On the Failure of Monotonicity in Uniform-Price Auctions

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Abstract

Except for well-studied special cases in which bidders have single-unit demand or bidders are risk-neutral with independent private values, equilibria of uniform-price auctions with private values need not possess familiar monotonicity properties. In particular, equilibria in weakly undominated strategies may exist in which some bidders bid strictly less on some units when they have strictly higher values for every unit.

Key words: Uniform-price auction, monotonicity, independence, affiliation, risk-neutrality, risk-aversion.

1 Introduction

The uniform-price auction has received much attention in the academic literature in recent years. Cripps and Swinkels [1] and Reny and Perry [4] show that equilibrium bidding in the uniform-price auction provides a micro-foundation for price formation in competitive markets when there are numerous bidders.

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In many applications, however, there are few enough participants that individual bidders have market power. See e.g. Wolfram [5] on evidence of market power in electricity auction markets.

Unfortunately, bidding behavior in uniform-price auctions with only a few bidders is still relatively poorly understood, although certain special cases have been well studied. For example, Fudenberg, Mobius, and Szeidl [2] and Reny and Perry [4] have made substantial progress in the case in which bidders have single-unit demand, while McAdams [3] considers the case in which bidders have multi-unit demand but are risk-neutral with independent types. These papers establish existence of a monotone pure strategy equilibrium, meaning that each bidder never bids strictly less on any unit given weakly higher values for every unit. Indeed, in the benchmark case with risk-neutrality and independent private values, McAdams shows that *every* equilibrium of the uniform-price auction must be monotone. [1]

This paper's contribution is to show that independence and risk-neutrality are essential to this monotonicity result. Non-monotone equilibria can exist when bidders have multi-unit demand and we relax *either* independence *or* risk-neutrality. Indeed, I provide examples in which all equilibria in weakly undominated strategies are non-monotone.

Model. Two identical objects ("units") are sold to two bidders in a uniform third-price auction. Each bidder has private marginal value schedule ("values") of the form $\mathbf{v}_i = (v_{i,1}, v_{i,2}) \geq \underline{0}$, where $v_{i,1} \geq v_{i,2}$. (The joint distribution of

[1] McAdams [3] shows that every mixed strategy equilibrium must be "ex post allocation-equivalent" and "interim expected payment-equivalent" to some monotone pure strategy equilibrium.

$(\mathbf{v}_1, \mathbf{v}_2)$ will be specified in each example.) Each bidder submits a unit-bid schedule ("bid") $\mathbf{b}_i = (b_{i,1}, b_{i,2}) \geq \underline{0}$, where $b_{i,1} \geq b_{i,2}$. The highest two unit-bids win, and each bidder pays a per-unit price equal to the third-highest unit-bid. Note that bidder 1 wins a first unit whenever $b_{1,1} > b_{2,2}$ and wins a second unit whenever $b_{1,2} > b_{2,1}$, and vice versa for bidder 2. For simplicity I ignore ties, which play no role in the analysis.

Given a profile of bids $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2)$, let $q_i(\mathbf{b})$ denote the quantity that bidder i wins and $z_i(\mathbf{b})$ his total payment. Bidder i's ex post surplus is $\sum_{x=1}^{q_i(\mathbf{b})} v_{i,x} - z_i(\mathbf{b})$ and ex post utility is $\pi_i(\mathbf{v}_i, \mathbf{b}) = u_i\left(\sum_{x=1}^{q_i(\mathbf{b})} v_{i,x} - z_i(\mathbf{b})\right)$, where u_i is a weakly increasing and concave function of surplus.

A pure strategy $\mathbf{b}_i(\cdot)$ is *monotone* when $b_{i,1}(\mathbf{v}'_i) \geq b_{i,1}(\mathbf{v}_i)$ and $b_{i,2}(\mathbf{v}'_i) \geq b_{i,2}(\mathbf{v}_i)$ for all $\mathbf{v}'_i, \mathbf{v}_i$ satisfying $v'_{i,1} \geq v_{i,1}$ and $v'_{i,2} \geq v_{i,2}$, and *non-monotone* otherwise. In each of the examples to follow, I establish existence of a (Bayesian Nash) equilibrium in weakly undominated pure strategies and show that some bidder adopts a non-monotone strategy in all such equilibria. [2]

