The gain achievable by network coding can grow unboundedly with the size of the network in directed graphs. [11]

Only the source's and sink's neighborhood matters: Unicast and multicast on random graphs and hypergraphs.

Our use of hypergraphs allows us to consider the broadcast nature of wireless networks.

However, unlike those results we find that it is the min-cut of the source or the destination, rather than effects in the core of the network, that determine

Jérôme Casse, *Student Member, IEEE* and Muriel Médard, *Fellow, IEEE*

***Abstract*—We study the maximum flow on random weighted directed graphs and hypergraphs that generalize Erdös-Rényi graphs. We show that, for a single unicast connection chosen at random, its capacity, determined by the max-flow between source and sink, converges in probability to the capacity around the source or sink. Using results from network coding, we generalize this result to different types of randomly selected multicast connections, whose capacity is given by the max-flow between the source(s) and all sinks. Our results indicate that the capacity of unicast and multicast connections ~~using network coding are, with high probability,~~ unaffected by network size in random networks. Our results generalize to networks with random erasures,** which can be represented by weights on hyperedges.

***Index Terms*—Flow, multicast, network coding, random graph, random hypergraph, unicast.**

Our results apply to various generalized multicast scenarios.

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I. Introduction

THE recent development of network coding [1] has shown that the capacity of multicast connections is given by the min-cut max-flow upper bound between source(s) and sinks, thus generalizing the unicast results of Ford-Fulkerson to multicast connections. Indeed, network coding can be used to show Ford-Fulkerson flow from an algebraic point of view [2]. The work of [3] generalized network coding from graphs to hypergraphs, which are useful for representing wireless broadcast links, as well as to networks with random erasures.

For a single source and sink, the problem of determining the behavior of the max-flow in random graphs was first envisaged in [4] and in [5]. These papers present results on undirected complete graphs with capacities that are randomly selected with a distribution that does not depend on node distance. Reference [6] provides, without proof, results for the problem of max-flow for directed random graphs, where arcs can only exist in a single direction between two nodes. While the above references consider only point-to-point connections without the use of network coding, the capacity of random graphs using network coding was first considered in [7]. In that article, the first random graphs considered are directed random graphs built over complete graphs, where the existence of an edge from one node to another implies the existence of a reverse edge of equal capacity. Moreover, the probability of the edge's existence is constant. The second model presented in [7] is the geometric random graph. ~~Unfortunately, there seems to be a mistake in the proof. Nevertheless, the idea can be reused to improve the class of random graphs for which the theorem holds, as is done in this article. Furthermore, this work begins to describe this larger class which excludes random geometric graphs. Moreover,~~ we improve the result of [7] by ~~having~~ the edge-existence probability that goes to 0 as the number of nodes goes to infinity.

Moreover we expand the graphs to consider hypergraphs, which model the broadcast nature of wireless transmission. Our model also allows the all-erasures, which commonly affect wireless networks.

The result that we establish can be interpreted as the maximum flow of a network. For the graph model, our result may represent the maximum flow in a wired network. For the hypergraph model, it may represent the maximum flow in a wireless network. ~~This is all the more true since the hyperedges are built to model a broadcast channel with erasures for the receiver nodes and wireless connections~~ are directed.

Without network coding, the multicast problem becomes a Steiner packing tree problem, which is ~~more intricate to solve~~ generally harder. Nevertheless, some results, such as the Kriesell conjecture, place bounds on network coding capacity gain. The Kriesell conjecture [8] states that network coding in undirected graphs cannot achieve more than twice the capacity of Steiner trees. It was subsequently proved in [9] that the gain ratio is limited to 6.5 rather than 2. ~~Indeed, when network coding~~ allows a max-flow ~~F, this is equivalent to say that the graph is F-connected. Now, the efficiency of network coding holds in directed graph.~~ As stated in [10], allowing network coding is roughly ~~equivalent~~ to replacing directed edges by undirected edges, with a gain ratio that is bounded between two finite constants. ~~The difference between the directed and undirected cases is crucial since the gain obtained is arbitrarily large [11].~~ The above results ~~could~~ generalize to hypergraphs. However, to our knowledge, they do not extend to networks with random erasures. Indeed, even for a single-source single-sink flow, max-flow results have not been shown to apply in the presence of random erasures since several copies of the initial packet have to be send until one is received. Therefore, tree packing does not extend to random-erasure networks. However, for the gains from coding

Coding gains in the above results do not clearly extend to graphs that are random, let alone hypergraphs or graphs with erasures.

Our contribution is different from the scaling laws presented by Gupta and Kumar in [12]. The problem studied by Gupta and Kumar is a case where the number of connections increases with the number of nodes. Moreover, these connections are disjoint unicast. In this article, we consider a finite number of connections and an increasing network size. Hence, the scaling law results do not apply. The ~~obtained results~~ conclusions are also very different since the bottleneck in our work is the source and the sink nodes, whereas the bottleneck in [12] consists of relay nodes. On the other hand, our results are closer to the percolation results of [13] about the connectivity of random graphs. In our work, we further characterize the strength of connections between two nodes in a large random graph. ~~The main contribution of this article is the extension of the~~ result on ~~directed~~ random graphs to directed random hypergraphs.

Our work allows (1), ~~which is interesting~~ for which allows to consider systems where connections become challenging.

while a flaw in the proof renders the results inapplicable, the class of random graphs is ~~of great~~ interest and is the basis for an investigation in this paper.

Give a bullet-point set of ~~[illegible]~~ contributions

The rest of this article is organized as follows. In Section II, we define our random graph model and flows on graphs. In Section III, we establish the convergence in probability of the max-flow of our random graphs. In Section IV, we define our random hypergraph model and flows on hypergraphs. In Section V, we establish the convergence in probability of the max-flow of our random hypergraphs. Finally, we conclude in Section VI.

II. Model of Wired Network : Random Weighted Directed Graph

A. Definitions and Notation

1) A Weighted Directed Graph: In this ~~article~~ paper, wired networks will be modeled by weighted directed graphs. First, a wired network can be modeled by a directed graph if the capacities of its links are ignored.

Definition 1: A directed graph $G = (N, E)$ is a pair of which the first element is the set of nodes N and the second is the set of edges E, a subset of $N \times N$.

The users of the network will be represented by the nodes of the directed graph, and the links by the directed edges. A weight function is added to the model whereby, for each edge, a weight is assigned corresponding to the capacity of the link in the network.

Definition 2: A weighted directed graph $(G = (N, E), W)$ is a pair where the first element is a directed graph G and the second is a non-negative function from $N \times N$, with the constraint that $W_e = 0$ if $e \notin E$. The value W_e is called the weight of the edge e.

2) Flow in a Weighted Directed Graph: Similarly to capacities of links that are modeled by weights of edges, the way the information is routed on a network between two points will be modeled by the flow between these two points on the graph. Hence, the capacity between two nodes of a network will be the maximum value of flows on the corresponding graph.

