Volatility of Power Grids under Real-Time Pricing

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***Abstract*—The paper proposes a framework for modeling and analysis of the dynamics of supply, demand, and clearing prices in power systems with real-time retail pricing and information asymmetry. Characterized by passing on the real-time wholesale electricity prices to the end consumers, real-time pricing creates a closed-loop feedback system between the physical layer and the market layer of the system. In the absence of a carefully designed control law, such direct feedback can increase sensitivity and lower the system's robustness to uncertainty in demand and generation. It is shown that price volatility can be characterized in terms of the system's *maximal relative price elasticity,* defined as the maximal ratio of the *generalized price-elasticity* of consumers to that of the producers. As this ratio increases, the system may become more volatile. Since new demand response technologies increase the price-elasticity of demand, and since increased penetration of distributed generation can also increase the uncertainty in price-based demand response, the theoretical findings suggest that the architecture under examination can potentially lead to increased volatility. This study highlights the need for assessing architecture systematically and in advance, in order to optimally strike the trade-offs between volatility/robustness and performance metrics such as economic efficiency and environmental efficiency.**

***Index Terms*—Real-Time Pricing, Marginal Cost Pricing, Volatility, Lyapunov Analysis.**

I. INTRODUCTION

THE increasing demand for energy along with growing environmental concerns have led to a national agenda for engineering modern power grids with the capacity to integrate renewable energy resources at large scale. In this paradigm shift, demand response and dynamic pricing are often considered as means of mitigating the uncertainties and intermittencies of renewable generation and improving the system's efficiency with respect to economic and environmental metrics. The idea is to allow the consumers to adjust their consumption in response to a signal that reflects the wholesale market conditions, possibly the real-time prices. However, this real-time or near real-time coupling between supply and demand creates significant challenges for guaranteeing reliability and robustness of future power systems. The challenges are in part, due to the uncertainties and complexities in the dynamics of consumption, particularly the dynamics of load-shifting and storage, as well as uncertainty in consumer behavior, preferences, time-varying and private valuation for electricity, and consequently, uncertainty in response to real-time prices.

Various forms of dynamic retail pricing of electricity have been studied in economic and engineering literature. In [1], Borenstein et. al. investigate both the theoretical and the practical implications of different dynamic pricing schemes such as *Critical Peak Pricing* (CPP), *Time-of-Use Pricing* (TUP), and *Real-Time Pricing* (RTP). They argue in favor of real-time pricing, characterized by passing on a price that best reflects the wholesale market prices to the end consumers, and conclude that real-time pricing delivers the most benefits in the sense of reducing the peak demand and flattening the load curve. In [2], Hogan identifies dynamic pricing, particularly real-time pricing as a priority for implementation of demand response in organized wholesale energy markets. Similar conclusions can be found in several independent studies including but not limited to the MIT Future of the Grid report [3], and a study conducted by Energy Futures Australia [4]. Real-world implementations of various forms of dynamic pricing have begun to emerge as well [5].

The viewpoint adopted in this paper is that directly linking price sensitive consumers to the wholesale electricity markets fundamentally changes the architecture of the system from an open-loop system in which demand is an exogenous input, to a closed-loop feedback dynamical system. In the absence of a well-designed control law, such direct feedback may lead to increased volatility, decreased robustness to disturbances, and new fragilities that increase the risk of a systemic failure.

The factors contributing to dynamics in the system—in addition to the supply-side dynamics—are at least twofold. The first is the time delay between market clearing and consumption decision, which necessitates a prediction (of demand or price) step. One may also consider this as a form of *information asymmetry* among consumers and system operators. Predicting price sensitive demand can be particularly challenging [6], [7], [3], and the errors in the prediction step and uncertainty in demand response contribute to volatility. This challenge is naturally more profound if consumers have access to highly variable distributed generation. The second factor is the inherent dynamics of consumption induced by storage and time-shifting of deferrable loads [7], [8]. In this paper, we abstract away the internal dynamics of consumers and develop a model with only the Locational Marginal Prices as the state variables. This leads to an abstract model that sheds light on the important macro parameters that influence the behavior of the system. The level of granularity at which such a complex multi-layered network must be modeled for design purposes and for guaranteeing reliability, robustness, and efficiency is an open question deserving dedicated research.

We introduce a notion of *generalized* price-elasticity, and use Lyapunov theory [9] and contraction analysis [10] to show that price volatility can be upperbounded by a function of the system's Maximal Relative Price-Elasticity (MRPE), defined

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as the maximal ratio of the generalized price-elasticity of consumers to the generalized price-elasticity of producers. As this ratio increases, the system may become more volatile, while no meaningful upper bound on volatility can be provided when the MRPE exceeds one.

While it is possible to stabilize the system and mitigate volatility by proper design of a control law that regulates the interaction of wholesale markets and retail consumers, limitations of performance and the tradeoffs between various performance and robustness properties must be carefully considered. The limitations of performance and the tradeoffs between performance and robustness are well studies in the context of classical control [11], [12], [13], [14], and have been more recently studied in the context of networked systems (see, e.g., [13], [15], [16], [17], [18] and the references therein). In view of these results, we posit that different pricing mechanisms, i.e., different control laws, pose different tradeoffs on performance metrics such as economic or environmental efficiency, and system reliability metrics such as volatility, robustness to disturbances, and fragility. We do not consider design issues and the associated tradeoffs in this paper and focus on analysis of the system under direct feedback. The intended message is that the design of a real-time pricing mechanism must take price and demand volatility issues into consideration, and that successful design and implementation of such a mechanism entails careful modeling and analysis of consumer behavior in response to price signals, and the tradeoffs between appropriate robustness and efficiency metrics.

Other existing literature related to our work can be found in the context of stability of power markets. These include some earlier works by Alvarado [19], [20] on dynamic modeling and stability, and more recent works by Watts and Alvarado [21] on the influence of future markets on price stability, and Nutaro and Protopopescu [22] on the impact of market clearing time and price signal delay on power market stability. The model adopted in this paper differs from those of [19], [20], [21], and [22] in that we analyze the global properties of the full nonlinear model as opposed to the first-order linear differential equations examined in these papers. In addition, the price updates in our paper occur at discrete time intervals, and are the outcome of marginal cost pricing in the wholesale market by an Independent System Operator (ISO), which is consistent with the current practice in deregulated electricity markets. Furthermore, beyond stability, we are interested in providing a characterization of the impacts of uncertainty in consumer behavior on price volatility and the system's robustness to uncertainties.

II. Preliminaries

A. Notation

The set of positive real numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$), and nonnegative real numbers (integers) by $\overline{\mathbb{R}}_+$ ($\overline{\mathbb{Z}}_+$). The class of real-valued functions with a continuous n-th derivative on $X \subset \mathbb{R}$ is denoted by $\mathcal{C}^n X$. For a vector $v \in \mathbb{R}^l$, v_k denotes the k-th element of v, and $\|v\|_p$ denotes the standard p-norm: $\|v\|_p \overset{\text{def}}{=} \left(\sum_{i=1}^l |v_i|^p\right)^{1/p}$. Also, we will use $\|v\|$ to denote any p-norm when there is no ambiguity. For a differentiable function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, we use $\dot{f}$ to denote the derivative of f with respect to its argument: $\dot{f}(x) = df(x)/dx$. Finally, for a measurable set $X \subset \mathbb{R}$, $\mu_L(X)$ is the Lebesgue measure of X.

B. Basic Definitions

Definition 1: **Scaled Incremental Mean Volatility (IMV):** Given a signal $h : \mathbb{Z} \mapsto \mathbb{R}^l$, and a function $\rho : \mathbb{R}^l \mapsto \mathbb{R}^m$, the *$\rho$-scaled incremental mean volatility* measure of $h(\cdot)$ is defined as

$$\overline{\mathcal{V}}_\rho(h) = \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \|\rho(h(t+1)) - \rho(h(t))\| \tag{1}$$

where, to simplify the notation, the dependence of the measure on the norm used in (1) is dropped from the notation $\overline{\mathcal{V}}_\rho(h)$.

To quantify volatility for fast-decaying signals with zero IMV, e.g., state variables of a stable autonomous system, we will use the notion of scaled *aggregate* volatility, defined as follows.

Definition 2: **Scaled Incremental Aggregate Volatility (IAV):** Given a signal $h : \mathbb{Z} \mapsto \mathbb{R}^l$, and a function $\rho : \mathbb{R}^l \mapsto \mathbb{R}^m$, the *$\rho$-scaled incremental aggregate volatility* measure of $h(\cdot)$ is defined as

$$\mathcal{V}_\rho(h) = \sum_{t=0}^{\infty} \|\rho(h(t+1)) - \rho(h(t))\|. \tag{2}$$

In particular, we will be interested in the log-scaled incremental volatility as a metric for quantifying volatility of price, supply, or demand in electricity markets.

Remark 1: The notions of incremental volatility presented in Definitions 1 and 2 accentuate the fast time scale, i.e., high frequency characteristics of the signal of interest. Roughly speaking, the scaled IMV or IAV are measures of the mean deviations of the signal from its *moving average*. In contrast, sample variance or CV (coefficient of variation, i.e., the ratio of standard deviation to mean) provide a measure of the mean deviations of the signal from its average, without necessarily emphasizing the high-frequency characteristics. Since we are interested in studying the fast dynamics of spot prices and supply/demand in electricity markets from a reliability perspective, the scaled IMV and IAV as defined above are appropriate metrics for volatility.

The notion of stability used in this paper is the standard notion of asymptotic stability and it applies to both price and quantity.

Definition 3: Consider the system

$$x(t+1) = \psi(x(t)) \tag{3}$$

where $\psi(\cdot)$ is an arbitrary map from a domain $X \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The equilibrium $\bar{x} \in X$ of (3) is *stable in the sense of Lyapunov* if all trajectories that start sufficiently close to $\bar{x}$ remain arbitrarily close to it, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0) - \bar{x}\| < \delta \Rightarrow \|x(t) - \bar{x}\| < \varepsilon, \quad \forall t \geq 0$$

The equilibrium is *globally asymptotically stable* if it is Lyapunov stable, and for all $x(0) \in X$ we have: $\lim_{t\to\infty} x(t) = \bar{x}$.

C. Market Structure

We begin with developing an electricity market model with three participants: 1. The suppliers, 2. The consumers, and 3. An Independent System Operator (ISO). The suppliers and the consumers are price-taking, profit-maximizing agents. The ISO is an independent, profit-neutral player in charge of clearing the market, that is, matching supply and demand subject to the network constraints with the objective of maximizing the social welfare. The details are as follows.

1) The Consumers and the Producers: Let D and S denote the sets of consumers and producers respectively. Each consumer $j \in D$ is associated with a value function $v_j : \overline{\mathbb{R}}_+ \mapsto \mathbb{R}$, where $v_j(x)$ can be thought of as the monetary value that consumer j derives from consuming x units of the resource, electricity in this case. Similarly, each producer $i \in S$, is associated with a production cost function $c_i : \overline{\mathbb{R}}_+ \mapsto \overline{\mathbb{R}}_+$.

Assumption I: For all $i \in S$, the cost functions $c_i(\cdot)$ are in $\mathcal{C}^2(0, \infty)$, strictly increasing, and strictly convex. For all $j \in D$, the value functions $v_j(\cdot)$ are in $\mathcal{C}^2(0, \infty)$, strictly increasing, and strictly concave.

