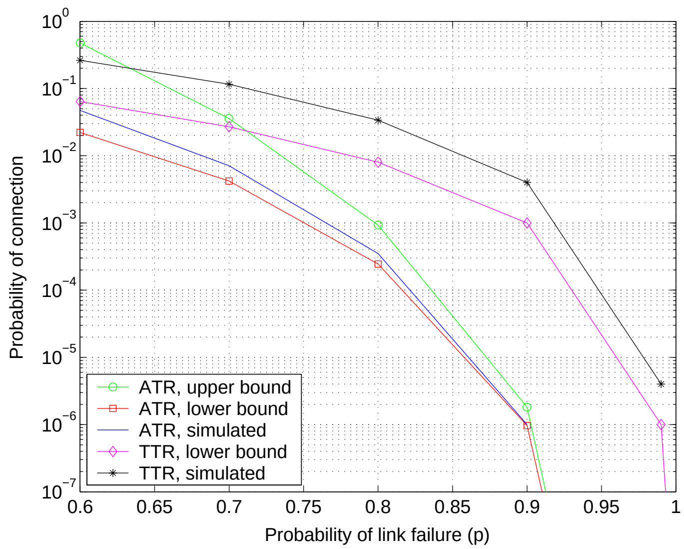

Fig. 7. Probability of graph connection and worst-case probability of node pair connection versus p for $H(10,3)$. "ATR, lower bound" refers to (9), "ATR, upper bound" refers to (10), and "TTR, lower bound" refers to (18).

Lemma 2 *The number of spanning trees in the degree Δ Harary graph is:*

$$t(G) = \begin{cases} \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h}\sin^2(ji\pi/n)\right], & \text{if } \Delta \text{ is even,} \\ \frac{1}{n}\prod_{i=1}^{n-1}\left[4\sum_{j=1}^{h-1}\sin^2(ji\pi/n) - (-1)^i + 1\right], & \\ \quad \text{if } \Delta \text{ is odd.} \end{cases}$$

The quality of these bounds is illustrated in Figure 7 for the ten node, degree three Harary graph.

In general, it appears that Harary graphs have fewer spanning trees than many of its circulant counterparts with the same number of nodes and edges. For example, the Harary graph $H(10,4)$ possesses 30250 spanning trees, whereas the circulant $C_{10}\langle 1,3\rangle$ possesses 40500 spanning trees. For values of p very close to unity this translates to a probability of connection for $H(10,4)$ which is smaller than that of $C_{10}\langle 1,3\rangle$ by approximately $10250(1-p)^9$.

F. Two-terminal reliability when p is high

When the probability of link failure p is high, we bound the probability of node pair connection using the technique outlined in Section IV-D. This technique requires knowledge of the edge-disjoint path lengths between nodes s and d. We consider Harary graphs of even degree only, as the case of odd degree is considerably more complex. Let d_{sd} denote the node separation of s and d. Define the parameter h as $\min(d_{sd}, n-d_{sd})$. By inspecting the structure of even degree Harary graphs, the length of path i for $i=1,\ldots,\min(h,\Delta/2)$ is found to be:

$$l_i = \left\lceil \frac{h-i+1}{\Delta/2} \right\rceil + 1 - \delta_1(i)$$

where the function $\delta_x(i)$ equals unity when its argument i equals x and is otherwise equal to zero. If $\Delta/2 > h$, then the length of path i for $i = h+1,\ldots,\Delta/2$ is given by:

$$l_i = \left\lceil \frac{i-h}{\Delta/2} \right\rceil + 1.$$

Finally, the length of path i for $i=\Delta/2+1,\ldots,\Delta$ is given by:

$$l_i = \left\lceil \frac{n-h-i+1}{\Delta/2} \right\rceil + 1 - \delta_{\Delta/2+1}(i).$$

These path lengths can now be substituted into (11) to obtain a lower bound for $P_c^{sd}(G,p)$.

When p is high, $P_c^{sd}(G,p)$ is minimized for node pairs which are most distantly placed in G. For even degree Harary graphs, such node pairs have indices which differ by $\lceil (n-1)/2 \rceil$. The diameter of even degree Harary graphs is thus $\left\lceil \frac{2}{\Delta}\left\lceil\frac{n-1}{2}\right\rceil\right\rceil$. For odd degree Harary graphs, most distantly placed nodes can be shown to have indices which differ by $\lceil (n+\Delta-3)/4 \rceil$, with a resulting graph diameter of $\left\lceil \frac{2}{\Delta-1}\left\lceil\frac{n+\Delta-3}{4}\right\rceil\right\rceil$. Thus, using (12), we have the following lower bound for $\min_{s,d}\left[P_c^{sd}(G,p)\right]$ for Harary graphs:

$$(1-p)^{k(G)} \le \min_{s,d}\left[P_c^{sd}(G,p)\right] \tag{18}$$

where,

$$k(G) = \begin{cases} \left\lceil \frac{2}{\Delta}\left\lceil\frac{n-1}{2}\right\rceil\right\rceil, & \text{if } \Delta \text{ is even,} \\ \left\lceil \frac{2}{\Delta-1}\left\lceil\frac{n+\Delta-3}{4}\right\rceil\right\rceil, & \text{if } \Delta \text{ is odd.} \end{cases}$$