Definition 3 (Flow): A flow from the source node i to the sink node j in a weighted directed graph is a function f on edges that satisfy these conditions:

1) the flow is less than the weight, i.e., for all nodes u,v,
$$f((u,v)) \leq W_{(u,v)}; \tag{1}$$
2) there is no incoming flow to i and outgoing flow from j, i.e., for all nodes u
$$f((u,i)) = f((j,u)) = 0; \tag{2}$$
3) the outgoing flow from i is equal to the incoming flow to j and has value F
$$\sum_{\substack{v \in N \\ (i,v) \in E}} f((i,v)) = \sum_{\substack{u \in N \\ (u,j) \in E}} f((u,j)) = F; \tag{3}$$
4) conservation: for each node except i and j, the incoming flow is equal to the outgoing flow, i.e., for all $u \neq i, j$
$$\sum_{\substack{v \in N \\ (v,u) \in E}} f((v,u)) = \sum_{\substack{v' \in N \\ (u,v') \in E}} f((u,v')). \tag{4}$$

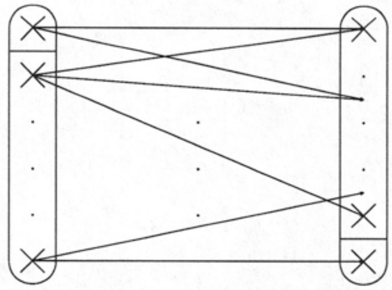

Fig. 1. Min-cut from the set of the right nodes to the set of the left nodes.

Definition 4 (Max-flow): The max-flow from i to j is a flow with the maximal value. We will denote $F^G_{(i,j)}$ the value of this flow.

Our aim will be to evaluate this max-flow in large random graphs.

3) Cut in a Weighted Directed Graph: An easy way to study the max-flow of a graph is to study its min-cut.

Definition 5 (Cut): A cut from the set of nodes N_0 to the set of nodes N_1 is a set S of edges such that if the edges in S are removed, then there is no directed path from u to v for any $u \in N_0$ and $v \in N_1$. The value of a cut is the sum of weights of its edges.

Definition 6 (Min-cut): The min-cut from the set of nodes N_0 to the set of nodes N_1 is a cut whose value is minimum. We denote this value $C^G_{(N_0,N_1)}$.

The following theorem gives the value of the min-cut from a subset N_0 of nodes to its complementary N_0^c. It is illustrated in ~~by the~~ figure 1.

Theorem 7: For any graph G and any subset N_0 of N, we have
$$C^G_{(N_0,N_0^c)} = \sum_{u \in N_0} \sum_{v \in N_0^c} W_{(u,v)}. \tag{5}$$

Proof: On one side, we have
$$C^G_{(N_0,N_0^c)} \leq \sum_{u \in N_0} \sum_{v \in N_0^c} W_{(u,v)} \tag{6}$$
since the edges $((u,v))_{u \in N_0, v \in N_0^c}$ are a cut from N_0 to N_0^c.

On the other side, for each $u \in N_0$ and each $v \in N_0^c$, we have to remove the directed edge (u,v) (if not, there exists the directed path (u,v) that links them and, thus, it cannot be a cut). Then, a cut from N_0 to N_0^c must contains the edges (u,v). Thus,
$$C^G_{(N_0,N_0^c)} \geq \sum_{u \in N_0} \sum_{v \in N_0^c} W_{(u,v)}. \tag{7}$$
■

The link between the max-flow and the min-cut of a graph is done by the min-cut max-flow theorem that was proven for the first time by Menger on unweighted undirected graphs. A proof for weighted directed graphs can be found in [14].

Theorem 8 (Min-cut max-flow theorem): For any weighted directed graph G, the max-flow from i to j is equal to the min-cut from $\{i\}$ to $\{j\}$, i.e.,
$$F^G_{(i,j)} = C^G_{(\{i\},\{j\})}. \tag{8}$$

A corollary of this theorem links the max-flow from i to j to the min of the min-cuts between all 2-partitions of nodes where i and j are not in the same. This will be useful since it is easier to evaluate.

Theorem 9: For any weighted directed graph G, we have

$$F^G_{(i,j)} = \min_{N_0 \subset N \backslash \{j\}, i \in N_0} C^G_{(N_0, N_0^c)}. \tag{9}$$

Proof: Clearly,

$$F^G_{(i,j)} \leq \min_{N_0 \subset N \backslash \{j\}, i \in N_0} C^G_{(N_0, N_0^c)} \tag{10}$$

since the cut from N_0 to N_0^c is a cut from i to j. Therefore, we can apply the min-cut max-flow theorem.

On the other side, the max-flow min-cut theorem gives us a set of edges S such that without them there is no directed path from i to j. We denote G' the graph G without the edges in S. Hence, we can define the two sets of nodes N_i and N_j such that

$$N_i = \{v \in N | \exists \text{ a directed path from } i \text{ to } v \text{ in } G'\}, \tag{11}$$

$$N_j = \{u \in N | \exists \text{ a directed path from } u \text{ to } j \text{ in } G'\}. \tag{12}$$

Since there is no directed path from i to j, we have $N_i \cap N_j = \emptyset$. We denote $N' = N \backslash (N_i \cup N_j \cup \{i,j\})$, the set of nodes disconnected of i and j. We have that S (that is the min-cut between i from j in G) is also a cut between $N_0 = N_i \cup \{i\}$ and $N_0^c = N' \cup N_j \cup \{j\}$ and, hence,

$$F^G_{(i,j)} \geq C^G_{(N_0, N_0^c)}. \tag{13}$$

■

B. Studied Random Weighted Directed Graphs

As in many problems on random graphs, our results hold only for random graphs that satisfy some conditions. Therefore, in this article, results established will concern only this type of random graphs.

Definition 10: Random weighted directed graphs studied in this article satisfy these conditions:

1) an edge exists with probability p_l ;
2) the weight of an edge is distributed as a random variable of density function f_W and of mean μ, i.e., for all nodes u,v,

$$P\left(W_{(u,v)} \geq w\right) = \begin{cases} 1 & \text{if } w = 0, \\ p_l \int_w^\infty f_W(x)\,\mathrm{d}x & \text{else;} \end{cases} \tag{14}$$

3) for each subset N_0 of nodes, the edges implied in the min-cut from N_0 to N_0^c are independent, i.e., for all N_0 subset of N,

$$\left(W_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent.} \tag{15}$$

Theorems established in this paper are proved only for this type of random graphs. ~~Maybe, they can hold for others conditions on random graphs, but it is not proven here.~~ In this class of random graphs, there exist several simpler subclasses. We ~~will~~ present four of them, the last three are special cases of the first one and are already studied in the literature.

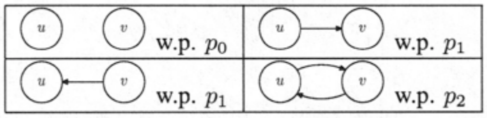

Fig. 2. The distribution of the existence of the edges of two nodes u and v.