Let $d_j : \mathbb{R}_+ \mapsto \overline{\mathbb{R}}_+$, $j \in D$, and $s_i : \mathbb{R}_+ \mapsto \overline{\mathbb{R}}_+$, $i \in S$ be demand and supply functions mapping price to consumption and production, respectively. In this paper, the producers and consumers are price-taking, utility-maximizing agents. Therefore, letting λ be the price per unit of electricity, we have

$$\begin{aligned} d_j(\lambda) &= \arg\max_{x \in \overline{\mathbb{R}}_+} \; v_j(x) - \lambda x, \qquad j \in D, \\ &= \max\{0, \{x \mid \dot{v}_j(x) = \lambda\}\} \end{aligned} \tag{4}$$

and

$$\begin{aligned} s_i(\lambda) &= \arg\max_{x \in \overline{\mathbb{R}}_+} \; \lambda x - c_i(x), \qquad i \in S. \\ &= \max\{0, \{x \mid \dot{c}_i(x) = \lambda\}\} \end{aligned} \tag{5}$$

For convenience in notation and in order to avoid cumbersome technicalities, we will assume in the remainder of this paper that $d_j(\lambda) = \dot{v}_j^{-1}(\lambda)$ is the demand function, and $s_i(\lambda) = \dot{c}_i^{-1}(\lambda)$ is the supply function. This can be mathematically justified by assuming that $\dot{v}(0) = \infty$, and $\dot{c}(0) = 0$, or that $\lambda \in [\dot{c}(0), \dot{v}(0)]$.

a) Consumers with Uncertain Value Functions: We will consider two models of uncertainty in consumer behavior.

– Multiplicative Perturbation Model: The uncertainty in consumer's value function is modeled as

$$\tilde{v}_j(x,t) = \alpha_j(t)\, v_o\left(\frac{x}{\alpha_j(t)}\right), \qquad j \in D, \tag{6}$$

where $v_o : \overline{\mathbb{R}}_+ \mapsto \mathbb{R}$ is a nominal value function and $\alpha_j : \overline{\mathbb{Z}}_+ \mapsto \mathbb{R}_+$ is an exogenous signal or disturbance. Given a price $\lambda(t) > 0$, under the multiplicative perturbation model (6) we have

$$d_j(\lambda, t) = \alpha_j(t)\, \dot{v}_o^{-1}(\lambda(t)) \tag{7}$$

Thus, the same price λ may induce different consumptions at different times, depending on the type and composition of the load.

– Additive Perturbation Model: The uncertainty in consumer's value function is modeled as

$$\tilde{v}_j(x,t) = v_o(x - u_j(t)), \qquad j \in D, \tag{8}$$

where $u_j : \overline{\mathbb{Z}}_+ \mapsto \mathbb{R}_+$ is exogenous. Thus, given a price $\lambda(t) > 0$, under the additive perturbation model (8), the demand function is

$$d_j(\lambda, t) = u_j(t) + \dot{v}_o^{-1}(\lambda(t)) \tag{9}$$

– Aggregation of Several Consumers: The aggregate response of several consumers (or producers) to a price signal may be modeled as the response of a single representative agent [23], although explicit formula for the utility of the representative agent may sometimes be too complicated to find [24], [23]. For the case of N identical consumers with value functions $v_j = v_o, \;\; j \in D$, it can be verified that the aggregate demand is equivalent to the demand of a representative consumer with value function [24]:

$$v(x) = N v_o\left(\frac{x}{N}\right) \tag{10}$$

Suppose now, that the consumer behavior can be modeled via (6)–(7). Let

$$\bar{\alpha}(t) = \sum_{j=1}^{N} \alpha_j(t),$$

and suppose that there exists a nominal value $\bar{\alpha}_0$, such that

$$\bar{\alpha}(t) = \bar{\alpha}_0 + \Delta\bar{\alpha}(t) = \bar{\alpha}_0(1 + \delta(t))$$

where $\delta(t) = \Delta\bar{\alpha}(t)/\bar{\alpha}_0$ satisfies $|\delta(t)| < 1$. Define $v(x) = \bar{\alpha}_0 v_o(x/\bar{\alpha}_0)$. It can be then verified that the aggregate demand can be modeled as the response of a representative agent with value function

$$\begin{aligned} \tilde{v}(x,t) &= \bar{\alpha}(t)\, v_o\left(\frac{x}{\bar{\alpha}(t)}\right) \\ &= (\bar{\alpha}_0 + \Delta\bar{\alpha}(t))\, v_o\left(\frac{x}{\bar{\alpha}_0 + \Delta\bar{\alpha}(t)}\right) \\ &= (1 + \delta(t))\, v\left(\frac{x}{1 + \delta(t)}\right) \end{aligned} \tag{11}$$

The aggregate response is then given by

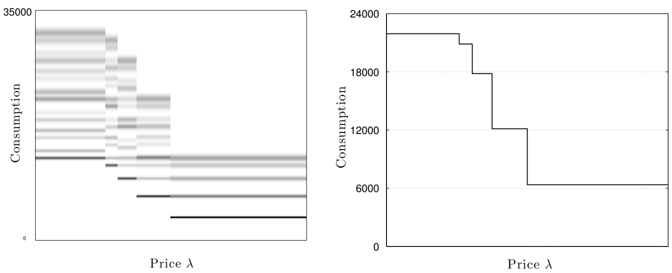
$$d(\lambda(t), t) = (1 + \delta(t))\, \dot{v}^{-1}(\lambda(t)). \tag{12}$$

Similarly, under the additive perturbation model the aggregate behavior can be represented by

$$\tilde{v}(x,t) = v(x - u(t)) \tag{13}$$

$$d(\lambda(t), t) = u(t) + \dot{v}^{-1}(\lambda(t)) \tag{14}$$

where $v(\cdot)$ is given by (10) and $u(t) = \sum u_j(t)$. The interpretation of (13) and (14) is that at any given time t, the demand comprises of an inelastic component $u(t)$ which is exogenous, and an elastic component $\dot{v}^{-1}(\lambda(t))$. Another interpretation is that $\dot{v}^{-1}(\lambda(t))$ represents the demand of those consumers who are subject to real-time pricing, and $u(t)$ represents the demand of the non-participating consumers. More importantly, the inelastic component may include contributions from distributed generators owned by the consumers, and thus, it may be subject to high variability and high



(a) (b)

Fig. 1. Aggregate consumption under load-shifting [7] (a) Variation of consumption as the price history varies. Darker regions indicate more frequent presence of the price-consumption function out of $100,000$ price history points. (b) A sample price-consumption function at an arbitrary time

uncertainty, making it more challenging to predict the total demand as a function of price.

Remark 2: The concave utility maximizing agent models have been used in many engineering and economic contexts. Recent results [7] on emerging aggregate behaviors from optimal load-shifting by individual consumers give theoretical justification to adoption of this model in the context of electricity consumers. In this model, individual consumers defer their flexible demands up to a deadline in order to minimize their total cost of consumption in the presence of time-varying exogenous prices. While the demand of an individual consumer may not be a monotonic function of price (because time may take priority over price), the aggregate consumption of a large population at each instance of time becomes a non-increasing function of price. However, as shown in Figure 1 this function is not a static function and it depends on both materialization of random exogenous demands and history of prices. More accurately, consumption is a function of the state (backlogged demand) which evolves dynamically in time. Interested readers may consult [7] for more information. In this paper, we abstract away the state and adopt a time-varying memoryless concave utility maximizing model for the consumer. More discussion will follow in Section V-C.

2) The Independent System Operator (ISO): The ISO is a non-for-profit entity whose primary function is to optimally match supply and demand subject to network and operational constraints. The constraints include power flow constraints, transmission line and generator capacity constraints, local and system-wide reserve capacity requirements, and possibly some other constraints specific to the ISO [25], [26], [27]. For real-time market operation, the constraints are linearized near the steady-state operating point and the ISO optimization problem is reduced to a convex—typically linear—optimization problem often referred to as the *Economic Dispatch Problem (EDP), or the Optimal Power Flow Problem*. A set of Locational Marginal Prices (LMP) emerge as the shadow cost of the nodal power balance constraints. These prices vary from location to location as they represent the marginal cost of supplying electricity to a particular location. We refer the interested reader to [28], [27], [26], and [29] for more details. However, we emphasize that the spatial variation in the LMPs is a consequence of congestion in the transmission lines. When there is sufficient transmission capacity in the system, a uniform price will materialize for the entire network. With this observation in sight, we make the following assumptions:

1) Resistive losses are negligible.
2) The line capacities are high enough, (i.e., no congestion)
3) There are no generator capacity constraints.
4) The discrepancy between forecast load and actual load is resolved through reserve generation/demand capacity, and the ex-ante energy price is used to settle the discrepancy between forecast load and actual load.

Under the first two assumptions, the network parameters become irrelevant in the supply-demand optimal matching problem. The third and fourth assumptions are made in the interest of keeping the development in this paper simple and focused. They could, otherwise, be relaxed at the expense of a somewhat more involved technical analysis. A thorough investigation of the effects of network constraints and reserve capacity markets, whether they are stabilizing or destabilizing, does not fall within the scope of this paper. The interested readers may consult [20], [30], [29] for an analysis of dynamic pricing in electricity networks with transmission line and generator capacity constraints.

a) Real-Time System Operation and Market Clearing: Consider the case of real-time market operation and assume that price-sensitive retail consumers do not bid in the real-time market. In other words, they do not provide their value functions to the system operator (or any intermediary entity in charge of real-time pricing). Though, they may adjust their consumption in response to a price signal, which is assumed in this paper, to be the ex-ante wholesale market clearing price. In this case, the demand is assumed to be inelastic over each *short* pricing interval, and supply is matched to demand. Thus, the ISO's problem reduces to meeting the fixed demand at minimum cost:

$$\begin{aligned} \min \quad & \sum_{i \in S} c_i(s_i) \\ \text{s.t.} \quad & \sum_{i \in S} s_i = \sum_{j \in D} \hat{d}_j \end{aligned} \tag{15}$$

where $\hat{d}_j$ is the predicted demand of consumer j for the next time period. We assume that the system operator solves (15) and sets the price to the marginal cost of production at the minimum cost solution. As stated in the assumptions, we do not include reserve capacity in our model and instead, assume that the ex-ante energy price is used to settle the discrepancy between the forecast load and the actual load that materializes. More details are presented in the next Section.

As we will see in the sequel, the prediction step in the ISO's optimization problem and the discrepancy between the scheduled generation and the materialized demand lead to a variant of cobweb-like dynamics [31], which in general describes cyclical fluctuations of supply and demand in markets where the quantity to produce must be decided before prices are revealed. In the model developed in this paper, the price associated with the supplied quantity must be decided before

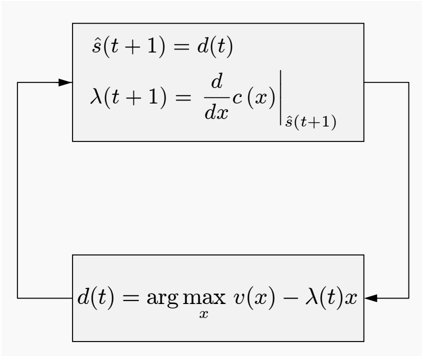


Fig. 2. Ex-ante Priced Supply/Demand Feedback

demand (which is equal to the supplied quantity) is revealed. We will see in Section IV that qualitatively similar fluctuations arise in this case. The higher the relative sensitivity of demand to supply, the more difficult it is to correctly predict the demand at the next time step, and tame the fluctuations.

