TITLE: Bounding Revenue Comparisons across Multi-Unit Auction Formats under ε-Best Response

AUTHORS:

James T.E. Chapman
Department of Banking Operations
Bank of Canada
234 Wellington Street
Ottawa, Ontario K1A 0G9
CANADA
Telephone: (613) 782-8461
Facsimile: (613) 782-8751
E-Mail: JChapman@BankofCanada.ca

David McAdams
Sloan School of Management
MIT
E52-448
50 Memorial Drive
Cambridge, MA 02142
Telephone: (617) 253-1306
Facsimile: (617) 258-6855
E-Mail: mcadams@mit.edu

Harry J. Paarsch, **corresponding author**
Department of Economics
University of Iowa
108 Pappajohn Business Building
21 East Market Street
Iowa City, Iowa 52242-1944
Telephone: (319) 335-0936
Facsimile: (319) 335-1956
E-Mail: hjp@paarsch.biz.uiowa.edu

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Bounding Revenue Comparisons across Multi-Unit Auction Formats Under ε-Best Response

By James T.E. Chapman, David McAdams, and Harry J. Paarsch *

Expected revenue and allocative efficiency are two of the most important considerations in auction design. The Revenue Equivalence Theorem provides conditions under which many single-object auctions generate the same expected revenue, but theory provides little guidance when more than one object is sold. In such settings, the question of which auction format or pricing rule generates the most expected revenue or the most efficient allocation remains largely an empirical one.

In Canada, the central bank (the Bank of Canada) currently uses a multi-unit, sealed-bid auction format in conjunction with a discriminatory pricing rule to invest excess government cash in term-deposits at a select set of financial institutions. Winners at these auctions pay their tendered bids, so participants shave their bids in equilibrium and winning allocations can be inefficient. We consider what would happen to expected revenues and allocational efficiency were the Bank of Canada to switch from the discriminatory pricing rule to the generalized Vickrey auction (GVA) pricing rule. The GVA is an interesting benchmark because, given private values, an equilibrium exists in which bids are truthful and allocations are efficient.

In both the discriminatory auction and the GVA, each bidder i's allocation and payment depends on his bid schedule and residual supply (total supply minus the bids of other participants). For example, in Figure 1, bidder i's payment in the discriminatory auction equals the area $0ABQ_i^D$, while his payment in the GVA equals the area $0CQ_i^V$. Revenue is higher in the discriminatory auction for any given bids, but bids are higher in the GVA. Thus, *a priori*, either format could generate more revenue for the seller.

The research reported here builds on that in James T.E. Chapman, David McAdams, and Harry J. Paarsch (2005) which in turn builds on the pioneering research of Ali Hortaçsu (2002). In Chapman et al., we developed a framework within which to bound the marginal values of bidders under the joint hypothesis of best response and non-increasing marginal values, and then to test that hypothesis. Below, we implemented this framework using the same data. While some observed bids in the Bank of Canada's auctions cannot be explained as best responses given non-increasing marginal values, almost all of them can be explained as being "close" to a best response given such values, hence the term "ε-best response" in our paper's title.

In this paper, we employ the bounds on bidder values developed in our previous work for counterfactual analysis. In particular, assuming that bidders would play their weakly undominated (truthful) strategies in the GVA, we compute a lower bound on the seller's expected revenue in the GVA as well as a lower bound on the efficiency gains associated with switching to the GVA. The fact that some observed bids in the discriminatory auction fail to be best-responses complicates this analysis. For such bids, we infer bounds on bidder values by assuming that the bids are weakly undominated. That is, we impose the conservative assumption that the bidder's true marginal value for each unit is greater than or equal to his bid on that unit.