1) For the first class, for each pair of nodes $\{u,v\}$, we associate $p_{0,\{u,v\}}$ the probability for two nodes to be not linked (i.e., (u,v) and (v,u) do not exist), we denote $p_{1,\{u,v\}}$ the probability to have the edge (u,v) (resp. (v,u)) without (v,u) (resp. (u,v)) and $p_{2,\{u,v\}}$ to have the two edges (u,v) and (v,u) (as illustrated figure 2) such that $p_{1,\{u,v\}} + p_{2,\{u,v\}} = p_l$. Then, the capacities $W_{(u,v)}$ and $W_{(v,u)}$ can be independent and distributed like a random variable of density function f_W or can be the same $W_{\{u,v\}}$ distributed like a random variable of density function f_W.
2) If, for all nodes u,v, $p_{1,\{u,v\}} = 0$ and $W_{(u,v)} = W_{(v,u)}$, then the model obtained is the one discussed in [7] where edges are two-way edges and each two-way edge has the same capacity ~~on the two~~ directions. This could be easily transformed on an weighted undirected graph.
3) If, for all nodes u,v, $p_{2,\{u,v\}} = 0$, then we obtain the model discussed in [6], where edges are one-sided. This can be seen like a random weighted undirected graph where sides of directed edges are chosen independently and uniformly.
4) If, for all nodes u,v, $p_{1,\{u,v\}} = p_l(1-p_l)$, $p_{2,\{u,v\}} = p_l^2$ and $W_{(u,v)}$ and $W_{(v,u)}$ are independent, then we obtain what we could ~~call~~ an Erdös-Rényi weighted random graph, since all directed edges are independent in this case.

III. Unicast and Multicast Types on Random Graphs

A. Unicast

The unicast problem consists in determining the maximum of information that can be sent from a source node i to a sink node j in a network. In our model of wired network, it is equivalent to evaluate the value of the max-flow $F^G_{(i,j)}$ from a node i to a node j. The aim of this section will be to evaluate the value of the max-flow $F^G_{(i,j)}$ in a large random graph G as defined before. We do not really care how these two nodes are chosen ~~on account~~ of the symmetry ~~of the graph.~~

~~Some~~ results about ~~the~~ unicast on random graph already exist. Grimett and Welsh, in [4], established results about particular type of random graphs when the probability p_l is fixed. Suen, in [6], established, for random graphs where an edge between two nodes is unique and has a unique direction, a more precise result since p_l can converge quickly to 0 but he did not ~~give~~ the proof, ~~neither reference~~. More recently, Ramamoorthy et al. in [7] established some results also for random graphs where edges between two nodes are inevitably two in opposite directions and for a fixed p_l. This two kinds of random graphs will be two sub-classes of random graphs we study here. The aim of this section is to give and prove a theorem that improves these results, as follows.

Theorem 11: We take a random weighted directed graph with $n+1$ nodes. We take i and j to be two nodes, i is the source and j the sink. If

$$\frac{np_l}{\ln n} \to \infty \tag{16}$$

and, for all subset of nodes N_0 of N such that $i \in N_0$ and $j \notin N_0$,

$$\left(W_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent} \tag{17}$$

then

$$\frac{F^G_{(i,j)}}{np_l\mu} \xrightarrow{p} 1. \tag{18}$$

In particular, the min-cut is around the source i or the sink j. Therefore, in a random network, the capacity is limited by what happens locally around the source and the sink and not in the rest of the network. → This is the condition le …

We ~~can probably do better for the condition~~ (16) since, to have ~~the connexity~~ connectivity of the random graph, we just need ~~that~~ there exists $c > 1$ such that $\frac{np_l}{\ln n} \to c$ and, moreover, for a constant weight, this condition is just there exists $c > 1$ such that $\frac{np_l}{\ln n} \to 32c$ as we can see in the following proof.

B. Proof

The proof follows what ~~we can see in~~ [4] and [7].

Lemma: $P\left(\frac{FijG}{np_l\mu} \leq 1 - \epsilon\right) \to 0$. We ~~will~~ carry out a proof by steps. First, we will prove the result for f_W that is a Dirac delta function (i.e., the random variable for the weight is a Bernouilli of parameter p_l). Then, we generalize for f_W that is a finite sum of Dirac delta functions. Finally, we conclude the proof by approximating a general f_W by a sum of Dirac delta functions.

a) For f_W a Dirac Function: We look, first, ~~a repartition~~ ~~of~~ at $f_W = \delta_\mu$ as a Dirac delta function (i.e., if the edge (u, v) exists then its weight is the value of this Dirac delta function μ). We will assume that $\mu = 1$ (we can do that since μ is independent of n and multiplying all the edges by μ multiply the flow by μ).

First, a lemma is needed. It is about the probability for the min-cut $C^G_{(N_0,N_0^c)}$ to be less than $np_l\mu$ when $N_0 \neq \{i\}$ and $N_0^c \neq \{j\}$.

Lemma 12: For any N_0 subset of N such that $i \in N_0$, but $N_0 \neq \{i\}$, and $j \in N_0^c$, but $N_0^c \neq \{j\}$, with $|N_0| = k+1$ (we remark that $1 \leq k \leq n-2$), we have

$$P\left(C^G_{(N_0,N_0^c)} \leq np_l\mu\right) \leq \exp\left(-\frac{k(n-k-1)}{8}p_l\right). \tag{19}$$

Proof: We have

$$C^G_{(N_0,N_0^c)} = \sum_{u \in N_0} \sum_{v \in N_0^c} W_{(u,v)}. \tag{20}$$

Moreover, ~~like~~ since $\left(W_{(u,v)}\right)_{u \in N_0, v \in N_0^c}$ are independent and identically distributed as Bernouilli variables, we have that $C^G_{(N_0,N_0^c)}$ ~~is distributed like~~ has a binomial distribution of mean $|N_0|(n+1-|N_0|)E\left[W_{(u,v)}\right] = (k+1)(n-k)p_l$. Hence,

$$\begin{aligned}
&P\left(C^G_{(N_0,N_0^c)} \leq np_l\right) \\
&= P\left(C^G_{(N_0,N_0^c)} \leq E\left[C^G_{(N_0,N_0^c)}\right] - \left(E\left[C^G_{(N_0,N_0^c)}\right] - np_l\right)\right) && (21)\\
&= P\left(C^G_{(N_0,N_0^c)} \leq E\left[C^G_{(N_0,N_0^c)}\right] - ((k+1)(n-k) - n)p_l\right) && (22)\\
&= P\left(C^G_{(N_0,N_0^c)} \leq E\left[C^G_{(N_0,N_0^c)}\right] - k(n-k-1)p_l\right) && (23)\\
&\leq \exp\left(-\frac{(k(n-k-1)p_l)^2}{2(k+1)(n-k)p_l}\right) && (24)\\
&\qquad \text{(see [15],p.12 or [16],p.26)} \\
&\leq \exp\left(-\frac{k(n-k-1)}{8}p_l\right). && (25)
\end{aligned}$$

■

Then, we continue the proof by looking what happens for the min-cut not around the source.

$$\begin{aligned}
&P\left(\min_{N_0 \subset N\backslash\{j\}, i \in N_0, N_0 \neq \{i\}, N_0^c \neq \{j\}} C^G_{(N_0,N_0^c)} \leq np_l\right) \\
&= P\left(\exists N_0 \subset N\backslash\{j\}, i \in N_0, N_0 \neq \{i\}, N_0^c \neq \{j\}, C^G_{(N_0,N_0^c)} \leq np_l\right) && (26)\\
&\leq \sum_{k=1}^{n-2} \binom{n-1}{k} \exp\left(-\frac{k(n-k-1)}{8}p_l\right) && (27)\\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \beta^{(n-1)\frac{k}{n-1}(1-\frac{k}{n-1})} - 2 && (28)\\
&\leq 2(1+\sqrt{\beta})^{n-1} - 2 \quad \text{(see [7])} && (29)
\end{aligned}$$

where $\beta = \exp\left(-\frac{(n-1)p_l}{8}\right)$.