III. Dynamic Models of Supply-Demand under Real-time Pricing

In this section, we develop dynamical system models for the interaction of wholesale supply and retail demand. These models are based on the current practice of marginal cost pricing in most wholesale electricity markets, with the additional feature that the retail consumers adjust their usages in response to the real-time wholesale market prices. We assume that the consumers do not bid in the market, i.e., they do not provide their value/demand functions to the ISO. The real-time market is cleared at discrete time intervals and the prices are calculated and announced for each interval. The practice of defining the clearing price corresponding to each pricing interval based on the predicted demand at the beginning of that interval is called ex-ante pricing. As opposed to this, ex-post pricing refers to the practice of defining the clearing price for each pricing interval based on the materialized consumption at the end of the interval. Here, we only present the dynamics of the ex-ante case. Under similar assumptions, the dynamics and the results for the ex-post case are analogous and can be found in [32], [24]. It is also possible to consider dynamic models arising from ex-ante pricing complemented with ex-post adjustments, see for instance [33].

A. Price Dynamics under Ex-ante Pricing

Let $\lambda(t)$ denote the ex-ante price corresponding to the consumption of one unit of energy in the time interval $[t, t+1]$. Let $d(t) = \sum_{j \in D} d_j(t)$ be the actual aggregate consumption during this interval:

$$d(t) = \sum_{j \in D} d_j(t) = \sum_{j \in D} v_j^{-1}(\lambda(t)). \tag{16}$$

Since $v_j(\cdot)$ is known only to consumer j, at time t, only an estimate of $d(t)$ is available to the ISO, based on which, the price $\lambda(t)$ is calculated. The price $\lambda(t)$ is therefore, the marginal cost of predicted supply that matches the predicted demand for the time interval $[t, t+1]$. We assume that the predicted demand/supply for each time interval is based on the actual demand at the previous intervals: $\hat{s}_{t+1} = \hat{d}(t+1) = \phi(d(t), \cdots, d(t-T))$, $T \in \mathbb{Z}$. The following equations describe the dynamics of the market:

$$\sum_{i \in S} \dot{c}_i^{-1}(\lambda(t+1)) = \hat{s}(t+1) = \hat{d}(t+1) \tag{17}$$

$$\hat{d}(t+1) = \phi(d(t), \cdots, d(t-T)) \tag{18}$$

$$\sum_{j \in D} \dot{v}_j^{-1}(\lambda(t-k)) = d(t-k), \qquad \forall k \leq T \tag{19}$$

where (19) follows from (16), and $\lambda(t+1)$ in (17) is the Lagrangian multiplier associated with the balance constraint in optimization problem (15) solved at time $t+1$, i.e., with $\sum_{j \in D} \hat{d}_j = \hat{d}(t+1)$.

The prediction step (18) may be carried through by resorting to linear auto-regressive models, in which case, we will have:

$$\phi(d(t), \cdots, d(t-T)) = \sum_{k=0}^{T} \alpha_k d(t-k), \qquad \alpha_k \in \mathbb{R}. \tag{20}$$

When $\phi(\cdot)$ is of the form (20), equations (17)−(19) result in:

$$\sum_{i \in S} \dot{c}_i^{-1}(\lambda(t+1)) = \sum_{k=0}^{T} \alpha_k \sum_{j \in D} \dot{v}_j^{-1}(\lambda(t-k)) \tag{21}$$

A special case of (18) is the so called *persistence model* which corresponds to the case where the predicted demand for the next time step is assumed to be equal to the demand at the previous time step, i.e., $\phi(d(t), \cdots, d(t-T)) = d(t)$. In this case, equations (17)−(19) result in:

$$\sum_{i \in S} \dot{c}_i^{-1}(\lambda(t+1)) = \sum_{j \in D} \dot{v}_j^{-1}(\lambda(t)). \tag{22}$$

If all the producers can be aggregated into one representative producer with a convex cost function $c(\cdot)$, and all the consumers can be aggregated into one representative consumer with a concave value function $v(\cdot)$, then (21) and (22) reduce, respectively, to:

$$\lambda(t+1) = \dot{c}\left(\sum_{k=0}^{T} \alpha_k \dot{v}^{-1}(\lambda(t-k))\right) \tag{23}$$

and

$$\lambda(t+1) = \dot{c}\left(\dot{v}^{-1}(\lambda(t))\right). \tag{24}$$

B. Demand Dynamics

We could alternatively write dynamical system equations for the evolution of demand. Under ex-ante pricing we will have:

$$\dot{v}_j(d_j(t+1)) = \dot{c}_i(s_i(t)) \quad \forall i \in S,\ j \in D \tag{25}$$

$$\sum_{i \in S} s_i(t) = \phi\left(\sum_{j \in D} d_j(t), \cdots, \sum_{j \in D} d_j(t-T)\right). \tag{26}$$

Assuming representative agent models, (25)−(26) reduce to

$$\dot{v}(d(t+1)) = \dot{c}(\phi(d(t), \cdots, d(t-T))) \tag{27}$$

Finally, under the persistence model for prediction we have:

$$\dot{v}(d(t+1)) = \dot{c}(d(t)) \tag{28}$$

In what follows, we will develop a theoretical framework that is suitable for analysis of dynamical systems described by implicit equations. Such systems arise in many applications which incorporate real-time optimization in a feedback loop, several instances of which were developed in this section. As we will see, this framework is useful for studying the dynamics of electricity markets, robustness to disturbances, and price volatility under real-time pricing.

IV. Theoretical Framework

A. Stability Analysis

In this section, we present several stability criteria based on Lyapunov theory [9] and contraction analysis [10], and examine stability properties of the clearing price dynamics formulated in Section III.

Theorem 1: Let $\mathcal{S}$ be a discrete-time dynamical system described by the state-space equation

$$\begin{aligned} \mathcal{S} \quad : \quad x(t+1) &= \psi(x(t)) \\ x_0 &\in X_0 \subset \mathbb{R}_+ \end{aligned} \tag{29}$$

for some function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Then, $\mathcal{S}$ is stable if there exists a pair of continuously differentiable functions $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying

$$g(x(t+1)) = f(x(t)) \tag{30}$$

and

$$\text{(i)}: \quad \theta^* = \inf\left\{\theta \mid \left|\dot{f}(x)\right| \le \theta\,|\dot{g}(x)|, \quad \forall x\right\} \le 1 \tag{31}$$

$$\text{(ii)}: \quad \mu_L(\{x \mid \dot{f}(x) = \dot{g}(x)\}) = 0 \tag{32}$$

and either:

$$\text{(iii)}: \quad \dot{g}(x) \ge 0,\ \forall x,\ \text{and}\ \lim_{x\to\infty}\{f(x) - g(x)\} < 0 \tag{33}$$

or

$$\text{(iii)}': \quad \dot{g}(x) \le 0,\ \forall x,\ \text{and}\ \lim_{x\to\infty}\{f(x) - g(x)\} > 0 \tag{34}$$

Proof: See the Appendix. ■

Remark 3: The monotonicity conditions (33) – (34) in Theorem 1 can be relaxed at the expense of more involved technicalities in both the statement of the theorem and its proof. As we will see, these assumptions are naturally satisfied in applications of interest to this paper. Therefore, we will not bother with the technicalities of removing the condition.

There are situations in which a natural decomposition of discrete-time dynamical systems via functions f and g satisfying (30) is readily available. This is often the case for applications that involve optimization in a feedback loop. For instance, for the price dynamics (24), we have $\psi = \dot{c} \circ \dot{v}^{-1}$, and the decomposition is obtained with $g = \dot{c}^{-1}$, and $f = \dot{v}^{-1}$. However, f and g obtained in this way may not readily satisfy (31). We present the following corollaries.

Corollary 1: Consider the system (29) and suppose that continuously differentiable functions $f, g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying (30) and (32)−(34) are given. Then, the system is stable if there exist $\theta \le 1$ and a strictly monotonic, continuously differentiable function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying

$$\left|\dot{\rho}(f(x))\,\dot{f}(x)\right| \le \theta\,|\dot{\rho}(g(x))\,\dot{g}(x)|,$$

for all $x \in \mathbb{R}_+$.

Proof: If f and g satisfy (30), then so do $\rho \circ f$ and $\rho \circ g$ for any $\rho \in \mathcal{C}^1(0, \infty)$. Furthermore, under the assumptions of the corollary, both $\rho \circ f$ and $\rho \circ g$ satisfy (31)−(34). The result then follows from Theorem 1. ■

Corollary 2: **Market Stability I:** The system (24) is stable if there exists a strictly monotonic, continuously differentiable function $\rho : \mathbb{R}_+ \mapsto \mathbb{R}$ satisfying

$$\left|\dot{\rho}\left(\dot{v}^{-1}(\lambda)\right)\frac{\partial \dot{v}^{-1}(\lambda)}{\partial \lambda}\right| \le \theta \left|\dot{\rho}\left(\dot{c}^{-1}(\lambda)\right)\frac{\partial \dot{c}^{-1}(\lambda)}{\partial \lambda}\right| \tag{35}$$

for all $\lambda \in \mathbb{R}_+$.

Similarly, the system (28) is stable if

$$|\dot{\rho}(\dot{c}(x))\,\ddot{c}(x)| \le \theta\,|\dot{\rho}(\dot{v}(x))\,\ddot{v}(x)| \tag{36}$$

for all $x \in \mathbb{R}_+$.

Proof: The statements follow from Corollary 1 with $f = \dot{v}^{-1}$ and $g = \dot{c}^{-1}$ for (35), and $f = \dot{c}$ and $g = \dot{v}$ for (36), and the fact that under Assumption I, all of the conditions required in Corollary 1 are satisfied. ■

Elasticity is often defined as a measure of how one variable responds to a change in another variable. In particular, *price-elasticity of demand* is defined as the percentage change in the quantity demanded, resulting from one percentage change in the price, and is viewed as a measure of responsiveness, or sensitivity of demand to variations in the price. *Price-elasticity of supply* is defined analogously. Herein, we generalize the standard definitions of elasticity as follows.

Definition 4: **Generalized Elasticity:** The quantity

$$\epsilon_D^{\mathrm{p}}(\lambda, l) = \left(\frac{\lambda}{\dot{v}^{-1}(\lambda)}\right)^l \frac{\partial \dot{v}^{-1}(\lambda)}{\partial \lambda}, \quad l \ge 0$$

is the generalized price-elasticity of demand at price λ. Similarly,

$$\epsilon_S^{\mathrm{p}}(\lambda, l) = \left(\frac{\lambda}{\dot{c}^{-1}(\lambda)}\right)^l \frac{\partial \dot{c}^{-1}(\lambda)}{\partial \lambda}, \quad l \ge 0$$

is the generalized price-elasticity of supply at price λ. Note that these notions depend on the exponent l. For $l = 1$, we obtain the standard notions of elasticity. We define the *market's relative generalized price-elasticity* as the ratio of the generalized price-elasticities:

$$\epsilon_{\mathrm{rel}}^{\mathrm{p}}(\lambda, l) = \frac{\epsilon_D^{\mathrm{p}}(\lambda, l)}{\epsilon_S^{\mathrm{p}}(\lambda, l)}. \tag{37}$$

The *market's maximal relative price-elasticity* (MRPE) is defined as

$$\theta^*(l) = \sup_{\lambda \in \mathbb{R}_+} \left|\epsilon_{\mathrm{rel}}^{\mathrm{p}}(\lambda, l)\right|. \tag{38}$$

The notions of generalized demand-elasticity of price and generalized supply-elasticity of price are defined analogously:

$$\epsilon_p^{\mathrm{d}}(x,l) = x^l \frac{\ddot{v}(x)}{\dot{v}(x)^l}, \quad \epsilon_p^{\mathrm{s}}(x) = x^l \frac{\ddot{c}(x)}{\dot{c}(x)^l}$$

When $l = 1$, these notions coincide with the Arrow-Pratt coefficient of Risk Aversion (RA) [34], [35], and we will adopt the same terminology in this paper. The market's *relative generalized risk aversion factor* is defined as:

$$\epsilon_{\mathrm{rel}}^{\mathrm{RA}}(x,l) = \frac{\epsilon_p^{\mathrm{s}}(x,l)}{\epsilon_p^{\mathrm{d}}(x,l)} = \frac{\ddot{c}(x)}{\ddot{v}(x)} \left(\frac{\dot{v}(x)}{\dot{c}(x)}\right)^l$$

Finally, the *market's maximal relative risk-aversion* (MRRA) is defined as

$$\eta^*(l) = \sup_{x \in \mathbb{R}_+} \left|\epsilon_{\mathrm{rel}}^{\mathrm{RA}}(x,l)\right|. \tag{39}$$

With a slight abuse of notation, when $l = 1$, we write $\epsilon_D^{\mathrm{p}}(\lambda)$ instead of $\epsilon_D^{\mathrm{p}}(\lambda, 1)$, and θ^* instead of $\theta^*(1)$, etc.