2 Relaxing independence

When bidders' values are affiliated, each bidder expects to face stiffer competition when he has higher values. This effect may induce some bidders to bid *less* aggressively on some units given higher values. For instance, in Ex-

[2] Monotone pure strategy equilibria also exist, but only in weakly dominated strategies. For instance, it is an equilibrium in both examples for bidder 1 to bid $(5, 5)$ and for bidder 2 to bid $(0, 0)$, regardless of their values. Such constant strategies are trivially monotone.

ample 1, bidder 1 faces an opponent who wants only one unit. Since bidder 2 will always bid zero on a second unit (in any weakly undominated strategy), bidder 1 can guarantee himself one unit at zero price by bidding zero on a second unit. Or, bidder 1 could compete for a second unit and, in doing so, raise the price that he pays for a first unit. As bidder 1's values increase, he expects bidder 2 to bid more aggressively. This increases the cost to bidder 1 of competing for a second unit, so much so that bidder 1 prefers to concede a unit to his opponent when he has high enough values for both units.

Example 1. Bidders are risk-neutral with *affiliated* private values. Bidder 1's values take the form $\mathbf{v}_1 = (v_1, v_1)$ while bidder 2's values take the form $\mathbf{v}_2 = (v_2, 0)$. v_1, v_2 are affiliated, with joint distribution generated as follows. With probability 50%, v_1, v_2 are independent with $v_1 \sim U[2,3]$ and $v_2 \sim U[0,1]$; otherwise, v_1, v_2 are independent with $v_1 \sim U[3,4]$ and $v_2 \sim U[4,5]$. [3]

In any weakly undominated strategy, each bidder must bid truthfully on the first unit and bid no more than his value on a second unit; hence $b_{1,1}(v_1, v_1) = v_1$, $b_{1,2}(v_1, v_1) \leq v_1$, $b_{2,1}(v_2, 0) = v_2$, and $b_{2,2}(v_2, 0) = 0$ for all v_1, v_2. All that remains is to determine what bidder 1 bids on a second unit. If he bids $b > v_2$, then bidder 1 wins both units at price v_2; if he bids $b < v_2$, then each bidder wins one unit at price b. Thus, in any best response, bidder 1 sets his second unit-bid to maximize the objective

$$\Pr(b > v_2 | v_1)(2v_1 - 2E[v_2 | v_1 \text{ and } b > v_2]) + \Pr(b < v_2 | v_1)(v_1 - b) \qquad (1)$$

[3] The fact that $v_2|v_1$ has changing support is not essential. It is possible to perturb this example so that (say) (v_1, v_2) has a continuous pdf on full support $[0,5]^2$, while maintaining the key non-monotonicity that $b_{1,2}(2,2) \geq 1$ and $b_{1,2}(4,4) = 0$ in all equilibria in weakly undominated strategies.

Suppose that $v_1 \in [2,3)$. $v_2|v_1$ is uniform over $[0,1]$ and (1) reduces to $2v_1 - 1$ for all $b \geq 1$ and $b(2v_1 - b) + (1-b)(v_1 - b) = (1+b)v_1 - b$ for all $b \in [0,1]$. Since $v_1 > 1$, we conclude that $b_{1,2}(v_1, v_1) \geq 1$ in any best response for all $v_1 \in [2,3)$.

Suppose that $v_1 \in (3,4]$. $v_2|v_1$ is uniform over $[4,5]$ and bidder 1 can not win the second unit with positive probability without bidding more than his value for that unit. Thus, bidder 1's second unit-bid never wins and always sets the price. Consequently, $b_{1,2}(v_1, v_1) = 0$ is bidder 1's unique best response for all $v_1 \in (3,4]$.

Finally, suppose that $v_1 = 3$. $v_2|v_1$ is uniformly distributed over $[0,1] \cup [4,5]$. In this case, it is easy to check that bidder 1's unique best response is to bid one on the second unit.

We have shown by construction that an equilibrium exists in weakly undominated strategies. Since $b_{1,2}(2,2) \geq 1 > 0 = b_{1,2}(4,4)$, bidder 1's strategy is non-monotone in all such equilibria.

3 Relaxing risk-neutrality

When a bidder is risk-averse, his incremental utility from a second unit decreases as his willingness to pay for the first unit increases. Consequently, a risk-averse bidder may prefer to bid less on a second unit given higher values on both units. For instance, in Example 2, bidder 1 bids his value for a second unit given relatively low values but bids zero on a second unit given relatively high values.