Then, since $\frac{(n-1)p_l}{\ln(n-1)} \to \infty$, there exists N such that for all $n \geq N$,

$$(n-1)p_l \geq 32\ln(n-1). \tag{30}$$

Moreover, → Give the conditions for (31) to hold

$$(1+\sqrt{\beta})^{n-1} \sim \exp\left((n-1)\exp\left(-\frac{(n-1)p_l}{16}\right)\right). \tag{31}$$

Since for all $n \geq N$,

$$\begin{aligned}
&(n-1)\exp\left(-\frac{(n-1)p_l}{16}\right) \\
&\leq (n-1)\exp\left(-\frac{32\ln(n-1)}{16}\right) && (32)\\
&= \frac{n-1}{(n-1)2} && (33)\\
&= \frac{1}{n-1} \to 0. && (34)
\end{aligned}$$

Therefore,

$$2(1+\sqrt{\beta})^{n-1} - 2 \to 2 - 2 = 0. \tag{35}$$

Finally, we obtain

$$P\left(\min_{N_0\subset N\setminus\{j\}, i\in N_0, N_0\neq\{i\}, N_0^c\neq\{j\}} C^G_{(N_0,N_0^c)} \leq np_l\right) \to 0. \tag{36}$$

Now, around the source i and the sink j, we have, by the law of large number

$$P\left(C^G_{(\{i\},\{i\}^c)} \leq (1-\epsilon)np_l\right) \to 0, \tag{37}$$

$$P\left(C^G_{(\{j\}^c,\{j\})} \leq (1-\epsilon)np_l\right) \to 0. \tag{38}$$

Then, by ~~lemma 37~~ in the appendix, we obtain

Put the lemma in the text.

$$P\left(F^G_{(i,j)} \leq (1-\epsilon)np_l\right) \to 0. \tag{39}$$

The case $\mu \neq 1$ is deduced by observing that $F^G_{(i,j)} = \mu F^{G'}_{(i,j)}$ where G' is the same graph as G with links capacity 1 instead of μ, then

$$P\left(\frac{F^G_{(i,j)}}{\mu} = F^{G'}_{(i,j)} \leq (1-\epsilon)np_l\right) \to 0. \tag{40}$$

b) For f_W a sum of Dirac delta functions: We suppose that $f_W = \sum_{k=1}^m q_k\delta_{\mu_k}$ is a sum of Dirac delta functions. We can assume that $\mu_1 < \cdots < \mu_m$. Clearly,

$$\mu = \sum_{k=1}^m q_k\mu_k. \tag{41}$$

We split the graph in m subgraphs G^k, where the edges (u,v) exists and have weight μ_k if it is the case in the original graph G. The subgraphs G^k are all random graphs with $p_l^k = p_l q_k$ and with $f_W^k = \delta_{\mu_k}$, hence, the previous result can be applied. This split implies

$$F^G_{(i,j)} \geq \sum_{k=1}^m F^{G^k}_{(i,j)}. \tag{42}$$

Indeed, we ~~can~~ take the union of the edges given by the right term, ~~this is~~ Corollary a flow for the original graph. Hence,

$$\begin{aligned}
&P\left(F^G_{(i,j)} < (1-\epsilon)np_l\mu\right)\\
&\leq P\left(\sum_{k=1}^m F^{G^k}_{(i,j)} < (1-\epsilon)np_l\mu\right) && (43)\\
&= P\left(\sum_{k=1}^m F^{G^k}_{(i,j)} < \sum_{k=1}^m (1-\epsilon)np_l q_k\mu_k\right) && (44)\\
&\leq P\left(\exists k\ F^{G^k}_{(i,j)} < (1-\epsilon)np_l q_k\mu_k\right) && (45)\\
&\leq \sum_{k=1}^m P\left(F^{G^k}_{(i,j)} < (1-\epsilon)np_l q_k\mu_k\right) && (46)\\
&\to \sum_{k=1}^m 0 = 0. && (47)
\end{aligned}$$

c) For a general f_W: We ~~will~~ approximate a general f_W by a finite sum of Dirac delta functions. We have a first lemma about the approximation of the infinite tail of the distribution.

Lemma 13: There exists M such that, for all $x \geq M$,

$$\int_x^\infty (t-x)f_W(t)\,\mathrm{d}t < \epsilon. \tag{48}$$

Proof: For all $m \geq 0$, we have

$$\begin{aligned}
\int_x^\infty (t-x)f_W(t)\,\mathrm{d}t &\leq \int_x^\infty t f_W(t)\,\mathrm{d}t && (49)\\
&= \mu - \int_0^x t f_c(t)\,\mathrm{d}t && (50)\\
&\to \mu - \mu = 0. && (51)
\end{aligned}$$

■

Thus, we approximate the function f_W by a sum of Dirac delta functions $\tilde{f_W}$ ~~like that~~ as follows

$$\begin{aligned}
\tilde{f_W} = &\sum_{k=0}^{m=\lceil M/\epsilon\rceil-1}\left(\int_{n\epsilon}^{(n+1)\epsilon} f_W(x)\,\mathrm{d}x\right)\delta_{n\epsilon}\\
&+\left(\int_{(m+1)\epsilon}^\infty f_W(x)\,\mathrm{d}x\right)\delta_{(m+1)\epsilon}. && (52)
\end{aligned}$$

We have

$$\mu - 2\epsilon \leq \tilde{\mu} = \int_0^\infty \tilde{f}_w(x)\,\mathrm{d}x \leq \mu. \tag{53}$$

Thanks to this approximation, we can conclude the proof in the general case. Let $\frac{3}{4} > \epsilon > 0$. We denote $\eta = \frac{\epsilon}{4(1-\epsilon)}\mu$ and $\epsilon' = 1 - \frac{\mu}{\mu-\eta}(1-\epsilon) = 1 - \frac{1-\epsilon}{1-\frac{\epsilon}{4(1-\epsilon)}} > 0$ since $0 < \epsilon < \frac{3}{4}$. We denote $\tilde{G}$ the η-approximation of G. We have $F^{\tilde{G}}_{(i,j)} \leq F^G_{(i,j)}$ since $\tilde{G}$ is the same graph with less capacity. Hence,

$$P\left(\frac{F^G_{(i,j)}}{np_l\tilde{\mu}} < 1-\epsilon'\right) \leq P\left(\frac{F^{\tilde{G}}_{(i,j)}}{np_l\tilde{\mu}} < 1-\epsilon'\right) \to 0. \tag{54}$$

Now,

$$\begin{aligned}
\frac{\tilde{\mu}}{\mu}(1-\epsilon') &\geq \frac{\mu-\eta}{\mu}(1-\epsilon') && (55)\\
&\geq 1-\epsilon. && (56)
\end{aligned}$$

Therefore,

$$P\left(\frac{F^G_{(i,j)}}{np_l\mu} < 1-\epsilon\right) \to 0. \tag{57}$$

By this part of the proof, we learn that the probability for the min-cut to be less that the cut around the source or the sink goes to 0 as n goes to infinity. This concludes the proof of the Lemma N.