Whereas expected revenue per auction in the observed discriminatory auctions was $153,440.15, we estimate that expected revenue per auction in the truthful equilibrium of the GVA would have been at least $152,401.33, for a loss of at most $1,038.82 per auction. (In our application, we define revenue as the aggregate interest payment that the Bank receives when the auctioned term deposits mature.) On the other hand, the discriminatory auction leads to efficiency losses estimated to be at least $0.0045 per auction, while the GVA leads to an efficient allocation. (We infer an ex post inefficiency in only 1.88 percent of the observed discriminatory auctions, and these inefficiencies are relatively small.)

I. Auctions of Receiver General Term-Deposits

As the fiscal agent of the Canadian federal government, the Bank of Canada manages the government's day-to-day cash and foreign exchange reserves. The Bank of Canada's main instruments for conducting cash management are the so-called *Receiver General* (RG) auctions, conducted in the morning and afternoon. In these auctions, excess government cash is invested in term-deposits held by a select group of financial institutions that are members of Canada's Large Value Transfer System (LVTS).

The Bank of Canada pays its bank rate on all positive LVTS account balances and charges the bank rate plus 50 basis points on all negative LVTS account balances. This policy effectively imposes a 50 basis-point band on the market-clearing interest rate in RG auctions.

The size of an RG auction is based on the daily operational needs of the Canadian federal government as well as daily monetary-policy operations. Thus, it is reasonable to view the supply of funds as exogenous. The interest rate on these deposits is determined by sealed bids in a multi-unit, discriminatory auction. In particular, members of the LVTS must submit bids consisting of up to four (price,quantity) pairs. (These prices and quantities are discrete, with prices specified in terms of basis points determined at an annual rate of interest and quantities specified in millions CAD, with a minimal quantity of $5 million CAD.)

The Bank of Canada transforms these bid vectors into an aggregate demand curve, allocates the available quantity to the highest bidders, and these winners pay an interest rate equal to their bid. Alternative payment rules include those of the uniform price auction—see, for example, Hortacsu (2002)—and the generalized Vickrey auction described by Vijay Krishna (2002). In the GVA, with S units for sale, each bidder's allocation and payment are determined as follows: Let $\mathbf{V}_i$, $\mathbf{V}$, and $\mathbf{V}_{-i}$ denote the reported ordered valuations of bidder i, of all bidders, and of all bidders other than i, respectively. Units are allocated to the bidders with the S highest reported valuations in $\mathbf{V}$, with bidder i paying the $(S-q)^{\text{st}}$ highest reported valuation in $\mathbf{V}_{-i}$ for the q^{th} unit if he wins that unit, for all q.

II. Theoretical Model

Here, we provide an abbreviated description of our theoretical model; see Chapman et al. (2005) for additional details. At any auction, denote by S the total supply. A bid by bidder i is a vector $\mathbf{b}_i$ of *unit-bids* $(b_{i,q} : q \in \mathcal{Q})$, specifying a *permissible price* $b_{i,q} \in \mathcal{P}$ for every *permissible quantity* $q \in \mathcal{Q}$, where $\mathcal{P}$ is $(p_{\text{out}}, \underline{p}, p_2, \ldots, \bar{p})$ and $\mathcal{Q}$ is $(\underline{q}, q_2, \ldots, \bar{q})$. (Bidding p_{out} for a q^{th} unit denotes an unwillingness to win that unit, even at the minimal price $\underline{p}$.) A *permissible bid* is any bid satisfying two requirements:

i) *non-increasing*—i.e, $b_{i,q} \geq b_{i,q'}$ for all $q < q'$;

ii) *at most $\bar{K}$ steps*—there do not exist quantities $q_1 < \ldots < q_{\bar{K}+1}$ such that $b_{i,q_1} > \ldots > b_{i,q_{\bar{K}+1}} \geq \underline{p}$.

In our empirical application, $\bar{K}$ is four and the maximum individual quantity $\bar{q}$ varies across bidders. For notational parsimony, we suppress this dependence and consider the special case in which $\mathcal{Q}$ equals $\{1, \ldots, S\}$. For simplicity, we also ignore the four-step constraint in the exposition. (This constraint only binds on about 2.5 percent of observed bids in our sample.)