The following corollary relates stability to the market's relative price-elasticity $\epsilon_{\mathrm{rel}}^{\mathrm{p}}(\lambda, l)$, and relative risk-aversion $\epsilon_{\mathrm{rel}}^{\mathrm{RA}}(x,l)$.

Corollary 3: **Market Stability II:** The system (22) is stable if the market's MRPE is less than one for some $l \geq 0$, that is:

$$\exists l \geq 0: \quad \theta^*(l) = \sup_{\lambda} \left|\epsilon_{\mathrm{rel}}^{\mathrm{p}}(\lambda,l)\right| < 1 \tag{40}$$

The system (28) is stable if the market's MRRA is less than one for some $k \geq 0$, that is:

$$\exists l \geq 0: \quad \eta^*(l) = \sup_{x} \left|\epsilon_{\mathrm{rel}}^{\mathrm{RA}}(x,l)\right| < 1 \tag{41}$$

Proof: The results are obtained by applying Corollary 2, criteria (35) and (36), with $\rho(z) = \log(z)$ for $l = 1$, and $\rho(z) = z^{-l+1}$ for $l \neq 1$. ■

When the cost and value functions are explicitly available, conditions (36) or (41) are more convenient to check, whereas, when explicit expressions are available for the supply and demand functions, it is more convenient to work with (35) or (40).

Example 1: Consider (24) with $c(x) = x^\beta$, and $v(x) = (x-u)^{1/\alpha}$, where $\alpha, \beta > 1$ and $u \geq 0$ is a constant. First, consider the $u = 0$ case. Then, we have

$$\lambda(t+1) = \beta(\alpha\lambda(t))^{\frac{\alpha\beta-\alpha}{1-\alpha}}$$

$$\dot{v}(x) = \alpha^{-1} x^{\frac{1-\alpha}{\alpha}}, \ \ddot{v}(x) = (1-\alpha)\alpha^{-2} x^{\frac{1-2\alpha}{\alpha}}$$

$$\dot{c}(x) = \beta x^{\beta-1}, \ \ddot{c}(x) = \beta(\beta-1) x^{\beta-2}$$

It can be verified that $\theta^*(0) = \infty$. However, by invoking (41) with $k = 1$, we have:

$$\eta^* = \frac{|\ddot{c}(x)|}{|\ddot{v}(x)|} \frac{|\dot{v}(x)|}{|\dot{c}(x)|} = \frac{(\beta-1)}{(\alpha-1)\alpha^{-1}} < 1$$

Hence, the system is stable if

$$\beta < 2 - \alpha^{-1}$$

It can be shown that the condition is also necessary and the system diverges for $\beta > 2 - \alpha^{-1}$. Moreover, invoking (40) with $l = 1$ yields exactly the same result, though, this need not be the case in general. Consider now the same system with $\alpha = \beta = 2$ and $u > 0$. Simulations show that the system is not stable in the asymptotic sense for $u < 1/4$. The following table summarizes the results of our analysis.

Table I

	$u = 0.25$	$u = 0.3$	$u = 0.5$
$\theta^*(1) =$	2	2	2
$\theta^*(1.5) =$	1	0.872	0.595
$\theta^*(2) =$	1.299	0.988	0.459

Thus, when $u = 1/4$, the system is at least marginally stable. Furthermore, the above analysis highlights the importance of the notion of generalized elasticity (cf. Definition 4), as $\theta^*(1)$, which is associated with the standard notion of price elasticity, can be greater than one while the system is stable and it's stability can be proven using the MRPE for some $l \geq 0$.

Remark 4: The main stability results (Theorem 1, and Corollaries 1, 2, and 3) provide criteria for analysis of nonlinear discrete time systems. These results can be readily applied to analysis of discrete-time systems described by implicit equations which arise from optimization in feedback loop in general, and to systems with cobweb-like dynamics in particular. The notions of "Generalized Elasticity" and "Generalized Risk Aversion" and the associated global stability criteria (i.e., Corollary 3) are new contributions to this domain, and more generally, to nonlinear systems analysis. In Section IV-B we extend these results to systems with autoregressive prediction, while in Section IV-C we go beyond stability by providing a characterization of volatility induced by uncertainty in the consumer's response. Relating the notions of generalized elasticity and generalized risk aversion to incremental volatility is a contribution of this paper.

Remark 5: We assumed that the network has high enough capacity and thereby removed congestion constraints to obtain a uniform price across the network. Numerical simulations of a DC power flow model with congestion constraints reported in [29] suggest that the qualitative behavior does not change significantly when transmission constraints and power flow equations are fully considered. For systems with relatively homogeneous consumers and producers, it appears that volatility is determined mostly by the relative price elasticity of consumers to the producers, as opposed to the network parameters. The network parameters become more important as heterogeneity of agents in the network increases.

B. Invariance Analysis

The analysis in the preceding sections is based on applying the results of Theorem 1 and Corollary 3 to systems of the form (22) (or (28)), which correspond to the persistence prediction model, whether it is demand prediction by the ISO in the ex-ante pricing case, or price prediction by the consumers in the ex-post pricing case. When functions of the form (18) are used for prediction of price or demand, the underlying dynamical system is not a scalar system. An immediate extension of Theorem 1 in its full generality to the multidimensional case, while possible, requires additional technicalities in both the proof and the application of the theorem. In what follows we take the middle way: we present a theorem that exploits the structure of the dynamical system that arises from autoregressive prediction models to both make the extension possible and to simplify the analysis.

Theorem 2: Let $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}$, be a real-valued sequence satisfying a state-space equation of the form:

$$g\left(x\left(t+1\right)\right) = f\left(x\left(t\right), x\left(t-1\right), \cdots, x\left(t-n\right)\right) \quad (42)$$

$$\left(x\left(0\right), ..., x\left(n\right)\right) \in X_0 \subset \mathbb{R}^{n+1},$$

for some continuously differentiable function $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}$, and a continuously differentiable monotonic function $g : \mathbb{R} \mapsto \mathbb{R}$, which satisfy

$$\left|\frac{\partial}{\partial y_k} f\left(y\right)\right| \leq \theta_k \left|\dot{g}\left(y_k\right)\right|, \quad \forall y \in \mathbb{R}^{n+1} \quad (43)$$

where

$$\sum_{k=1}^{n} \theta_k \leq 1 \quad (44)$$

Then, there exists a constant $\gamma_0 \geq 0$, which depends only on the first $n+1$ initial states $x\left(n\right), ..., x\left(0\right)$, such that the set

$$\Omega_0 = \left\{x \in \mathbb{R} \mid \exists z \in \mathbb{R}^n : \left|g\left(x\right) - f\left(x, z\right)\right| \leq \gamma_0\right\} \quad (45)$$

is invariant under (42), i.e.,

$$x\left(T-n\right), ..., x\left(T\right) \in \Omega_0 \Rightarrow x\left(t\right) \in \Omega_0, \ \forall t > T$$

Furthermore, if (44) holds with strict inequality, then the g-scaled IAV of x is bounded from above:

$$\mathcal{V}_g\left(x\right) = \sum_{t=1}^{\infty} \left|g\left(x\left(t+1\right)\right) - g\left(x\left(t\right)\right)\right| \leq \frac{\gamma_0}{1 - \sum_{k=1}^{n} \theta_k} \quad (46)$$

Proof: See the Appendix. ■

It follows from the proof of Theorem 2 that when the initial conditions are close to the equilibrium of (42), it is sufficient to satisfy conditions (43)–(44) only locally, over a properly defined subset of $\mathbb{R}^{n+1}$. This is summarized in the following corollary.

Corollary 4: Let $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}$, be a real-valued sequence satisfying (42), where f and g are continuously differentiable functions. Let

$$\widetilde{\Omega}_0 = \left\{\left(x, z\right) \in \mathbb{R} \times \mathbb{R}^n : \left|g\left(x\right) - f\left(x, z\right)\right| \leq \gamma_0\right\}$$

where γ_0 is given in (74)–(75). If

$$\left|\frac{\partial}{\partial y_k} f\left(y\right)\right| \leq \theta_k \left|\dot{g}\left(y_k\right)\right|, \quad \forall y \in \widetilde{\Omega}_0$$

where θ_k's satisfy (44), then $\widetilde{\Omega}_0$ is invariant under (42). Furthermore, when (44) holds with strict inequality, and the initialization vector $x_0 = \left[x\left(n\right), \ldots, x\left(0\right)\right]$ is an element of $\widetilde{\Omega}_0$, then (46) holds.

Theorem 2 and Corollary 4 can be applied to analysis of market dynamics under the generic autoregressive prediction models that were presented in Section III. The sets Ω_0 or $\widetilde{\Omega}_0$ being invariant implies that the difference between the predicted demand and the actual supply remains bounded.

1) Analysis of Market Dynamics under Linear Autoregressive Prediction Models: Consider the model (23), repeated here for convenience:

$$\dot{c}^{-1}\left(\lambda\left(t+1\right)\right) = \sum\nolimits_{k=0}^{n} \alpha_k \dot{v}^{-1}\left(\lambda\left(t-k\right)\right)$$

We apply Theorem 2 (alternatively Corollary 4) with

$$g\left(\lambda\right) = \rho\left(\dot{c}^{-1}\left(\lambda\right)\right) \quad (47)$$

and

$$f\left(\lambda_t, \ldots, \lambda_{t-n}\right) = \rho\left(\sum\nolimits_{k=0}^{n} \alpha_k \dot{v}^{-1}\left(\lambda_{t-k}\right)\right) \quad (48)$$