Lemma N+1: $P\left(\frac{F^G_{(i,j)}}{np_l\mu} \geq 1+\epsilon\right) \to 0$. ~~To finish the convergence in probability, we have to prove the other inequality.~~ we need the following lemma which For that, we consider directly any function f_W. We have $F^G_{(i,j)} \leq C^G_{(\{i\},\{i\}^c)}$, thus

$$P\left(\frac{F^G_{(i,j)}}{np_l\mu} \geq 1+\epsilon\right) \leq P\left(\frac{C^G_{(\{i\},\{i\}^c)}}{np_l\mu} \geq 1+\epsilon\right). \tag{58}$$

proves an inequality in the reverse direction of Lemma N.

However, $\frac{C^G_{(\{i\},\{i\}^c)}}{p_l}$ is the sum of n independent random variables whose mean is μ. Then we obtain, by the law of large numbers,

$$P\left(\frac{C^G_{(\{i\},\{i\}^c)}}{np_l\mu} \geq 1+\epsilon\right) \rightarrow 0. \tag{59}$$

~~That concludes the proof.~~

C. Multicast ~~Types~~ and Variations on Multicast

We obtained results for the unicast problem in random graphs. Thanks to network coding, we ~~will~~ extend them in different types of multicast. We refer to [2] for the definitions of different types of multicast that we recall below.

1) Multicast: First, we look the usual multicast that is between one source node i and r sink nodes $J = \{j_k\}_{k=1,\ldots,r}$ that want all the information. We denote $F^M_{(i,J)}$ the max-flow between the source and all this nodes. The result is

Theorem 14:

$$\frac{F^M_{(i,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{60}$$

This theorem ~~tell us~~ that the max-flow is only dependent, for the multicast, of the capacities around the source and the sinks.

Proof: For each sink node j_k, we have

$$\frac{F^G_{(i,j_k)}}{np_l\mu} \xrightarrow{p} 1. \tag{61}$$

Then, by lemma 37 ~~in appendix~~,

$$\frac{F^M_{(i,J)}}{np_l\mu} = \min_{k=1,\ldots,r} \frac{F^G_{(i,j_k)}}{np_l\mu} \xrightarrow{p} 1. \tag{62}$$

■

2) 2-layer Multicast:

Definition 15: In the 2-layer multicast problem, a source node has all the information and there ~~is~~ are two types of sink nodes. The first type wants just a part of the information whereas the second type wants all the information. For example, a ~~television channel sends a show~~ in a normal ~~quality~~ for the first type and in a high definition quality for the second type.

In the 2-layer multicast case, there is always one source node i and r sink nodes $J = \{j_k\}$ but one of them (j_1) does not want all information but just a fraction ϵ of it. We denote $F^M_{(i,J\setminus\{j_1\})}$, the maximal flow for the sink nodes $j_2, \ldots, j_r$. We have

Theorem 16:

$$\frac{F^M_{(i,J\setminus\{j_1\})}}{np_l\mu} \xrightarrow{p} 1 \tag{63}$$

$$\tag{64}$$

and ϵ can take any value between 0 and 1.

Proof: The same proof as before. ■

3) Disjoint Multicast:

Definition 17: In the disjoint multicast problem, one source node has all the information, but each sink node just wants a fraction of information disjoint from the information needed by each sink node.

In the disjoint multicast case, we have, always, one source and r sink nodes $J = \{j_k\}$, but each node j_k just wants a disjoint fraction ϵ_k of the total information sent by the source node. We denote $F^D_{(i,J)}$ the maximal flow that the source can send. We have

Theorem 18:

$$\frac{F^D_{(i,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{65}$$

Proof: For all I, subset of $\{1,\ldots,r\}$, we want

$$\max_{i\in I} C^G_{(\{i\},\{j_k|k\in I\})} \geq \sum_{i\in I} \epsilon_i F^D_{(i,J)}. \tag{66}$$

Dividing by $np_l\mu$ and taking the limit in probability, we obtain

$$1 \geq \left(\sum_{i\in I}\epsilon_i\right) \lim_p \frac{F^D_{(i,J)}}{np_l\mu}. \tag{67}$$

Then

$$\frac{F^D_{(i,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{68}$$

■

4) Multisource-Multicast:

Definition 19: In the multisource-multicast problem, the information is split between several source nodes (and not only one node) and each sink node wants all the information of each source node.

In the multisource-multicast problem, we have t independent source nodes $I = \{i_k\}_{k=1,\ldots,t}$ and r sink nodes $J = \{j_{k'}\}_{k'=1,\ldots,r}$. Each sink node wants all the information sent by the source nodes. We denote $F^M_{(I,J)}$ the maximal flow transmitted by all the sources (i.e., received by each node). We have

Theorem 20:

$$\frac{F^M_{(I,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{69}$$

Proof: The multisource problem with one sink node is the same problem as the disjoint multicast problem if the edges are inverted. Hence, we have, if we denote $F^D_{(I,j_{k'})}$ the maximal flow between all the source nodes I and the sink node $j_{k'}$

$$\frac{F^D_{(I,j_{k'})}}{np_l\mu} \xrightarrow{p} 1. \tag{70}$$

By $F^M_{(I,J)} = \min_{k'} F^D_{(I,j_{k'})}$ and lemma 37,

$$\frac{F^M_{(I,J)}}{np_l\mu} \xrightarrow{p} 1. \tag{71}$$

■

This section concludes the results about wired networks modeled by random graphs. We have seen that, for our class of random graphs, the min-cut is around the source or the sink. Therefore, in random graphs, the max-flow is local and independent from the rest of the graph. Moreover, network coding, in the case of multicast, is superior to routing. Indeed, it is only necessary to examine cuts around the source and the sink (i.e., local conditions) to determine the maximum amount of information that can be sent by using the random linear network coding developed in [17]. On the contrary, if routing is used, we have to study the whole random network (i.e., global condition) to determine how many Steiner spanning trees can be built.

~~Now, we~~ We ~~will~~ shall study the flow in random hypergraphs. To our knowledge, this work is the first proposal to extend the results from random graphs to hypergraphs. In the first section, we present the model of studied random hypergraphs and, in the second section, we establish asymptotic flows in some random hypergraphs.

IV. Model of Random Wireless Network : Random Weighted Directed Hypergraph

Wired networks were studied using random weighted directed graphs since a user in a wired network can send different information on his links. However, in a wireless network, a node broadcasts information to its neighbors. To model this, hypergraphs are associated to wireless networks.