Example 2. Bidders are *risk-averse* with independent private values. Bidder 1's values take the form $\mathbf{v}_1 = (v_1, v_1)$ while bidder 2's values take the form $\mathbf{v}_2 = (v_2, 0)$. v_1, v_2 are independent, where $v_1 \sim U[1, 3/2]$ and $v_2 \in [0,1]$ has pdf $f_2(v_2) = 2(1-v_2)$. Bidder 2 is risk-neutral while bidder 1 is satiated by one and a half units of surplus. That is, $u_1(x) = \max\{x, 3/2\}$ where x is bidder 1's ex post surplus.

In any weakly undominated strategy, $b_{1,1}(v_1, v_1) = v_1$, $b_{1,2}(v_1, v_1) \leq v_1$, $b_{2,1}(v_2, 0) = v_2$, and $b_{2,2}(v_2, 0) = 0$ for all v_1, v_2, so it remains only to determine bidder 1's second unit-bid. As in Example 1, bidder 1's expected utility is continuous in $b_{1,2}$, so a weakly undominated best response exists for bidder 1 for all v_1. Hence an equilibrium in weakly undominated strategies exists. To prove that bidder 1's strategy is non-monotone in any such equilibrium, it suffices to show that bidder 1 bids less on a second unit when $v_1 = 3/2$ than when $v_1 = 1$.

Bidder 1 bids $(1,1)$ *when* $v_1 = 1$. When $v_1 = 1$, $\{(1, \hat{b}) : \hat{b} \in [0,1]\}$ is the set of bidder 1's weakly undominated bids. When bidding $(1, \hat{b})$, bidder 1 is satiated with utility $3/2$ when $b_{2,1} = v_2 < \min\{\hat{b}, 1/4\}$, gets utility $2(1-v_2)$ when $1/4 < v_2 < \hat{b}$, and gets utility $1 - \hat{b}$ when $\hat{b} < v_2$. Thus, bidder 1 gets expected utility (2) or (3) from bid $(1, \hat{b})$, when $\hat{b} \leq 1/4$ or $\hat{b} \geq 1/4$, respectively.

$$\pi_1\left((1,1),(1,\hat{b})\right) = \int_0^{\hat{b}} \frac{3}{2}\,(2(1-v_2))\,\mathrm{d}v_2 + \int_{\hat{b}}^1 (1-\hat{b})\,(2(1-v_2))\,\mathrm{d}v_2 \qquad (2)$$

for all $\hat{b} \in [0, \frac{1}{4}]$ and

$$\pi_1\left((1,1),(1,\hat{b})\right) = \int_0^{1/4} \frac{3}{2}\,(2(1-v_2))\,\mathrm{d}v_2 + \int_{1/4}^{\hat{b}} 2(1-v_2)\,(2(1-v_2))\,\mathrm{d}v_2 + \int_{\hat{b}}^1 (1-\hat{b})\,(2(1-v_2))\,\mathrm{d}v_2 \qquad (3)$$

for all $\hat{b} \in [1/4, 1]$. As can be easily checked, (2) is strictly increasing in $\hat{b}$ for all $\hat{b} \in [0, 1/4]$ while (3) is strictly increasing in $\hat{b}$ for all $\hat{b} \in [1/4, 1]$. Thus, bidder 1's unique weakly undominated best response is to bid $(1, 1)$.

Bidder 1 bids $(3/2, 0)$ *when* $v_1 = 3/2$. When $v_1 = 3/2$, $\{(3/2, \hat{b}) : \hat{b} \in [0, 3/2]\}$ is the set of bidder 1's weakly undominated bids. By bidding $(3/2, 0)$, bidder 1 is certain to win one unit at price zero, guaranteeing that he will be satiated. When bidding $(3/2, \hat{b})$ for any $\hat{b} \in (0, 3/2]$, however, bidder 1 will not be satiated with positive probability: for all $\hat{b} \in (0, 1]$, bidder 1 gets surplus $3/2 - \hat{b} < 3/2$ when $v_2 \in (\hat{b}, 1]$; for all $\hat{b} \in [1, 3/2]$, bidder 1 gets surplus $2(3/2 - v_2) < 3/2$ when $v_2 \in (3/4, 1]$. Thus, bidder 1's unique weakly undominated best response is to bid $(3/2, 0)$.

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