The quality of this bound is illustrated in Figure 7 for the ten node, degree three Harary graph. Note that as the number of nodes n increases relative to the degree Δ, odd degree Harary graphs possess diameters which are approximately half as large as even degree Harary graphs. Furthermore, because Harary graphs are defined such that nodes are connected to their nearest neighbors, the diameter of Harary graphs are generally larger than other circulant graphs with the same number of nodes and edges. For example, the Harary graph $H(30,4)$ has diameter eight, whereas the circulant $C_{30}\langle 4,5\rangle$ has diameter four.

It is interesting to consider the relationship between a graph's diameter and its number of spanning trees. Although a smaller diameter does not necessarily imply a larger number of spanning trees, or vice versa, there does seem to exist an inverse correlation between these properties. The intuition behind this trend is that for the same number of nodes and edges, the nodes of a graph with a larger diameter are generally more distant from one another. The result is that there are fewer combinations of edges of the graph that could form spanning trees since there are more constraints on the edges in order that more distant nodes be connected. Hence, the number of spanning trees generally decreases with diameter when the number of nodes and edges is held constant. Thus, when p is high, graphs which have good all-terminal reliability performance generally have good two-terminal reliability performance, and vice versa.

VI. Conclusion

In this paper, we justified Harary graphs as candidate topologies for high-reliability applications by virtue of their excellent performance in the low p regime, and their attractiveness relative to present-day commercial architectures. We also established general reliability bounds which are useful in the design of communication networks. Our reliability study addressed the often neglected high p regime, in which network diameter and number of spanning trees were identified as the key figures of merit. Our reliability study was also specialized to Harary graphs, which yielded new results for this family of graphs.

Acknowledgements

The authors would to thank Professor Daniel Kleitman for his useful insights.

References

[1] C. J. Colbourn, "Reliability issues in telecommunications network planning," in *Telecommunications network planning*, P. Soriano and B. Sanso, Eds. Kluwer Academic Publishers, 1999, ch. 9, pp. 135–146.
[2] F. Harary, *Graph Theory*. Addison-Wesley, 1969.
[3] ——, "The maximum connectivity of a graph," *Proceedings of the National Academy of Sciences USA*, vol. 48, pp. 1142–1146, 1962.
[4] F. T. Boesch, X. Li, and C. Suffel, "On the existence of uniformly optimally reliable networks," *Networks*, vol. 21, pp. 181–194, 1991.
[5] G. Wang, "A proof of Boesch's conjecture," *Networks*, vol. 24, pp. 277–284, 1994.
[6] A. Satyanarayana, L. Schoppmann, and C. Suffel, "A reliability-improving graph transformation with applications to network reliability," *Networks*, vol. 22, pp. 209–216, 1992.
[7] F. T. Boesch, "Synthesis of reliable networks — a survey," *IEEE Transactions on Reliability*, vol. 35, no. 3, pp. 240–246, 1986.
[8] F. T. Boesch and J. Wang, "Super line-connectivity properties of circulant graphs," *SIAM J. Algebraic and Discrete Methods*, vol. 7, pp. 89–98, 1986.
[9] W. Mader, "Minimale n-fach kantenzusammenhängende graphen," *Math. Ann.*, vol. 191, pp. 21–28, 1971.
[10] F. T. Boesch and J. Wang, "Reliable circulant networks with minimum transmission delay," *IEEE Transactions on Circuits and Systems*, vol. 32, no. 12, pp. 1286–1291, Dec. 1985.
[11] J. F. Wang and C. S. Yang, "On the number of spanning trees in circulant graphs," *International Journal of Computer Mathematics*, vol. 16, pp. 229–241, 1984.
[12] M. N. S. Swamy and K. Thulasiraman, *Graphs, Networks, and Algorithms*. John Wiley & Sons, 1981.
[13] L. Weinberg, "Number of trees in a graph," *Proceedings of the IRE*, vol. 46, no. 12, pp. 1954–1955, Dec. 1958.
[14] R. Breyer and S. Riley, *Switched, Fast, and Gigabit Ethernet*. New Riders Publishing, 1998.
[15] R. Ramaswami and K. Sivarajan, *Optical Networks: A Practical Perspective*, 1st ed. San Francisco, California: Morgan Kaufmann, 1998.
[16] G. E. Weichenberg, "High-reliability architectures for networks under stress," S. M. thesis, Massachusetts Institute of Technology, 2003.