→ Need a short paragraph indicating that we shall connect weights to erasure probabilities

A. Definitions and Notation

1) Weighted Directed Hypergraph: In this section, the most general definition of directed hypergraphs and weighted directed hypergraphs are given, but the hypergraphs studied later are more specific and their properties are given later in this section.

Definition 21: A directed hypergraph $H = (N, E)$ is a pair where the first element N is a set of nodes and the second element E is a set of edges. An edge is a pair (U, V), where U and V are subsets of N.

Definition 22: A weighted directed hypergraph $(H = (N, E), W)$ is a pair where the first element is a directed hypergraph H and the second element is a non-negative function from $P(N) \times P(N) \times P(N)$ (where $P(N)$ is the set of all the subsets of N) with the constraint that $W_{(U,V,V')} = 0$ if $(U, V) \notin E$ or V' is not a subset of V.

In this work, we focus on the following sub-class of weighted directed hypergraphs to model wireless network.

Definition 23: Weighted directed hypergraphs ~~studied~~ have ~~this~~ the following properties

1) the edge has only one node u as sender, i.e., for all edge (U, V),
$$|U| = 1 \text{ (i.e., } U = \{u\}); \tag{72}$$
2) a sender u can send to only one set of receiver nodes U, i.e., for all node u
$$\left.\begin{array}{l}(\{u\}, U) \in E\\(\{u\}, U') \in E\end{array}\right\} \Rightarrow U = U'; \tag{73}$$
3) a weight $w_{(u,v)} \leq 1$ is associated to each pair (u, v) of nodes (this weight represents the probability for the node u to transmit well to the node v). Then we obtain the weight of the sub-edge $(\{u\}, V, V')$ ~~like that~~ as follows:
$$W_{(\{u\},V,V')} = \prod_{v' \in V'} \prod_{v \in V \setminus V'} w_{(u,v')}\left(1 - w_{(u,v)}\right). \tag{74}$$
The weight $W_{(\{u\},V,V')}$ is the probability that the node u transmits in a lossless fashion only to the nodes in the subset V' of V.

This model of hypergraphs corresponds to a network of wireless broadcast channels without interference and with independent packet erasures for the receiver nodes. The two notions of flow and cut are transportable to hypergraphs as shown below.

2) Flow:

Definition 24: The flow from i to j in an hypergraph is a function f on edges such that

1) it cannot send more than one bit per edge
$$f \leq 1; \tag{75}$$
2) j does not send information
$$f((\{j\}, J)) = 0; \tag{76}$$
3) for all nodes v except i, we have that the outgoing flow ~~is less than~~ is bounded by the incoming flow
$$f((\{v\}, V)) \leq \sum_{u \in N \setminus \{v\}} w_{(u,v)} f((u, U)). \tag{77}$$

The value F of the flow is the value of the incoming flow in j
$$F = \sum_{u \in N \setminus \{j\}} w_{(u,j)} f((\{u\}, U)). \tag{78}$$

Definition 25: The max-flow is a flow with a maximal value in the hypergraph. We denote this value $F^H_{(i,j)}$.

The max-flow as before corresponds to the maximum information that can be sent from the source to the sink.

3) Cut:

Definition 26: A cut from the set of nodes N_0 to the set of nodes N_1 is a set of sub-edges S such that if we delete these sub-edges, there is no directed path from a node in N_0 to a node in N_1. The value of the min-cut is the sum of the weights of the sub-edges in S.

Definition 27: The min-cut from N_0 to N_1 is a cut from N_0 to N_1 with the minimal value. We denote this value $C^H_{(N_0,N_1)}$.

As before, the min-cut max-flow theorem connects the notion of cut and flow in an hypergraph asserting that the max-flow from i to j is equal to the min-cut from $\{i\}$ to $\{j\}$ (i.e., $F^H_{(i,j)} = C^H_{(\{i\},\{j\})}$). As before, we have two theorems, which generalize theorems 7 and 9, about the min-cut in hypergraphs ~~that look like theorems 7 and 9 that hold for graphs~~.

Theorem 28: For any hypergraph H and any subset N_0 of N, we have
$$C^H_{(N_0,N_0^c)} = \sum_{u \in N_0} \left(1 - \prod_{v \in N_0^c} \left(1 - w_{(u,v)}\right)\right). \tag{79}$$

Theorem 29: For any weighted directed hypergraph H, we have

$$F_{(i,j)}^{H} = \min_{N_0 \subset N \setminus \{j\}, i \in N_0} C_{(N_0, N_0^c)}^{H}. \tag{80}$$

Proof ~~of theorem 28:~~ The proof holds like ~~in~~ as the graph case, ~~but we consider~~ the subedges $((\{u\}, \{v \in N_0^c | v \in U\})_{u \in N_0}$ where $(\{u\}, U) \in E$ ~~that~~ correspond to the edges $((u,v))_{u \in N_0, v \in N_0^c}$ in the graph case. These sub-edges are a cut from N_0 to N_0^c for the hypergraph, so

$$C_{(N_0,N_0^c)}^{H} \leq \sum_{u \in N_0} \left(1 - \prod_{v \in N_0^c} \left(1 - w_{(u,v)}\right) \right). \tag{81}$$

Re-moved

On the other side, we need to remove the directed edges (u,v) for all $u \in N_0$ and $v \in N_0^c$ and the minimum weight to remove all of that is to consider the sub-edges $((u, \{v \in N_0^c | v \in U\}))_{u \in N_0}$, and, so

$$C_{(N_0,N_0^c)}^{H} \geq \sum_{u \in N_0} \left(1 - \prod_{v \in N_0^c} \left(1 - w_{(u,v)}\right) \right). \tag{82}$$

■

Proof of theorem 29: Since a cut from N_0 to N_0^c is a cut from $\{i\}$ to $\{j\}$, we have

$$F_{(i,j)}^{H} \leq \min_{N_0 \subset N \setminus \{j\}, i \in N_0} C_{(N_0, N_0^c)}^{H}. \tag{83}$$

To prove the other side, the max-flow min-cut theorem gives us a set of sub-edges S such that, without them, there is not a directed path from i to j. N_0 is ~~built like~~ the set of nodes such that there exists a directed path from i to them in the hypergraph H' that is the hypergraph H in which we remove the sub-edges of S, i.e.,

Re-moved

$$N_0 = \{u \in N | \exists \text{ a directed path from } i \text{ to } u \text{ in } H'\} \cup \{i\}. \tag{84}$$

We denote $N_1 = N_0^c$. Then, the only sub-edges from N_0 to N_1 in H are in the cut S. Indeed, if not, there exists a directed path from a node $u \in N_0$ to a node $v \in N_1$ in H', but there is a directed path from i to u in H' (u is in N_0), therefore there exists a directed path from i to v in H' and v is in N_0 (contradiction). Then, S is a cut from N_0 to N_0^c and so

$$F_{(i,j)}^{H} \geq C_{(N_0,N_0^c)}^{H}. \tag{85}$$

■

B. Random Weighted Directed Hypergraph

Definition 30: We can associate a graph to the hypergraph that ~~we will~~ we study in ~~the following way~~ as follows. For ~~all~~ each node u, we create the edges $((u,v))_{v \in U, (\{u\},U) \in E}$ and the weight for the edge (u,v) is the weight $w_{(u,v)}$ as in figure 3.

