MATH 215A: Algebraic Topology

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Autumn 2022

Introduction

Professor Li can be reached in 380-383FF; office hours are Tuesdays and Thursdays from 6-7:30pm (or by appointment). The CA (Qianhe Qin)'s office hours start next week and are available on Canvas.

There will be nine homework assignments, released weekly and due each Wednesday at midnight. Grading will be 70 percent homework (with lowest two homework grades dropped) and 30 percent final take-home exam.

This is a course in algebraic topology and will follow Hatcher's book (it can be freely accessed online). That book has four chapters on the fundamental group, homology theory, cohomology theory, and homotopy theory; we will cover the first three. Some basic knowledge from topology and algebra will be assumed (like knowing what groups and group homomorphisms are).

1 September 27, 2022

Throughout the course, we will assume that all spaces are topological spaces and all maps are continuous. In this first week, we'll discuss homotopy, CW complexes, and some basic ways to construct spaces, and after that we'll get into the content previously mentioned.

Definition 1

Let $f, g : X \to Y$ be two (continuous) maps between topological spaces. We say that f and g are **homotopic** if there is a (continuous) map $H : I \times X \to Y$ (where I is the interval $[0, 1]$ with its standard topology) such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$. We denote this as $f \simeq g$.

We can think of the interval as parameterizing "time," so that we have a one-parameter (continuous) family of maps $h_t : X \to Y$ which is a deformation from f to g.

Example 2

Suppose $X = Y = \mathbb{R}$. Then the maps $f(x) = x$ and $g(x) = x^2$ are homotopic, because we have the homotopy $H(t, x)$ (alternatively written $h_t(x)$) given by

$$H(t, x) = (1 - t)x + tx^2.$$

It can directly be checked that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ and that H is continuous.

Example 3

More generally, if $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ (for $m, n \in \mathbb{Z}_{\geqslant 0}$) and f, g are two arbitrary maps $X \to Y$, we can always construct a homotopy between them given by the linear interpolation $h_t(x) = (1-t)f(x) + tg(x)$.

We can check that being homotopic is an **equivalence relation**, meaning that $f \simeq f$, $f \simeq g$ if and only if $g \simeq f$, and if $f \simeq g$ and $g \simeq h$ then $f \simeq h$.

Definition 4

Let X and Y be two topological spaces. We say that X and Y are **homotopy equivalent** if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. We denote this as $X \simeq Y$.

Example 5

Let $X = \{0\}$ and $Y = \mathbb{R}^2$. These two spaces are indeed homotopy equivalent, since we can have $f : X \to Y$ send 0 to $(0, 0)$ and have $g : Y \to X$ send everything to 0; indeed $g \circ f$ is the identity map on X and any two maps $\mathbb{R}^2 \to \mathbb{R}^2$ are homotopic.

Definition 6

A space is **contractible** if it is homotopy equivalent to $\{0\}$.

Example 7

For any $m, n \geqslant 0$, we can check that $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ are homotopy equivalent with the same argument as above.

Definition 8

Suppose $X \subseteq Y$, and we let $f : X \to Y$ be the canonical inclusion. A map $g : Y \to X$ is called a **retraction** if $g(x) = x$ for all $x \in X$ (or equivalently, $g|_X = \mathrm{id}_X$).

For example, the map g in Example 5 is a retraction. And since $X \subseteq Y$, retractions can be interpreted as maps $g : Y \to Y$ whose image is X, or equivalently maps where $g^2 = g$. We'll now relate this concept to that of homotopy equivalence:

Definition 9

Let $X \subseteq Y$, and let $g : Y \to Y$ (with $\mathrm{im}(g) = X$) be a retraction. g is called a **deformation retraction** if there is a homotopy $h_t : Y \to Y$ such that $h_0 = \mathrm{id}_Y$ and $h_1 = g$, and $h_t|_X = \mathrm{id}_X$ for all t.

In other words, we start from the identity map and deform it to a "projection" onto X, while keeping the map on X constant. And the map g in Example 5 is a deformation retraction if we view 0 as the origin in $\mathbb{R}^2$.