We examine (47)–(48) with $\rho\left(z\right) = \log\left(z\right)$ and $\rho\left(z\right) = z^{-l+1}$, $l \neq 1$. Conditions (43)–(44) then imply that the following conditions are sufficient (for some $l \geq 0$):

$$\left|\frac{\alpha_k \left.\dfrac{\partial \dot{v}^{-1}\left(\lambda\right)}{\partial \lambda}\right|_{\lambda=\lambda_{t-k}}}{\left[\sum\nolimits_{j=0}^{n} \alpha_j \dot{v}^{-1}\left(\lambda_{t-j}\right)\right]^l}\right| \leq \theta_k \left|\epsilon_S^{\mathrm{p}}\left(\lambda_{t-k}, l\right)\right| \quad (49)$$

$$\sum_{k=1}^{n} \theta_k \leq 1 \quad (50)$$

Conditions (49)–(50) are complicated and in general demand numerical computation for verification. However, examination of (49) near equilibrium is informative. Suppose that (23) converges to an equilibrium price $\bar{\lambda}$. Letting $\lambda_t = \lambda_{t-1} = \cdots = \lambda_{t-n} = \bar{\lambda}$, we observe that the following condition is implied by (49)–(50):

$$\exists l \geq 0 : \left|\sum\nolimits_{k=1}^{n} a_k\right| \left|\epsilon_D^{\mathrm{p}}\left(\bar{\lambda}, l\right)\right| \leq \left|\epsilon_S^{\mathrm{p}}\left(\bar{\lambda}, l\right)\right| \left|\sum\nolimits_{k=1}^{n} a_k\right|^l \quad (51)$$

where $\epsilon_D^{\mathrm{p}}\left(\bar{\lambda}, l\right)$ and $\epsilon_D^{\mathrm{p}}\left(\bar{\lambda}, l\right)$ are generalized elasticities as defined in Definition 4, evaluated at the equilibrium. It can be shown that (51) is equivalent to $\epsilon_{\mathrm{rel}}^{\mathrm{p}}\left(\bar{\lambda}, 1\right) \leq 1$, independently of l. Furthermore, for a large class of cost and value functions, namely power functions of the form $c\left(x\right) = x^\beta$ and $v\left(x\right) = x^{1/\alpha}$, $\alpha, \beta \geq 1$, the equilibrium relative elasticity $\theta\left(\bar{\lambda}\right) = \epsilon_{\mathrm{rel}}^{\mathrm{p}}\left(\bar{\lambda}, 1\right)$ is independent of the autoregressive coefficients a_k, $k = 1, \ldots, n$. Thus, if the closed-loop market is unstable under the persistent prediction model ($a_1 = 1$, $a_k = 0$, $k \neq 1$), then global stability cannot be verified for any linear autoregressive model of the form (23) using (49)–(50). Although this analysis is based on sufficient criteria, it suggests that it may be difficult to globally stabilize these systems via linear autoregressive prediction. Indeed, extensive simulations show that such models will not globally stabilize an unstable market, unless the MRPE is very close to one. For values of $\theta^* > 1.05$ global stabilization could not be achieved in our simulations. Local stabilization is, however, possible for moderate values of θ^*, namely, $\theta^* \lessapprox 3$.

2) Analysis of Market Dynamics under Exogenous Disturbances: This subsection provides theoretical results that will be used as the basis for volatility analysis in Subsection IV-C.

Theorem 3: Let $x : \overline{\mathbb{Z}}_+ \to \mathbb{R}$ and $u : \overline{\mathbb{Z}}_+ \to \mathbb{R}$ be real-valued sequences which satisfy a state-space equation of the form:

$$g\left(x\left(t+1\right)\right)=f\left(x\left(t\right),u\left(t\right)\right),\quad u\left(t\right)\in U \tag{52}$$
$$x\left(0\right)\in X_0\subset\mathbb{R}$$

for some continuously differentiable function $f:\mathbb{R}^2\to\mathbb{R}$ and a continuously differentiable monotonic function $g:\mathbb{R}\to\mathbb{R}$ satisfying

$$\left|\frac{\partial}{\partial u}f\left(x,u\right)\right|\leq 1,\quad\forall x\in\mathbb{R},\ u\in U \tag{53}$$

and

$$\left|\frac{\partial}{\partial x}f\left(x,u\right)\right|\leq\theta\left|\dot{g}\left(x\right)\right|,\quad\forall x\in\mathbb{R},\ u\in U, \tag{54}$$

where

$$U=\left\{u\in\mathbb{R}:|u|\leq\kappa\right\}$$

and $\kappa\in\left(0,\infty\right)$, and $\theta\in\left[0,1\right)$. Define

$$\zeta_\kappa\left(\theta\right)=\kappa\frac{1+\theta}{1-\theta}. \tag{55}$$

Then, the set

$$\Omega\left(\theta\right)=\left\{x:\Big|\left|f\left(x,\nu\right)-g\left(x\right)\right|-|\nu|\Big|\leq\zeta_\kappa\left(\theta\right),\ \forall\nu\in U\right\} \tag{56}$$

is invariant under (52). Furthermore, the g-scaled IMV of x is bounded from above:

$$\overline{\mathcal{V}}_g\left(x\right)=\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left|g\left(x\left(t+1\right)\right)-g\left(x\left(t\right)\right)\right|\leq\frac{2\kappa}{1-\theta} \tag{57}$$

Proof: See the Appendix. ∎

The following corollary is a local variant of Theorem 3, and is useful for scenarios in which there exists no positive number $\theta<1$ such that (54) is satisfied for all $x\in\mathbb{R}$, whereas it might be possible to satisfy the inequality locally over a subset that contains $\Omega\left(\theta\right)$.

Corollary 5: Let $x:\overline{\mathbb{Z}}_+\to\mathbb{R}$ and $u:\overline{\mathbb{Z}}_+\to\mathbb{R}$ be real-valued sequences satisfying (52). For $\theta<1$, define:

$$\widetilde{\theta}^*=\inf\left\{\widetilde{\theta}:\left|\frac{\partial}{\partial x}f\left(x,u\right)\right|\leq\widetilde{\theta}\left|\frac{\partial}{\partial x}g\left(x\right)\right|,\right.$$
$$\left.\forall x\in\Omega\left(\theta\right),\ u\in U\right\}$$

where $\Omega\left(\theta\right)$ is given in (56). Then $\Omega\left(\widetilde{\theta}^*\right)$ is invariant under (52) if

$$\widetilde{\theta}^*\leq\theta.$$

Furthermore, (57) holds with $\theta=\widetilde{\theta}^*$.

C. Volatility

Consider equation (52) or (42). When the functions g and f are ρ-scaled supply and demand functions, the minimal θ satisfying (54) or (43) will be the MRPE associated with these market models. When g and f are ρ-scaled marginal value and marginal cost functions respectively, the minimal θ satisfying the inequalities will be the associated MRRA. The following corollaries follow from Theorems 2 and 3, and explicitly relate the market's MRPE and MRRA to volatility.

Corollary 6: **Volatility I:** Let $\theta^*<1$ and $\eta^*<1$ be the MRPE and MRRA associated with the market model (52). Then, there exists a constant C, depending on the size of the disturbances only, such that the $\log$-scaled IMV of supply is upperbounded by $C/\left(1-\theta^*\right)$, i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left|\log\left(\dot{c}^{-1}\left(\lambda\left(t+1\right)\right)\right)-\log\left(\dot{c}^{-1}\left(\lambda\left(t\right)\right)\right)\right|\leq\frac{C}{1-\theta^*} \tag{58}$$

And the $\log$-scaled IMV of price is upperbounded by $C/\left(1-\eta^*\right)$, i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\left|\log\left(\lambda\left(t+1\right)\right)-\log\left(\lambda\left(t\right)\right)\right|\leq\frac{C}{1-\eta^*} \tag{59}$$

Corollary 7: **Volatility II:** Let $\theta^*<1$ and $\eta^*<1$ be the MRPE and MRRA associated with the market model (42). Then, there exists a constant C such that the $\log$-scaled IAV of supply is upperbounded by $C/\left(1-\theta^*\right)$, i.e.,

$$\sum_{t=1}^{\infty}\left|\log\left(\dot{c}^{-1}\left(\lambda\left(t+1\right)\right)\right)-\log\left(\dot{c}^{-1}\left(\lambda\left(t\right)\right)\right)\right|\leq\frac{C}{1-\theta^*} \tag{60}$$

And the $\log$-scaled IAV of price is upperbounded by $C/\left(1-\eta^*\right)$, i.e.,

$$\sum_{t=1}^{\infty}\left|\log\left(\lambda\left(t+1\right)\right)-\log\left(\lambda\left(t\right)\right)\right|\leq\frac{C}{1-\eta^*} \tag{61}$$

Remark 6: Generalized versions of the above corollaries can be formulated based on $\theta^*\left(l\right)$ and $\eta^*\left(l\right)$, in which case the scalings of the signals need to be defined accordingly: letting $\rho_l\left(x\right)=x^{-l+1}$ for $l\neq 1$, the ρ_l-scaled IMV of supply and price will be upperbounded by $C/\left(1-\theta^*\left(l\right)\right)$ and $C/\left(1-\eta^*\left(l\right)\right)$ respectively. Furthermore, when the prices remain bounded within an invariant set, e.g., when the conditions of Corollary 4 or Corollary 5 hold, one can replace $\theta^*\left(l\right)$ and $\eta^*\left(l\right)$ with local relative elasticity ratios $\widetilde{\theta}^*\left(l\right)$ and $\widetilde{\eta}^*\left(l\right)$.
In the remainder of this section, we apply Corollaries 6 and 7 to the two time-varying models of consumer behavior (14) and (12).

a) Multiplicative Perturbation: Consider the multiplicative perturbation model (12). Under this model, the market dynamics is given by

$$\dot{c}^{-1}\left(\lambda\left(t+1\right)\right)=\left(1+\frac{1}{2}\delta\left(t\right)\right)\dot{v}^{-1}\left(\lambda\left(t\right)\right),\quad\delta\left(t\right)\in\left[-\kappa,\kappa\right] \tag{62}$$

where the $1/2$ factor in front of $\delta\left(t\right)$ is simply a scaling factor. We invoke Corollary 6 (or Theorem 3) with

$$g\left(\lambda\right)=\log\left(\dot{c}^{-1}\left(\lambda\right)\right) \tag{63}$$

and

$$\begin{aligned}f\left(\lambda,\delta\right)&=\log\left(\left(1+\delta/2\right)\dot{v}^{-1}\left(\lambda\right)\right)\\&=\log\left(1+\delta/2\right)+\log\left(\dot{v}^{-1}\left(\lambda\right)\right).\end{aligned}$$

It can be verified that (53) and (54) are satisfied as long as $\kappa \leq 1$ and $\theta^* < 1$, where θ^* is the MRPE defined in (38). Furthermore, $\zeta_\kappa(\theta^*)$ is the upperbound on the size of the invariant set, where $\zeta_\kappa(\cdot)$ is defined in (55). In particular as $\theta^* \to 1$, small perturbations may induce extremely large fluctuations as measured by $\log$-scaled IMV of supply. The theoretical upperbound is $1/(1-\theta^*)$. When Corollary 5 is applicable, the size of the invariant set can be characterized by $\zeta_\kappa(\widetilde{\theta}^*)$, where $\widetilde{\theta}^*$ is the market's local relative price-elasticity. Furthermore, volatility can be characterized by $\widetilde{\theta}^*$ as well.

b) Additive Perturbation: Under the additive perturbation model (14), the market dynamics can be written as

$$\dot{c}^{-1}(\lambda(t+1)) = u_0 + \frac{1}{2}u(t) + \dot{v}^{-1}(\lambda(t)), \quad u(t) \in [-\kappa, \kappa] \tag{64}$$

where $u_0 \geq 1$ is a shifting factor, and $\kappa \leq u_0$, so that the demand is always at least $u_0/2$. Again, we invoke Corollary 6 (or Theorem 3) with (63) and

$$f(\lambda, u) = \log\left(u_0 + \frac{1}{2}u + \dot{v}^{-1}(\lambda)\right)$$

Then, under the above assumptions, (53) is satisfied. In a similar fashion to previous analyzes, (54) can be related to the MRPE. In this case, the price-elasticity of demand turns out to be:

$$\epsilon_D(\lambda) = \frac{\partial f(\lambda, u)}{\partial \lambda} = \frac{\lambda}{u_0 + u/2 + \dot{v}^{-1}(\lambda)} \frac{\partial \dot{v}^{-1}(\lambda)}{\partial \lambda}$$

The larger the minimum of the inelastic component (i.e., $u_0 - \kappa/2$), the smaller the price-elasticity of the overall demand will be. Under the assumptions made above, there is always a nonzero minimal demand $u_{\min}(t) = u_0/2$. Therefore, it is sufficient to verify (54) over $\lambda \geq \dot{c}(u_0/2)$ instead of all $\lambda > 0$. In conclusion, (54) reduces to:

$$\left|\frac{\partial \dot{v}^{-1}(\lambda)/\partial \lambda}{u_0/2 + \dot{v}^{-1}(\lambda)}\right| \leq \theta \left|\frac{\partial \dot{c}^{-1}(\lambda)/\partial \lambda}{\dot{c}^{-1}(\lambda)}\right|, \quad \forall \lambda \geq \dot{c}(u_0/2) \tag{65}$$

Let $\widetilde{\theta}^*$ be the minimal θ satisfying (65). Similar to the case with multiplicative uncertainty, in this case too, the upperbound on the size of the invariant set is given by $\zeta_\kappa(\widetilde{\theta}^*)$, where $\zeta_\kappa(\cdot)$ is given in (55). Moreover, the $\log$-scaled IMV of supply is upperbounded by $u_0/(1-\widetilde{\theta}^*)$.