Then we have a bijection between the set of graphs with weights less than 1 and the set of the hypergraphs studied.

Definition 31: The ~~random hypergraphs that we~~ study are just ~~the hypergraphs associated to the random graphs previously studied.~~ Therefore, the random hypergraphs studied will have these properties:

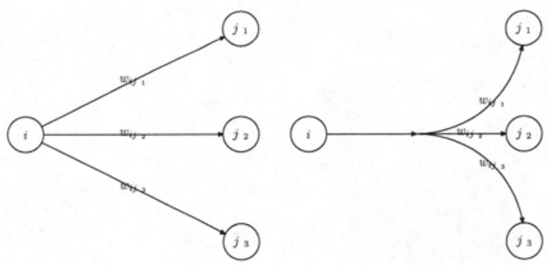

Fig. 3. On the left, there is a node with its outgoing links for the graph whose weights are less than 1. On the right, the corresponding hyperedge for the hypergraph where weights are the probability to the receiver node to get the information without error.

We cannot mention this relation between weight and probability of erasure only in the figure. It must be brought forth in the text.

1) for each node u the directed hyperedge $(\{u\}, U)$ is distributed ~~like~~ so that, for all node v,

$$P(v \in U) = p_l; \tag{86}$$

2) the weights over the edges is distributed ~~like~~ so that, for all nodes u,v

$$P\left(w_{(u,v)} \geq w\right) = \begin{cases} 1 & \text{if } w = 0, \\ p_l \int_w^\infty f_W(x)\, \mathrm{d}x & \text{else;} \end{cases} \tag{87}$$

3) ~~with some~~ the conditions of independence over the weights ~~that~~ are, that for ~~all~~ any N_0 subset of N,

$$\left(w_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent.} \tag{88}$$

V. Unicast and Multicast Types on Random Hypergraphs

A. Unicast

In this section, we ~~will~~ shall consider flows on random weighted directed hypergraphs. To the best knowledge of the authors, there is no mention of such hypergraphs in prior literature.

Theorem 32: We take a random weighted directed hypergraph with $n+1$ nodes. We take i and j two nodes, i is the source node and j the sink node. If

$$\frac{n p_l}{\ln n} \to \infty \tag{89}$$

and, for all N_0 subset of N such that $i \in N_0$ and $j \notin N_0$,

$$\left(w_{(u,v)}\right)_{u \in N_0, v \in N_0^c} \text{ are independent} \tag{90}$$

then

$$F_{(i,j)}^{H} \xrightarrow{p} 1. \tag{91}$$

This shows a similar result as for random graphs, i.e., the capacity is limited by the capacity of the source (and only the source here) and not by the rest of the hypergraph.

→ in two steps

write it out

~~B. Proof~~

We ~~will~~ prove ~~the two sides to show~~ the probability ~~convergence~~. First, we prove $P\left(F_{(i,j)}^{H} \leq 1 - \epsilon\right) \to 0$ ~~in two parts since we need a technical trick~~ to obtain the required result. In the second part, we prove the second convergence through the law of large numbers. The most important idea in this proof

by first considering a particular form of ... and later generalizing our results is that we ~~will~~ compare graphs and hypergraphs and we will show that for two corresponding cuts, the difference between the value of this cut and the cut around the source is bigger for hypergraphs than for graphs.

for which

1) $P\left(F^H_{(i,j)} \leq 1-\epsilon\right) \to 0$*:* To prove it, we need to consider, first, f_W ~~like that~~, there exists w_m such that, for all $w < w_m$,

Remember this condition,

$$f_W(w) = 0. \tag{92}$$

condition (x)

Then, we ~~will~~ shall generalize ~~for~~ to any f_W that satisfy condition (x).

a) ~~*For this particular*~~ for f_W*:* The proof begins by a lemma that ~~said~~ that, if a cut around a node is less than $1-\epsilon$ in the random hypergraph, then it is also the case in the associated random graph.

Lemma 33: For all ϵ, there exists $n_{\max}$ such that for all $n \geq n_{\max}$, for all $k \leq n-1$ and for all sequences $(w_q)_{q=1,\ldots,k}$, we have

$$1 - \prod_{q=1}^{k}(1-w_q) \leq 1-\epsilon \Rightarrow \frac{\sum_{q=1}^{k} w_q}{np_l\mu} \leq 1-\epsilon. \tag{93}$$

Proof: We prove, first, an easier result where k is fixed. For all ϵ, for all k and for all sequences $(w_q)_{q=1,\ldots,k}$, there exists $n_{\max,k}$ such that for all $n \geq n_{\max,k}$,

$$1 - \prod_{q=1}^{k}(1-w_q) \leq 1-\epsilon \Rightarrow \frac{\sum_{q=1}^{k} w_q}{np_l\mu} \leq 1-\epsilon. \tag{94}$$

That is clear since

- On the one hand, we have $1-\prod_{q=1}^{k}(1-w_q)$ ~~that is~~ constant.
- On the other hand, $\frac{\sum_{q=1}^{k} w_q}{np_l\mu}$ ~~that~~ converges to 0 as $n \to \infty$.

Now, we derive an upper-bound for k. To upperbound k, we need the special form of the distribution,

$$(1-w_m)^k \geq \prod_{q=1}^{k}(1-w_q) \geq \epsilon. \tag{95}$$

Thus,

$$k \leq \frac{\ln \epsilon}{\ln(1-w_m)}. \tag{96}$$

Reword

Therefore, now, we can switch $\forall k$ and $\exists n_{\max}$, by taking $n_{\max} = \max_{k \leq \frac{\ln\epsilon}{\ln(1-p_m)}} n_{\max,k}$. ∎

Now, we ~~will~~ prove that the probability for a node u to have a cut around it less than $1-\epsilon$ is less probable in the hypergraph than in the graph divided by $np_l\mu$. For all $v \in N\backslash\{u\}$, we denote $l_{(u,v)}$ the random variable that is 1 if the edge (u,v) exists and 0 else. Then, for all $n \geq n_{\max}$, for all subset N_0

indicator

of N, $i \in N_0$ with $|N_0| = k+1$,

$$\begin{aligned}
&P\left(C^H_{(\{u\},N_0^c)} \leq 1-\epsilon\right) \\
&= \int_{\substack{\{0,1\}^{n-k} \\ [p_m,1]^{n-k}}} \mathbf{1}_{C^H_{(\{u\},N_0^c)} \leq 1-\epsilon} \\
&\qquad \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} \left(\mathrm{d}f_W\right)_{v\in N_0^c} \quad (97) \\
&= \int_{\substack{\{0,1\}^{n-k} \\ [p_m,1]^{n-k}}} \mathbf{1}_{1-\prod_{v \in compN_0}(1-l_{(u,v)}w_{(u,v)}) \leq 1-\epsilon} \\
&\qquad \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} \left(\mathrm{d}f_W\right)_{v\in N_0^c} \quad (98) \\
&\leq \int_{\substack{\{0,1\}^{n-k} \\ [p_m,1]^{n-k}}} \mathbf{1}_{\frac{\sum_{v\in N_0^c} l_{(u,v)}w_{(u,v)}}{np_l\mu} \leq 1-\epsilon} \\
&\qquad \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} \left(\mathrm{d}f_W\right)_{v\in N_0^c} \quad (99) \\
&= \int_{\substack{\{0,1\}^{n-k} \\ [p_m,1]^{n-k}}} \mathbf{1}_{C^G_{(\{u\},N_0^c)} \leq 1-\epsilon} \\
&\qquad \left(\mathrm{d}p_{l_{(u,v)}}\right)_{v\in N_0^c} \left(\mathrm{d}f_W\right)_{v\in N_0^c} \quad (100) \\
&= P\left(C^G_{(\{u\},N_0^c)} \leq 1-\epsilon\right). \quad (101)
\end{aligned}$$