Example 10

Let $X = S^{n+1}$ be the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, given by $X = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$, and let $Y = \mathbb{R}^{n+1} \backslash \{0\}$.

We claim that there is a deformation retraction from Y to X (though there is none from $\mathbb{R}^n$ to X). Indeed, let $g : \mathbb{R}^{n+1} \backslash \{0\} \to S^{n+1}$ send y to the normalized vector $\frac{1}{||y||} y$. Indeed g is a retraction (since a unit-length vector is sent to itself), and the homotopy is given by the usual linear interpolation

$$h_t(y) = \left(1 - t + \frac{t}{||y||}\right) y.$$

Remark 11. *In Hatcher's book, the third condition $h_t|_X = id_X$ is required for us to have a deformation retraction, but in some other books this is called a "strong deformation retraction." Note that if we have a homotopy $h_t : Y \to Y$ between id_Y and a retraction g* **without** *that third condition, we* **already** *have $X \simeq Y$.*

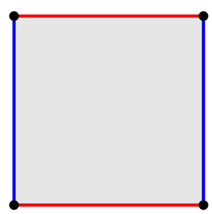
The idea is that homotopy equivalent spaces will have the same fundamental groups, homology groups, and cohomology groups, so they are "basically the same" when we consider them in those ways. But we'll come back to all of that later.

Our next topic will be **CW complexes**, and we'll start with some motivation:

Example 12

A circle can be broken up into two points connected by two disjoint arcs, and a square can be broken up into a face, four edges, and four vertices. Even with more complicated spaces, we can perform this process: for example, we can cut a torus along a curve of "longitude" and "latitude," and we end up with a square where two opposite edges are identified.

(For example, if we imagine gluing together the red lines and the blue lines below in the same orientation, we will recover a torus.)



The idea with CW complexes is that they are made up of various n-dimensional components, called ***n*-cells** (which are homeomorphic to closed n-balls), glued together so that higher-dimensional cells are glued to lower-dimensional cells at their boundary.

Definition 13

A **CW complex** is defined inductively in the following way: X^0 is a discrete set of points, and given X^{n-1}, we take a set of n-cells $\{D^n_\alpha\}_{\alpha \in I}$ and define a set of **characteristic maps** $\{\phi^n_\alpha : \partial D^n_\alpha \to X^{n-1}\}_{\alpha \in I}$. Then we define the ***n*-skeleton**

$$X^n = \left(X^{n-1} \amalg \bigcup_{\alpha \in I} D^n_\alpha\right) / \sim$$

with quotient induced by the gluing maps $(x \in \partial D^n_\alpha) \sim (\phi^n_\alpha(X) \sim X^{n-1})$ and equipping it with the quotient topology. Finally, take $X = \cup_{n \geqslant 0} X^n$.

Example 14

For a single square, X^0 would be a set of four vertices, X^1 would be the set of four vertices with edges between them (where we've specified which vertices are the boundaries of each edge), and X^2 is the full square.

To describe the topology on X more explicitly, we're looking at the **weak topology**. In particular, X^n is equipped with the quotient topology, and if $X = X^{n_0}$ for some n_0, then the weak topology is just the quotient topology. But more generally, if $X = X^0 \cup X^1 \cup X^2 \cup \cdots$, we say that a set $U \subseteq X$ is open if and only if $U \cap D^n_\alpha$ is open in D^n_α for any cell D^n_α. (So this is the weakest topology on which all of the inclusion maps are continuous.) It's also okay to require that $U \cap X^n$ is open in for all n.

Remark 15. *The name "CW-complex" comes from "closure finiteness" (any compact set intersects the interior of only finitely many cells) and "weak topology."*

Example 16

Consider the two-dimensional sphere S^2. We can decompose S^2 into the northern and southern hemisphere (each of which is a 2-cell) and an equator (which is S^1). We denote this $S^2 = S^1 \cup D^2_+ \cup D^2_-$, and more generally we can write $S^n = S^{n-1} \cup D^n_+ \cup D^n_-$.