The analysis confirms the intuition that participation of a small portion of the population in real-time pricing will not greatly impact the level of volatility in the system, as satisfying (65) for larger values of u_0 is easier. Increased volatility may materialize only when a large portion of the population is exposed to real-time pricing.

V. Discussion

A. Ramp Constraints

Cho and Meyn [36] have investigated the problem of volatility of power markets in a dynamic general equilibrium framework. Their model can be viewed as a full-information model in which the system operator has full information about the cost and value functions of the producers and consumers. Market clearing is instantaneous and supply and demand are matched with no time lag. The producer's problem is, however, subject to supply friction or a ramp constraint, i.e., a finite bound on the rate of change in the supply capacity. It is concluded that efficient equilibria are volatile and volatility is attributed to the supply friction. In the formulation of [36], the consumer's problem is not subject to ramp constraints. In our formulation, neither the consumer's nor the producer's problem is explicitly subject to ramp constraints, yet other factors are shown to contribute to volatility, namely, information asymmetry and subsequently, prediction errors, and relatively high price elasticity of demand. Interestingly, if we included ramp constraints in the consumer's problem it would have a stabilizing effect, as it would limit the consumer's responsiveness to price signals and reduce the elasticity of demand. This effect is, however, abstractly and qualitatively captured in our framework through the introduction of an inelastic component in the demand, which certainly limits the rate of change in the demand in a similar way to ramp constraints. However, uncertainty in the supply side, either in the available capacity or in the cost, works in the reverse direction: when supply is sufficiently volatile, a trade-off might exist and responsiveness and increased elasticity of demand might be desirable to some extent. The models developed in the paper do not include uncertainty in generation, and investigating these tradeoffs in a rigorous framework is an interesting direction for future research.

B. Learning

An interesting question that arises here is related to *learning* and can be posed as follows: can market participants *learn* to adjust their behavior in response to volatility in market prices and thereby, mitigate volatility? Remarkably, *learning* in the sense intended here and *ramp constraints* are closely related. Absence of ramp constraints implies that consumers can quickly adjust their consumption, hence, price volatility may not necessarily alter their consumption patterns or the way they (or autonomous devices on their behalf) react to prices. In other words, consumers' optimal strategy in response to price signals would be myopic. Therefore, in this case, the consumers may not have an incentive for altering their behavior to avoid aggravating volatility. Consumers have an incentive to reduce volatility only when they have ramp constraints, since they stand to loose in a volatile market if they cannot adjust their consumption fast enough. On the contrary, consumers with access to storage, or storage owners, have an incentive to increase price volatility, particularly if the market allows them to use storage as an arbitrage mechanism. This is because the economic value of storage increases with price volatility [8]. Whether learning in such a complex market with erratic outcomes can occur or not is an open question which does not fall within the scope of this paper. However, we note in passing that it has been shown that convergence of neural network learning in cobweb-like macro-economic models is related to the so-called E-Stability condition in Muth model (see, e.g., Heinemann [37] and Packalen [38]). The E-stability condition is a sensitivity-based sufficient condition for stability. The implication is that for unstable systems, neural network learning may not necessarily converge, and higher levels of sensitivity to disturbances make learning more challenging. Regulating consumer behavior and designing proper incentives might be a practical alternative to relying on learning. Nevertheless, even in the absence of ramp constraints on the demand side, volatility has an effect on the supply side and/or ISO side. Since the suppliers have ramp constraints, and since increased demand volatility increases the overall stress on the grid, the ISO has an incentive to reduce volatility. This may be done by externality pricing or by adopting pricing mechanisms other than direct marginal cost pricing.

C. Storage and Load Shifting

Access to storage, and the possibility to defer flexible loads induce an *internal state* or *memory* in the consumer model [7], [8]. The *state* is essentially the amount of backlogged or stored energy, which evolves dynamically in time. If the consumers are price responsive, the evolution of this state is dependent on the history of the materialized prices: an extended period of high prices is likely to induce a relatively large backlogged demand and a relatively small amount of stored energy; while an extended period of low prices has the reverse effect. Furthermore, the *price-responsiveness* or *elasticity* of consumers is not a memoryless function of the current price. Rather, it is a function of the past history of prices which can be summarized in the state; when there is a lot of backlogged demand in the system, a relatively low price will induce a large demand for power, whereas the same price may induce a relatively small demand when there is little or no backlogged demand in the system. Such effects will be even more exaggerated if the consumers implement *price threshold policies*—which are shown to be optimal for load-shifting under some technical assumptions [7]—for managing their consumption. As a result, *dynamics* is inherent to such systems even in the "full information" case in which the ISO has a complete model of the consumers. While bidding will change the dynamics of the system, it will not create a *memoryless* system because the bids themselves will be dynamic; they will depend on the internal state which itself is determined by the past history. This discussion on existence of a dynamic feedback due to storage and load-shifting also highlights additional challenges in learning the demand curve by the ISO, or learning the dynamics of the market by the consumers. Modeling, system identification, state estimation, and analysis of the dynamics of such systems are interesting and important directions for further research. Some recent results on identification of the aggregate dynamics that emerges from optimal load-shifting by a large group consumers can be found in [7].

D. Value of Information and Bidding

The above discussions lead to yet another interesting research direction: "quantifying the value of information in closed-loop electricity markets". Given the heterogeneous nature of consumers and time-varying uncertainty in their preferences, needs, and valuations for electricity, learning their value functions and predicting their response to a price signal in real-time is a challenging problem. Furthermore, as we discussed in Section V-C, load-shifting and storage lead to additional complexities by inducing dynamics in the demand model. Suppose that the consumers provide a real-time estimate of their inelastic and elastic consumption to the ISO, either directly or through bidding in the market. Such information will change the dynamics of the market. The important question is "How valuable will this real-time information be and what would its impact be on volatility, efficiency, robustness and fragility of the system?" Given the potentially significant costs and barriers associated with creating and obtaining such information in real-time, quantifying the value of information in this context seems an important and timely question with potentially significant impact the architecture of future power systems.

VI. Numerical Simulations

In this section we present the results of some numerical simulation. For the purpose of simulations, we use the following demand model:

$$D(t) = \mu_1 d_1(t) + \mu_2 (1 + \delta_2(t)) \dot{v}^{-1}(\lambda(t)) \tag{66}$$

where $d_1(t)$ is the exogenous, inelastic demand:

$$d_1(t) = a_0 + a_1 \sin(t) + a_2 \sin(2t) + \delta_1(t)$$

and $\delta_1(t) \sim \mathcal{N}(0, 0.1^2)$ and $\delta_2(t) \sim \mathcal{N}(0, 0.01^2)$ are random disturbances. The parameters μ_1 and μ_2 are adjusted, on a case-to-case basis, such that the average demand under real-time pricing (i.e., when $\mu_2 > 0$, $\mu_1 < 1$) remains nearly equal to the average demand in the open loop market ($\mu_2 = 0$, $\mu_1 = 1$), that is:

$$\sum_{t=1}^{N} D(t) \approx \sum_{t=1}^{N} d_1(t)$$

This normalization, takes out the effect of higher or lower average demand on price and allows for a fair comparison of volatility of prices in open-loop and closed-loop markets. The following parameters are chosen for all simulations in this section:

$$a_0 = 4 \text{ GW},\ a_1 = 1 \text{ GW},\ a_2 = 1 \text{ GW}$$

This puts the peak of the inelastic demand at 6 GW and the valley at 2 GW, modulo the random disturbance $\delta_1(t)$. All simulations are for a 24 hour period and prices are updated every 5 minutes. The average demand in all simulations is approximately 4 GW per five minutes for both open-loop and closed-loop markets. The metric for comparison in these simulation is the Relative Volatility Ratio (RVR), defined as the ratio of the $\log$-scaled IAV of the closed-loop market to the $\log$-scaled IAV of the open-loop market. The results of the first simulation are summarized in Figure 3. The prices are extremely volatile under real-time pricing (RVR $= 51.12$) and the system is practically unstable.

The results of the second simulation are summarized in Figure 4. Based on the chosen parameters, this market is less volatile than the one in the first simulation, yet, volatility of demand increases under real-time pricing (RVR=2.33). Since

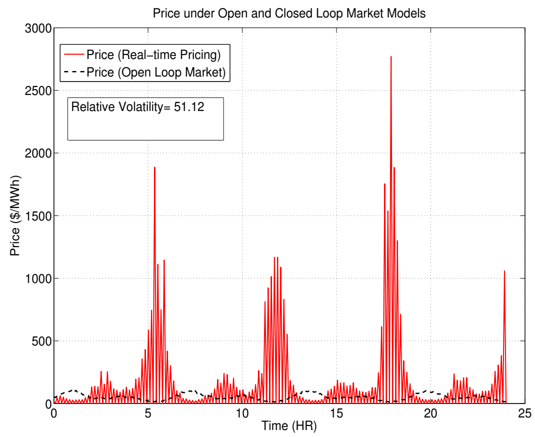


Fig. 3. Simulation of a market with quadratic cost function $c(x) = x^3$, value function $v(x) = \log(x)$, and demand function $D(t)$ given in (66) with $\mu_1 = 0.075$, $\mu_2 = 2$.

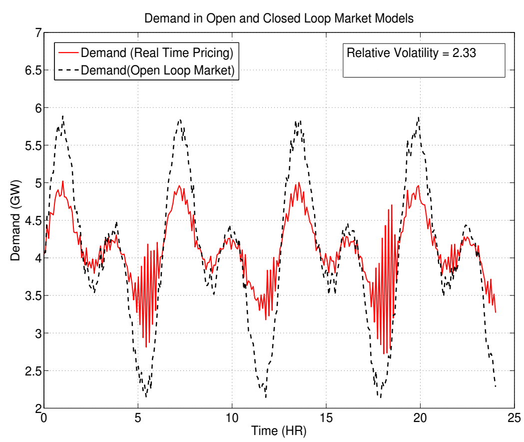


Fig. 4. Simulation of a market with quadratic cost function $c(x) = 3x^2$, value function $v(x) = \sqrt{x}$, and demand function $D(t)$ in (66) with $\mu_1 = 0.7$, $\mu_2 = 3 \times 10^3$.

in this simulation the cost is quadratic, the price (not shown) has a very similar pattern.

The third simulation is summarized in Figure 5. For each value of $\mu_1 \in [0, 1]$ (with 0.05 increments), the expected RVR was calculated by taking the average RVR of 50 randomized simulations. The random parameters are $\delta_1(t)$, $\delta_2(t)$, and the initial conditions. The experiment was repeated for four different value functions: $v(x) = x^{1/a}$, $a = 4, 4.5, 5, 5.5$. It is observed that volatility increases with decreasing a or μ_1, both of which increase the price-elasticity of demand.