Bidder i has private marginal value schedule ("values") $\mathbf{v}_i$, where $v_{i,q}$ denotes his marginal value for quantity $q = 1, \ldots, S$. We assume non-increasing marginal values (NIMV), i.e., that $v_{i,q} \geq v_{i,q'}$ for all i and all $q < q'$, and that $(\mathbf{v}_1, \ldots, \mathbf{v}_N)$ are independent, having been drawn from the same joint distribution at each auction.

Each bidder adopts a monotone pure strategy, with bid $\mathbf{b}_i(\mathbf{v}_i)$ specifying unit-bid $b_{i,q}(\mathbf{v}_i)$ for each quantity $q = 1, \ldots, S$ given values $\mathbf{v}_i$. Monotonicity means that, for all $\mathbf{v}'_i, \mathbf{v}_i$ such that $v'_{i,q} \geq v_{i,q}$ for all q, $b_{i,q}(\mathbf{v}'_i) \geq b_{i,q}(\mathbf{v}_i)$ for all q. (McAdams (forthcoming) has proven that every mixed-strategy equilibrium is outcome equivalent to a monotone pure-strategy equilibrium. Thus, restricting attention to monotone pure strategies is without loss of generality.)

Let $s_{i,q}(\mathbf{b}_{-i})$ (shorthand $s_{i,q}$) denote bidder i's *residual supply* of quantity q. ($s_{i,q}$ is a random variable since the bids of opponents are random.) In words, when opponents tender bids $\mathbf{b}_{-i}$, bidder i wins less than q units whenever his unit-bid on the q^{th} unit is less than $s_{i,q}(\mathbf{b}_{-i})$, and wins at least q units whenever this unit-bid is greater than $s_{i,q}(\mathbf{b}_{-i})$. Bidder i's expected payoff when bidding $\mathbf{b}_i$ given values $\mathbf{v}_i$ and the strategies of others takes the form $\sum_{q=1}^{S}(v_{i,q} - b_{i,q}) \Pr_{s_{i,q}}(b_{i,q} > s_{i,q})$.

Let $\mathcal{V}(\mathbf{b}_i)$ denote the set of NIMV schedules given which bid $\mathbf{b}_i$ is a best-response for bidder i. A joint distribution of bids is consistent with static Bayes–Nash equilibrium and NIMV if and only if, for all i, $\mathcal{V}(\mathbf{b}_i)$ is non-empty for all $\mathbf{b}_i$ in the support of bidder i's strategy. (We focus on static best-responses, ignoring any dynamic features of the game.)

III. Bounding Bidder Values

For a bid $\mathbf{b}_i$ to be a best response, every possible deviation must be unprofitable, including all local and global deviations. In Chapman, et al. (2005), we leverage this fact to construct bounds on bidders' marginal values and to derive testable implications. We summarize these results in the remainder of this section.

Consider a local deviation $\mathbf{b}_i^{q\uparrow}$ defined as follows:

$$b_{i,x}^{q\uparrow} = b_{i,x} \text{ if } x > q \text{ or } b_{i,x} \neq b_{i,q}$$

$$b_{i,x}^{q\uparrow} = b_{i,x} + \Delta \text{ if } x \leq q \text{ and } b_{i,x} = b_{i,q}.$$

Such a deviation is profitable for all small enough $\Delta > 0$ unless

$$\sum_{x \leq q : b_{i,x} = b_{i,q}} \frac{\mathrm{d}\left[(v_{i,x} - b_{i,x}) \Pr_{s_{i,x}}(b_{i,x} > s_{i,x})\right]}{\mathrm{d}b_{i,x}} \leq 0. \tag{1}$$