Justify this step

? ≤

b) For the general f_W*:* We now approximate for the general case. For that, we ~~will~~ delete all the edges whose weight is less than a certain w_m (we can choose any $w_m < \mu$). The new hypergraph will be denote $\check{H}$ and each previous quantity, x in the first hypergraph or graph associated will be denoted $\check{x}$ when we delete the edges whose weight is less than w_m. We have, for new probability for two nodes to be linked,

$$\check{p_l} = \left(1 - \int_{x=0}^{w_m} f_W(x)\,\mathrm{d}x\right) p_l. \tag{102}$$

Justify

However, we still have that $\frac{n\check{p_l}}{\ln n} \to \infty$. For this approximation we have that

$$P\left(C^H_{(\{u\},N_0^c)} < 1-\epsilon\right) \leq P\left(C^{\check{H}}_{(\{u\},N_0^c)} < 1-\epsilon\right) \tag{103}$$

since the weight ~~in~~ s of all in $\check{H}$ ~~is~~ are less than or equal ~~in~~ than those in H.

Now, it is easy to prove what we want. Indeed, ~~like~~ since

$$C^H_{(N_0,N_0^c)} = \sum_{u\in N_0} C^H_{(\{u\},N_0^c)}, \tag{104}$$

$$C^G_{(N_0,N_0^c)} = \sum_{u\in N_0} C^G_{(\{u\},N_0^c)}, \tag{105}$$

we obtain that

Again, justify

$$P\left(C^H_{(N_0,N_0^c)} < 1-\epsilon\right) \leq P\left(\frac{C^{\check{G}}_{(N_0,N_0^c)}}{np_l\mu} < 1-\epsilon\right). \tag{106}$$

By minimizing over all N_0 subset of N where $i \in N_0$ and $j \notin N_0$,

$$P\left(F^H_{(i,j)} < 1-\epsilon\right) \leq P\left(\frac{F^{\check{G}}_{(i,j)}}{np_l\mu} < 1-\epsilon\right) \to 0 \tag{107}$$

2) $P\left(F^H_{(i,j)} \geq 1+\epsilon\right) \to 0$*:* ~~Like~~ as the cut around the source i is a cut, we have

$$F^H_{(i,j)} \leq C^H_{(\{i\},\{i\}^c)} \leq 1. \tag{108}$$

Hence,

$$P\left(F^H(n) > 1+\epsilon\right) = 0. \tag{109}$$

This concludes the proof.

C. *Multicast Types in Random Hypergraph*

The proofs are the same as in the multicast types of random graphs and we omit here for brevity.

1) Multicast:

Theorem 34: We denote $F^M_{(i,J)}$, the maximal flow through the network from one source node i to r sink nodes $J = \{j_k\}_{k=1,\dots,r}$ that want all information. We have

$$F^M_{(i,J)} \xrightarrow{p} 1. \tag{110}$$

2) 2-layer Multicast:

Theorem 35: We denote $F^M_{(i,J\backslash\{j_1\})}$ the max-flow achievable for the multicast from the source node i to the $r-1$ sink nodes $J\backslash\{j_1\} = \{j_k\}_{k=2,\dots,r}$ when j_1 just wants a fraction ϵ of the total information. We have

$$F^M_{(i,J\backslash\{j_1\})} \xrightarrow{p} 1 \tag{111}$$

and ϵ can take any value between 0 and 1.

3) Disjoint Multicast:

Theorem 36: We denote $F^D_{(i,J)}$ the maximal flow that the source ~~can send where the source~~ node i sends ~~information~~ to r sink nodes $J = \{j_k\}_{k=1,\dots,r}$ that want, each, a fraction ϵ_k of information, and ~~all~~ disjoint from each other. We have

$$F^D_{(i,J)} \xrightarrow{p} 1. \tag{112}$$

We cannot obtain a result for the multisource-multicast by the same method used in the graph case since we cannot reverse the hyperedges as we have done in the graph case.

These results show that network coding is very useful in the random hypergraph case since, if only the capacity around the source is known, the maximal amount of information that can be sent through the random hypergraph for a multicast can be determined. Furthermore, it can be determined locally without prior knowledge of the whole hypergraph.

VI. Conclusion

e.g. cylind model
physical layer
broadcast nature.

We have shown that, for a large class of random graphs and hypergraphs, the capacity of the network can be easily known by looking at the cut around the source, ~~a local procedure~~. ~~This result generalizes a large number of results previously obtained about random graphs in [4], [6] and [7].~~ Moreover, to the best knowledge of the authors, ~~it is the first~~ we establish result about max-flows in random hypergraphs. ~~Our result can be applied to ad-hoc networks due to the properties of hyperedges that can model broadcast channels with independent erasures.~~

Flag results

Nevertheless, we use simple *geometryless* models for random graphs and hypergraphs. In addition, our results are asymptotic. Therefore, our work opens up interesting questions and areas of research. Primary amongst them is the extension of our results to random geometric graphs. Another important question for future work involves the determination of the minimal graph size that guarantees the validity of our results.

We furthermore generalize hypergraphs to allow for weights on hyperedges. These weights allow us to model random erasures in wireless networks.

Appendix

Lemma 37: Let $Y(n) = (Y_1(n), \dots, Y_l(n))$ be a sequence of random vectors and $(y_1, \dots, y_l)$ a vector of real numbers such that, for all i, $Y_i(n)$ converge in probability to y_i ($Y_i(n) \xrightarrow{p} y_i$). Then,

$$\min_i Y_i(n) \xrightarrow{p} \min_i y_i. \tag{113}$$

Proof: We can assume that y_1 is the minimum of the y_i. Let $\epsilon > 0$. We have,

$$P\left(\min_i Y_i(n) - y_1 > \epsilon\right) \leq P\left(Y_1(n) - y_1 > \epsilon\right) \tag{114}$$

$$\to 0. \tag{115}$$

Therefore,

$$P\left(\min_i Y_i(n) - y_1 < -\epsilon\right)$$

$$\leq \sum_{i=1}^{l} P\left(Y_i(n) - y_1 < -\epsilon\right) \tag{116}$$

$$\leq \sum_{i=1}^{l} P\left(Y_i(n) - y_i < -\epsilon\right) \tag{117}$$

$$\to 0 \tag{118}$$

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