So this inductively allows us to find a CW structure for S^n, but it is not the only one – for example, we can write S^2 as $D^2 \cup D^0$ (by imagining gluing the entire boundary of a disk to a single point, "closing it" into a sphere).

Definition 17

The **real projective space** $\mathbb{RP}^n$ is given by

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \backslash \{0\}) / \sim,$$

where $x \sim \lambda x$ for any nonzero scalar λ.

In other words, we can imagine that $\mathbb{RP}^n$ is the set of all lines passing through the origin in $\mathbb{R}^{n+1}$. We can give this a CW structure by noticing that on the unit sphere S^n, only the points x and $-x$ are related to each other, so we actually have $\mathbb{RP}^n = S^n / \sim$ under the relation $x \sim -x$. Thinking back to the decomposition $S^n = S^{n-1} \cup D^n_+ \cup D^n_-$, notice that points on the interior of D^n_+ and D^n_- are taken to each other, and what's left is "half of" S^{n-1}. So we can write the inductive relation

$$\mathbb{RP}^n = D^n_+ / \sim,$$

where $\sim$ is the relation that turns ∂D^n_+ into $\mathbb{RP}^{n-1}$ by identifying opposite points. The gluing map is then the projection map $\phi^n : \partial D^n_+ \to \mathbb{RP}^{n-1}$, and thus we are able to obtain $X^n = \mathbb{RP}^n$ from $X^{n-1} = \mathbb{RP}^{n-1}$.

Definition 18

Let Y be a CW complex. If $X \subseteq Y$ is the union of some cells of Y, then (Y, X) is a **CW-pair**.

We'll continue to cover some related ideas next time.

2 September 29, 2022

Last lecture, we discussed some basic ideas surrounding homotopy and CW complexes. (We won't cover everything in the book during lectures, but we can ask if we have any questions.) We'll continue on today with "homotopy stuff," starting with criteria for homotopy equivalence. Recall that $X \simeq Y$ (X and Y are "homotopy equivalent") if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$. We call the maps f and g **homotopy equivalences**.

Proposition 19

Let (X, A) be a CW-pair (recall that this means that A is a union of some cells of the CW complex X), and suppose A is contractible (meaning that A is homotopy equivalent to a single point). Then the natural quotient map $q : X \to X/A$ (in which all points in A are reduced to a single point) is a homotopy equivalence.

Example 20

Suppose $X = D^n$ is the n-dimensional disk, and $A = S^{n-1} = \partial D^n$. (This is not contractible – it's just an example to illustrate the quotient map.) Then it turns out $X/A = S^n$ – for example, we can imagine that identifying the boundary of a two-dimensional disk gives us a sphere in 3D space by "rolling up."

Example 21

Let T be a torus, and let u be a ring on the torus (a cross-section). Then we can compare $X = T/u$ with the space $Y = S^2 \vee S^1$ where we "wedge" S^2 and S^1 together (gluing the two spaces along at a single point).

These spaces may look different, but it turns out they are homotopy equivalent. Indeed, consider the space $Z = S^2 \cup \beta$ defined by starting with a 2-sphere and then adding an arc β connecting the north and south pole. Letting α be an arc contained within S^2 and also connecting N with S, we see that both α and β are contractible in Z. So by Proposition 19, Z should be homotopy equivalent to both Z/α and Z/β. But quotienting by α gives us $Y = S^2 \vee S^1$ (where β becomes S^1 and the sphere stays S^2), and quotienting by β gives us $X = T/u$ (where the north and south pole become the point that u contracts to).

We'll discuss the proof of Proposition 19 later, but for now we'll discuss another criterion for homotopy equivalence. We'll need the idea of **gluing spaces** for this:

Definition 22

Let $A \subseteq X$, and suppose $f : A \to Y$ is a map. Then we can form the space $X \amalg_f Y = X \amalg Y / \sim$ by identifying the point $a \in A$ with $f(A) \in Y$.

Example 23

If $A = \{x_0\} \in X$ is a single point and f maps x_0 to some $y_0 \in Y$, then we basically take X and Y together and glue them together at the point $x_0 = y_0$. This gives us the "wedge" $X \vee Y = X \amalg_f Y$.