This relationship implies an *upper bound* $\overline{v}_{i,q}(\mathbf{b}_i)$ on bidder i's marginal value for the q^{th} unit, defined implicitly by

$$\sum_{x \leq q : b_{i,x} = b_{i,q}} \frac{\mathrm{d}\left[(\overline{v}_{i,q}(\mathbf{b}_i) - b_{i,x}) \Pr_{s_{i,x}}(b_{i,x} > s_{i,x})\right]}{\mathrm{d}b_{i,x}} = 0. \tag{2}$$

If $v_{i,q} > \overline{v}_{i,q}(\mathbf{b}_i)$, then $v_{i,x} > \overline{v}_{i,q}(\mathbf{b}_i)$ for all $x \leq q$ by non-increasing marginal values, and condition (1) is violated. Similarly, a lower bound on $v_{i,q}$ is defined implicitly by

$$\sum_{x \geq q : b_{i,x} = b_{i,q}} \frac{\mathrm{d}\left[(\underline{v}_{i,q}(\mathbf{b}_i) - b_{i,x}) \Pr_{s_{i,x}}(b_{i,x} > s_{i,x})\right]}{\mathrm{d}b_{i,x}} = 0. \tag{3}$$

These bounds lead to two testable implications: First, $\mathbf{b}_i$ can only be a static best response if $\overline{v}_{i,q}(\mathbf{b}_i) \geq \underline{v}_{i,q}(\mathbf{b}_i)$ for all q. Otherwise, some local deviation must be profitable given *any* non-increasing marginal values. Second, since bidders play monotone pure strategies, $\underline{\mathbf{v}}_i(\mathbf{b}_i) \leq \mathbf{v}_i \leq \overline{\mathbf{v}}_i(\mathbf{b}_i)$ implies that bidder i's minimal best response given values $\underline{\mathbf{v}}_i(\mathbf{b}_i)$ must be weakly less than $\mathbf{b}_i$, and his maximal best response given values $\overline{\mathbf{v}}_i(\mathbf{b}_i)$ must be weakly greater than $\mathbf{b}_i$.

Thirty-one percent of all bids in our sample pass these two tests. For such bids, we use $\overline{v}_{i,q}(\mathbf{b}_i)$ and $\underline{v}_{i,q}(\mathbf{b}_i)$ as upper and lower bounds on $v_{i,q}$ given bid $\mathbf{b}_i$. For a bid that fails either of the tests, we use ∞ and $b_{i,q}$ as the upper and lower bounds on $v_{i,q}$.

IV. Counterfactual Analysis and Conclusions

We use the lower bounds on bidder values to compute a lower bound on the Bank of Canada's expected revenues, were the Bank to switch to the GVA and bidders to play their weakly-undominated (truthful) strategies. According to our estimates, expected revenue per auction would be at least $\$153,400.15$, for a loss of at most $\$1,038.22$ relative to the discriminatory format. Subsampling allows us to construct consistent confidence intervals for this revenue effect. A ninety-five percent confidence interval for this bound on the change in expected revenue is $[-1,506.51, -683.98]$.

To compute a lower bound on the efficiency gain from switching to the GVA with truthful bidding, we compare the *upper* bounds on marginal values associated with *winning* units in the discriminatory auction versus the *lower* bounds on marginal values associated with *losing* units in the discriminatory auction. (There must be an efficiency loss in the discriminatory auction if the lower bound on a loser's marginal value exceeds the upper bound on a winner's marginal value.) Overall, we estimate an efficiency gain of at least $0.0045, with a ninety-five percent confidence interval of $[0.0000, 0.6398]$ for the bound on this effect.

Since the efficiency gain from switching to the GVA may be small—and lost revenue relatively large—our results do not provide strong evidence that the Bank of Canada should make this switch.

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Lead Footnote

* James T.E. Chapman, Department of Banking Operations, Bank of Canada, JChapman@BankofCanada.ca; David McAdams, Sloan School of Management, MIT, mcadams@mit.edu; and Harry J. Paarsch, Department of Economics, University of Iowa, hjp@paarsch.biz.uiowa.edu

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Figure 1
Allocations and Revenues:
Discriminatory versus Generalized Vickrey Auction