VII. Conclusions and Future Work

We developed a theoretical framework to study the effects of real-time retail pricing on the stability and volatility of power systems. We highlighted that exposing the retail consumers to the real-time wholesale market prices creates a closed-loop feedback system. From a control system's perspective, it is intuitive that in the absence of a carefully designed control law, such direct feedback may lead to increased volatility and decreased robustness to external disturbances. While we

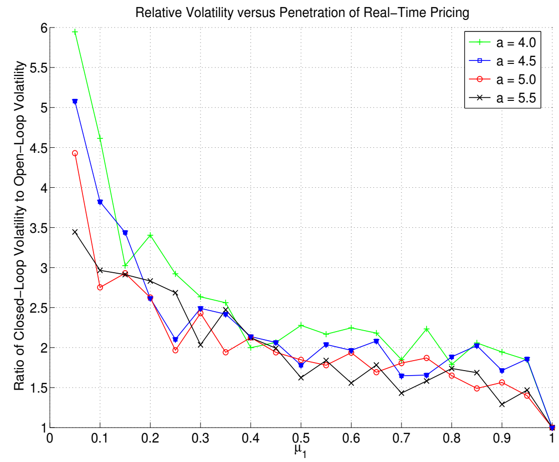


Fig. 5. Simulation of a market with quadratic cost function $c(x) = 3x^2$, value function $v(x) = x^{1/a}$, and demand function $D(t)$ given in (66) with $\mu_1 \in [0, 1]$, and μ_2 adjusted accordingly to keep the total demand constant. Decreasing a or μ_1 increase the price-elasticity of the overall demand and hence, increase volatility.

used a static model of consumers and derived a dynamical system model based on delay and information asymmetry, we pointed out that storage and load shifting induce dynamics and memory in the consumer model, with the state being the amount of stored or backlogged demand. Considering such models would lead to closed loop dynamical system models of power systems even when the consumers bid in the market. As a result, feedback and dynamics are inherent to power systems under real-time pricing. Rigorous analysis of the dynamics of the system in this case is part of the future work.

Under the assumptions of memoryless utility functions for the consumers and autoregressive prediction models for the ISO, we showed that scaled incremental volatility can be linked to a function of the market's maximal relative price-elasticity, defined as the maximal ratio of *generalized* price-elasticity of consumers to that of the producers. As this ratio increases, the system may become more volatile. As the penetration of new demand response technologies and distributed storage within the power grid increases, so does the price-elasticity of demand. Our theoretical analysis suggests that under current market and system operation practices, this technological change may to lead to increased volatility. In order to further substantiate these results, more experimental studies as well as simulations with more detailed models of consumer behavior and actual market data are needed.

While it is possible to design a a pricing mechanism, i.e., a control law that regulates the interaction of wholesale markets and retail consumers, limitations of performance and the trade-offs between various performance and robustness properties must be carefully considered. In light of this, systematic analysis of the implications of different pricing mechanisms, analysis of the inherent dynamics of the system induced by load-shifting and storage, quantifying the value of information and characterization of the fundamental trade-offs between volatility/robustness/reliability, and economic efficiency, and environmental efficiency are important directions of future research. In summary, more sophisticated models of demand, a deeper understanding of consumer behavior in response to real-time prices, and a thorough understanding of the implications of different market mechanisms and system architectures are needed for efficient and reliable implementation of real-time pricing schemes in large-scale.

Appendix A
Proofs

Before we proceed with providing the proofs, we present the following lemma, which will be used several times in this section.

Lemma 1: Let X be a subset of $\mathbb{R}$. Suppose that there exists a continuously differentiable function $f : X \mapsto \mathbb{R}$, a continuously differentiable monotonic function $g : X \mapsto \mathbb{R}$, and a constant $\theta \in [0, \infty)$ satisfying

$$\left|\dot{f}(x)\right| \le \theta \left|\dot{g}(x)\right|, \quad \forall x \in X \tag{67}$$

Then

$$\left|f(x) - f(y)\right| \le \theta \left|g(x) - g(y)\right|, \quad \forall x, y \in X \tag{68}$$

Furthermore, if (32) is satisfied, then

$$\left|f(x) - f(y)\right| < \left|g(x) - g(y)\right|, \quad \forall x, y \in X,\ x \neq y \tag{69}$$

Proof: We have

$$\begin{aligned} &\forall x, y \in X,\ x \neq y : \\ \left|f(x) - f(y)\right| &\le \left|\int_y^x \left|\dot{f}(\tau)\right| d\tau\right| \\ &\le \theta \left|\int_y^x \left|\dot{g}(\tau)\right| d\tau\right| = \theta \left|g(x) - g(y)\right| \end{aligned} \tag{70}$$

where the inequality in (70) follows from (67) and the subsequent equality follows from (33). Proof of (69) is similar, except that under the assumptions of the lemma, the non-strict inequality in (70) can be replaced with a strict inequality. ∎

We will now present the proof of Theorem 1.

Proof of Theorem 1: The key idea of the proof is that the function

$$V(x) = \left|f(x) - g(x)\right| \tag{71}$$

is strictly monotonically decreasing along the trajectories of (29). From Lemma 1 we have:

$$\begin{aligned} &V(x(t+1)) - V(x(t)) \\ &= \left|f(x(t+1)) - g(x(t+1))\right| - \left|f(x(t)) - g(x(t))\right| \\ &= \left|f(x(t+1)) - f(x(t))\right| - \left|g(x(t+1)) - g(x(t))\right| \\ &< 0. \end{aligned} \tag{72}$$

Therefore, $\{V(x(t))\}$ is a strictly decreasing bounded sequence and converges to a limit $c \ge 0$. We show that $c > 0$ is not possible. Note that the sequence $\{x(t)\}$ is bounded from below since the domain of ψ is $\mathbb{R}_+$. Furthermore, as long as $f(x(t)) < g(x(t))$, the sequence $\{g(x(t))\}$ decreases strictly. Therefore, (33) implies that

$$\forall x_0 : \exists\ M \in \mathbb{R},\ N \in \mathbb{Z}_+ : g(x(t)) \le M,\ \forall t \ge N. \tag{73}$$

It follows from (73), monotonicity and continuity of $g(\cdot)$ that the sequence $\{x(t)\}$ is bounded from above too (similar arguments prove boundedness of $\{x(t)\}$ when (34) holds). Hence, either $\lim_{t\to\infty} x(t) = 0$, or $\{x(t)\}$ has a subsequence $\{x(t_i)\}$ which converges to a limit $x^* \in \mathbb{R}_+$. In the latter case we have

$$\begin{aligned} \lim_{t\to\infty} V(x(t)) = \lim_{i\to\infty} V(x(t_i)) &= \left|\lim_{i\to\infty} \{f(x(t_i)) - g(x(t_i))\}\right| \\ &= \left|f(x^*) - g(x^*)\right| \end{aligned}$$

If $g(x^*) = g(\psi(x^*))$ then $c = \left|f(x^*) - g(\psi(x^*))\right| = 0$ (due to (30)). If $g(x^*) \neq g(\psi(x^*))$ then

$$\exists \delta, \varepsilon > 0, \text{ s.t. } \left|g(\psi(x)) - g(x)\right| \ge \varepsilon,\ \forall x \in \mathcal{B}(x^*, \delta)$$

Define a function $\tau : \mathcal{B}(x^*, \delta) \mapsto \overline{\mathbb{R}}_+$ according to

$$\tau : x \mapsto \frac{\left|f(\psi(x)) - f(x)\right|}{\left|g(\psi(x)) - g(x)\right|}$$

Then it follows from 72 that $\tau(x) < 1$ for all $x \in \mathcal{B}(x^*, \delta)$. Furthermore, the function is continuous over the compact set $\mathcal{B}(x^*, \delta)$ and achieves its supremum $\overline{\tau}$, where $\overline{\tau} < 1$. Since $x(t_i)$ converges to x^*, there exists $\tilde{t} \in \mathbb{N}$, such that $x(t) \in \mathcal{B}(x^*, \delta)$. Then

$$\begin{aligned} &V(x(t+1)) - \overline{\tau} V(x(t)) \\ &= \left|f(x(t+1)) - f(x(t))\right| - \overline{\tau}\left|g(x(t+1)) - g(x(t))\right| \\ &\le 0, \quad \forall t \ge \tilde{t} \end{aligned}$$

Since $\overline{\tau} < 1$, this proves that $c = 0$. Finally,

$$\lim_{t\to\infty} f(x(t)) = \lim_{t\to\infty} g(x(t)) = g(x^*)$$

$$x^* = g^{-1}(\lim_{t\to\infty} f(x(t))) = \lim_{t\to\infty} g^{-1} \circ f(x(t)) = \lim_{t\to\infty} x(t)$$

This completes the proof of convergence for all initial conditions. Proof of Lyapunov stability is based on standard arguments in proving stability of nonlinear systems (see, e.g., [9]), while using the same Lyapunov function defined in (71). ∎

Proof of Theorem 2: For simplicity and convenience in notation, we prove the theorem for the $n = 1$ case. The proof for the general case is entirely analogous. Define the function $V : \mathbb{R}^2 \mapsto \mathbb{R}_+$ according to

$$V(x, z) = \left|g(x) - f(x, z)\right| \tag{74}$$

Let

$$\gamma_0 = V(x(1), x(0)) + \left|g(x(1)) - g(x(0))\right| \tag{75}$$

To prove that Ω_0 is invariant under (42), it is sufficient to show that

$$V(x(T+1), x(T)) \le \gamma_0,\ \forall T \in \mathbb{Z}_+ \tag{76}$$

To simplify the notation, define $\Delta f_t = f(x(t+1), x(t)) - f(x(t), x(t-1))$, and $\Delta g_t = g(x(t+1)) - g(x(t))$.

We have:

$$
\begin{aligned}
&V\left(x\left(t+1\right),x\left(t\right)\right)-V\left(x\left(t\right),x\left(t-1\right)\right)\\
&\quad=\left|g\left(x\left(t+1\right)\right)-f\left(x\left(t+1\right),x\left(t\right)\right)\right|\\
&\qquad\qquad-\left|g\left(x\left(t\right)\right)-f\left(x\left(t\right),x\left(t-1\right)\right)\right|\\
&\quad=\left|f\left(x\left(t\right),x\left(t-1\right)\right)-f\left(x\left(t+1\right),x\left(t\right)\right)\right|\\
&\qquad\qquad-\left|g\left(x\left(t\right)\right)-g\left(x\left(t+1\right)\right)\right|\\
&\quad\leq\left|f\left(x\left(t\right),x\left(t-1\right)\right)-f\left(x\left(t\right),x\left(t\right)\right)\right|\\
&\qquad+\left|f\left(x\left(t\right),x\left(t\right)\right)-f\left(x\left(t+1\right),x\left(t\right)\right)\right|-\left|\Delta g_t\right|\\
&\quad\leq\theta_2\left|\Delta g_{t-1}\right|+\left(\theta_1-1\right)\left|\Delta g_t\right| \qquad (77)
\end{aligned}
$$

where the first inequality is obtained by applying the triangular inequality, and (77) follows from (43) and Lemma 1. By summing up both sides of (77) from $t=0$ to $t=T$ we obtain:

$$
\begin{aligned}
&V\left(x\left(T+1\right),x\left(T\right)\right)\leq V\left(x\left(1\right),x\left(0\right)\right)\\
&\quad+\left(\theta_1+\theta_2-1\right)\sum_{t=1}^{T}\left|\Delta g_t\right|+\theta_2\left(\left|\Delta g_0\right|-\left|\Delta g_T\right|\right) \qquad (78)
\end{aligned}
$$