(If X and Y are path-connected, then different choices of points x_0, y_0 are homeomorphic. But otherwise we may need to specify x_0 and y_0 more specifically.)

Definition 24

The **mapping cylinder** M_f of a map $f : X \to Y$ is the space

$$M_f = (X \times I) \amalg_f Y,$$

where we think of f as a map from $X \times \{1\} \to Y$.

The idea is that we imagine the image of X as a circle (so that $X \times I$ looks like a cylinder, and we attach one end of the cylinder to Y.

Definition 25

The **mapping cone** C_f of $f : X \to Y$ is the space

$$C_f = (X \times I / X \times \{0\}) \amalg_f Y.$$

(This is very similar to the previous case, but we identify the other end of the cylinder so that we just have a cone sticking out of Y. We call $CX = X \times I / X \times \{0\}$ the **cone over X**.)

Example 26

Suppose $X = S^n$. Then the cone over X is the disk D^{n+1} (we can imagine that the cone fills in the interior until we get to the origin, which is the tip of the cone). Thus, for any map $f : S^{n-1} \to Y$, the mapping cone C_f is $CS^{n-1} \amalg_f Y$, which is just $D^n \amalg_f Y$. Thus the mapping cone glues an n-dimensional ball to Y along the boundary given by f, and in particular this allows us to glue an n-cell to the skeleton in a CW complex. Specifically, if Y is a single point, there is a unique map $f : S^{n-1} \to Y$ and the mapping cone gets us $C_f = S^n$ again.

Notice that for an arbitrary space X, the cone CX is **always contractible** by shrinking along the time-direction. So now if we have a CW pair (X, A) with contractible A, and $\iota : A \hookrightarrow X$ is the inclusion, then we can glue the cone CA to X along A, and thus Proposition 19 allows us to identify (by homotopy equivalence) the mapping cone $C_\iota = X \cup CA$ with X/A, whether or not A is contractible. And we're now ready to state the second criterion:

Proposition 27

Let (X, A) be a CW pair, and suppose $f, g : A \to Y$ are homotopic. Then we have homotopy equivalence of the spaces $X \amalg_f Y \simeq X \amalg_g Y$.

Example 28

Suppose (X, A) is a CW pair with A contractible. Then we can consider the inclusion map $\iota : A \hookrightarrow X$, as well as the map $f : A \to \{a_0\}$ for some fixed point $a_0 \in A \subseteq X$.

Since A is contractible, these maps are homotopic, so applying Proposition 27 tells us that $C_\iota \simeq C_f$. But $\boxed{X/A} \simeq C_\iota$ as discussed, while C_f can be thought of $\boxed{X \vee SA}$, which is wedging X with the **suspension** of A:

Definition 29

Let X be a space. Then the **suspension** of X is

$$SX = (X \times [-1, 1] / X \times \{1\}) / X \times \{-1\}.$$

In the example above, quotienting at 1 corresponds to the mapping cone of A, and quotienting at -1 corresponds to gluing all points in A to $a_0 \in X$.

If we take $S^0 \subseteq S^2$ (which we can take to be the north and south pole), then S^0 is not contractible (because it is a set of two disjoint points) but it is homotopic to a single point in S^2. So we can still work with the maps $\iota : S^0 \hookrightarrow S^2$ and $f : S^0 \to \{N\}$, which are still homotopic, and the same conclusion still holds: $C_\iota \simeq S^2/S^0$ is homotopy equivalent to $C_f \simeq S^2 \vee SS^0$. But the suspension of S^0 is S^1, so we are saying that $S^2/S^0 \simeq S^2 \vee S^1$, recovering the result from earlier this lecture. And more generally, if $S^n \subseteq S^m$ and $m > n$, we have $S^m/S^n \simeq S^m \vee S^{n+1}$.

We can now start to discuss how to prove these two criteria for homotopy equivalence, and an important ingredient will be **homotopy extension**. The idea is the following: let $A \subseteq X$, and consider maps $f_t : A \to Y$ for $t \in [0, 1]$, forming a homotopy. Additionally, suppose f_0 extends to a map $\tilde{f}_0 : X \to Y$. Then we may ask when f_t etxends as well:

Definition 30

A pair (X, A) has the **homotopy extension property** if any homotopy $f_t : A \to Y$ extends to $\tilde{f}_t : X \to Y$ whenever f_0 extends to $\tilde{f}_0$.