The inequality (76) then follows from (78) and (44). When (44) holds with strict inequality, (46) follows from (78) and nonnegativity of $V\left(x\left(T+1\right),x\left(T\right)\right)$ for all $T\in\mathbb{Z}$. ∎

Proof of Theorem 3: Define

$$V\left(x\right)=\sup_{\nu\in U}\left\{\Big|\left|f\left(x,\nu\right)-g\left(x\right)\right|-\left|\nu\right|\Big|\right\}-\zeta_\kappa\left(\theta\right).$$

It is sufficient to show that there exists $\tau\geq0$, such that:

$$V\left(x\left(t+1\right)\right)-\tau V\left(x\left(t\right)\right)\leq0,\ \forall t\in\mathbb{Z}_+.$$

To simplify the notation, define $\Delta f_t = f\left(x\left(t+1\right),u\left(t+1\right)\right) - f\left(x\left(t\right),u\left(t\right)\right),$ and $\Delta g_t=g\left(x\left(t+1\right)\right)-g\left(x\left(t\right)\right).$ Then

$$
\begin{aligned}
&V\left(x\left(t+1\right)\right)-\tau V\left(x\left(t\right)\right)\\
&\quad=\sup_{\nu\in U}\left\{\Big|\left|f\left(x\left(t+1\right),\nu\right)-g\left(x\left(t+1\right)\right)\right|-\left|\nu\right|\Big|\right\}\\
&\qquad-\tau\sup_{\nu\in U}\left\{\Big|\left|f\left(x\left(t\right),\nu\right)-g\left(x\left(t\right)\right)\right|-\left|\nu\right|\Big|\right\}\\
&\qquad\qquad+\zeta_\kappa\left(\theta\right)\left(\tau-1\right)\\
&\quad\leq\sup_{\nu\in U}\left\{\Big|\left|f\left(x\left(t+1\right),\nu\right)-f\left(x\left(t\right),u\left(t\right)\right)\right|-\left|\nu\right|\Big|\right\}\\
&\qquad\qquad-\tau\left|\Delta g_t\right|+\tau\kappa+\zeta_\kappa\left(\theta\right)\left(\tau-1\right) \qquad (79)\\
&\quad\leq\sup_{\nu\in U}\left|f\left(x\left(t+1\right),\nu\right)-f\left(x\left(t\right),\nu\right)\right|\\
&\qquad+\sup_{\nu\in U}\left\{\Big|\left|f\left(x\left(t\right),\nu\right)-f\left(x\left(t\right),u\left(t\right)\right)\right|-\left|\nu\right|\Big|\right\}\\
&\qquad\qquad-\tau\left|\Delta g_t\right|+\tau\kappa+\zeta_\kappa\left(\theta\right)\left(\tau-1\right) \qquad (80)\\
&\quad\leq\left(\theta-\tau\right)\left|\Delta g_t\right|+\left(1+\tau\right)\kappa+\zeta_\kappa\left(\theta\right)\left(\tau-1\right) \qquad (81)
\end{aligned}
$$

where (79) follows from the choice of $\nu=u\left(t\right)$ and $\left|u\left(t\right)\right|\leq\kappa$, (80) follows from the triangular inequality, and (81) follows from (53)–(54) and Lemma 1. The desired result follows from the fact that the right-hand side of (81) will be non-positive for $\tau=\theta$, and $\zeta_\kappa\left(\theta\right)$ defined in (55). To prove (57), let $\tau=1$ in (81) to obtain

$$V\left(x\left(t+1\right)\right)-V\left(x\left(t\right)\right)\leq\left(\theta-1\right)\left|\Delta g_t\right|+2\kappa \qquad (82)$$

Summing both sides of (82) over all $t=0,1,..,T$ results in:

$$V\left(x\left(T+1\right)\right)\leq V\left(x\left(0\right)\right)+\left(\theta-1\right)\sum_{t=1}^{T}\left|\Delta g_t\right|+2T\kappa \qquad (83)$$

It follows from (83) and non-negativity of $V\left(x\left(T+1\right)\right)+\zeta_\kappa\left(\theta\right)$ that

$$\left(1-\theta\right)\sum_{t=1}^{T}\left|\Delta g_t\right|\leq2T\kappa+V\left(x\left(0\right)\right)+\zeta_\kappa\left(\theta\right). \qquad (84)$$

The desired result (57) then follows immediately from (83) by dividing by T and taking the limit as $T\rightarrow\infty$. ∎

References

[1] S. Borenstein, M. Jaske, and A. Rosenfeld, "Dynamic pricing, advanced metering and demand response in electricity markets," *UC Berkeley: Center for the Study of Energy Markets.*, 2002.

[2] W. Hogan, "Demand response pricing in organized wholesale markets," *IRC Comments, Demand Reponse Notice of Proposed Rulemaking, FERC Docket RM10-17-000*, May 2010.

[3] J. G. Kassakian and R. Schmalensee, *(Study Co-chairs)*, "The future of the electric grid: An interdisciplinary MIT study," tech. rep., Massachusetts Institute of Tech., 2011.

[4] D. Crossley, "The role of advanced metering and load control in supporting electricity networks," Tech. Rep. No 5 Task XV, International Energy Agency Demand Side Management Programme. Energy Futures Australia PTY LTD, Australia, 2008.

[5] California Public Utilities Commission, "Docket #: A.06-03-005. http://docs.cpuc.ca.gov/published/news_release/85953.htm."

[6] J. L. Mathieu, D. S. Callaway, and S. Kiliccote, "Examining uncertainty in demand response baseline models and variability in automated responses to dynamic pricing," in *Decision and Control (CDC), 50th IEEE Conf. on*, pp. 4332–4339, Dec. 2011.

[7] M. Roozbehani, M. Ohannessian, D. Materassi, and M. A. Dahleh, "Load-shifting under perfect and partial information: Models, robust policies, and economic value," *Submitted, 2012, available at* http://web.mit.edu/mardavij/www/.

[8] A. Faghih, M. Roozbehani, and M. A. Dahleh, "On the economic value and price-responsiveness of ramp-constrained storage," *Submitted, 2012, available at* http://web.mit.edu/mardavij/www/.

[9] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 2002.

[10] W. Lohmiller and J. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[11] I. M. Horowitz, *Synthesis of Feedback Systems*. London, U.K.: Academic Press, 1963.

[12] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *Autom. Control, IEEE Trans. on*, vol. 26, pp. 301 – 320, apr 1981.

[13] J. S. Freudenberg and D. P. Looze, *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*. Springer, 1988.

[14] J. Chen, "Sensitivity integral relations and design trade-offs in linear multivariable feedback systems," *Automatic Control, IEEE Transactions on*, vol. 40, pp. 1700 –1716, oct 1995.

[15] N. C. Martins, M. A. Dahleh, and J. C. Doyle, "Fundamental limitations of disturbance attenuation in the presence of side information," *Autom. Control, IEEE Trans. on*, vol. 52, pp. 56–66, Jan. 2007.

[16] N. Martins and M. Dahleh, "Feedback control in the presence of noisy channels: Bode-like fundamental limitations of performance," *Autom. Control, IEEE Trans. on*, vol. 53, pp. 1604–1615, Aug. 2008.

[17] S. Tonetti and R. Murray, "Limits on the network sensitivity function for homogeneous multi-agent systems on a graph," in *American Control Conf. (ACC), 2010*, pp. 3217–3222, June 30 - July 2, 2010.

[18] F. Chandra, G. Buzi, and J. C. Doyle, "Glycolytic oscillations and limits on robust efficiency," *Science*, vol. 333, pp. 187–192, 2011.

[19] F. L. Alvarado, "The dynamics of power system markets," Tech. Rep. PSERC-97-01, Power Systems Engineering Research Consortium (PSERC), Mar. 1997.

[20] F. Alvarado, "The stability of power system markets," *Power Syst., IEEE Trans. on*, vol. 14, pp. 505–511, May 1999.

[21] D. Watts and F. Alvarado, "The influence of futures markets on real time price stabilization in electricity markets," in *Proc. of the 37th Annu. Hawaii Int. Conf. on Syst. Sciences.*, HICSS '04, Jan. 2004.

[22] J. Nutaro and V. Protopopescu, "The impact of market clearing time and price signal delay on the stability of electric power markets," *Power Syst., IEEE Trans. on*, vol. 24, pp. 1337–1345, Aug. 2009.

[23] J. E. Hartley, *The Representative agent in macroeconomics*. Routledge, 1997.

[24] M. Roozbehani, M. A. Dahleh, and S. K. Mitter, "On the stability of wholesale electricity markets under real-time pricing," in *Decision and Control (CDC), 2010 49th IEEE Conf. on*, pp. 1911–1918, Dec. 2010.

[25] The New England Independent System Operator (ISO-NE), "http://www.iso-ne.com/."

[26] The PJM Independent System Operator, "http://pjm.com/markets-and-operations.aspx."

[27] T. Zheng and E. Litvinov, "Ex-post pricing in the co-optimized energy and reserve market," *Power Syst., IEEE Trans. on*, vol. 21, pp. 1528–1538, Nov. 2006.

[28] F. C. Schweppe, M. C. Caramanis, R. D. Tabors, and R. E. Robn, *Spot Pricing of Electricity*. Kluwer, 1998.

[29] M. Roozbehani, M. Dahleh, and S. K. Mitter, "Dynamic pricing and stabilization of supply and demand in modern electric power grids," in *Proc. of the 1st IEEE Int. Conf. on Smart Grid Commun.*, pp. 543 –548, Oct. 2010.

[30] G. Wang, M. Negrete-Pincetic, A. Kowli, E. Shafieepoorfard, S. Meyn, and U. Shanbhag, "Dynamic competitive equilibria in electricity markets," in *Control and Optimization Theory for Electric Smart Grids* (A. Chakrabortty and M. Illic, eds.), Springer, 2011.

[31] N. Kaldor, "The cobweb theorem," *Quarterly Journal of Economics*, vol. 52, pp. 255–280, February 1938.

[32] M. Roozbehani, M. A. Dahleh, and S. K. Mitter, "Volatility of power grids under real-time pricing," *Technical Report: http://arxiv.org/abs/1106.1401*, June 2011.

[33] M. Roozbehani, M. Rinehart, M. A. Dahleh, S. K. Mitter, D. Obradovic, and H. Mangesius, "Analysis of competitive electricity markets under a new model of real-time retail pricing," in *European Energy Markets (EEM), 2011 8th Int. Conf. on*, pp. 250–255, May. 2011.

[34] K. J. Arrow, *The theory of risk aversion*. 1970.

[35] P. J. W., "Risk aversion in the small and in the large.," *Econometrica*, vol. 32, no. 1, pp. 122–136, 1964.

[36] I.-K. Cho and S. P. Meyn, "Efficiency and marginal cost pricing in dynamic competitive markets with friction," *Theoretical Economics*, vol. 5, no. 2, pp. 215–239, 2010.

[37] M. Heinemann, "Adaptive learning of rational expectations using neural networks," *Journal of Economic Dynamics and Control*, vol. 24, pp. 1007–1026, June 2000.

[38] M. Packalen, "Adaptive learning of rational expectations: a neural network approach," *Issue 447 of Kansantaloustieteen laitoksen keskustelualoitteita / Helsingin yliopisto. Dept. of Economics, University of Helsinki, ISBN 951458018*, 1998.

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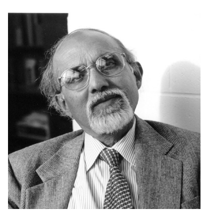

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