Recalling that a homotopy on X is a map on $X \times I \to Y$, we can imagine that $X \times I$ is a cylinder that we would like to map to Y, where we already have the map on $A \times I$ (a smaller cylinder contained inside) and on $X \times \{0\}$ (the base of the larger cylinder). Homotopy extension is then equivalent to the following: any map $(X \times \{0\}) \cup (A \times I) \to Y$ extends to a map $X \times I \to Y$.

Lemma 31

The homotopy extension property is equivalent to $(X \times \{0\}) \cup (A \times I)$ being a retraction of $X \times I$ (meaning that there is a map $r : X \times I \to X \times I$ such that $\text{im}(r) = (X \times \{0\}) \cup (A \times I)$, and such that $r|_{(X \times \{0\}) \cup (A \times I)}$ is the identity map on that space).

Proof. If a retraction r $X \times I \to (X \times \{0\}) \cup (A \times I)$ does exist, we can define $\tilde{f} = f \circ r$ to be the composition of the retraction and the map f, which gives us a map $X \times I \to Y$ and thus satisfying the homotopy extension property. On the other hand, if (X, A) satisfies the homotopy extension property, then we can extend the identity map on $(X \times \{0\}) \cup (A \times I)$ (that is, taking Y to be the space itself), and then the property yields the desired retraction r. □

In general, it's easier to construct a retraction and write the map explicitly, and that will automatically show homotopy extension directly.

Example 32

If (X, A) has the homotopy extension property, then so does $(X \times Y, A \times Y)$ for any Y (by constructing a retraction and then multiplying everything by Y).

Recall that both Proposition 19 and Proposition 27 involve CW complexes, and this is actually an important detail:

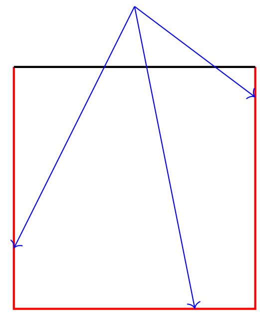
Proposition 33

If (X, A) is a CW pair, then (X, A) satisfies the homotopy extension property.

For example, we'll see that $(D^n, \partial D^n)$ satisfies the homotopy extension, because there is a deformation retraction $D^n \times I \to (D^n \times \{0\}) \cup (\partial D^n \times I)$. So we do have this property for a single cell.

Example 34

D^1 is a line segment, so $D^1 \times I$ is a square. Then we want to retract the square onto three of its edges, which we do by projecting downward as shown below.



Applying this to an arbitrary CW pair (X, A), we find that because X^n is obtained from $X^{n-1} \cup A^n$ by gluing n-cells, $X^n \times I$ is obtained from $(X^n \times \{0\}) \cup ((X^{n-1} \cup A^n) \times I)$ by gluing $D^n_\alpha \times I$ along $(D^n_\alpha \times \{0\}) \cup (\partial D^\alpha_n \times I$. So on each cell $D^n_\alpha \times I$, we have a retraction, and thus we can (deformation) retract the entire n-skeleton $X^n \times I$ to $(X^n \times \{0\}) \cup (X^{n-1} \cup A^n) \times I$.

3 October 4, 2022

Last lecture, we established two criteria for homotopy equivalence: if (X, A) is a CW-pair and A is contractible, then the quotient map $q : X \mapsto X/A$ is a homotopy equivalence, and if $f, g : A \to Y$ are homotopic maps, then the "glued spaces" $X \amalg_f Y \simeq X \amalg_g Y$ are homotopy equivalent as well. We proved last time that any (X, A) has the **homotopy extension property** – specifically, there is a retraction $X \times I \to (X \times \{0\}) \cup (A \times I)$, and the existence of such a retraction is sufficient, and for a CW pair we actually have a **deformation** retraction because we can do this cell by cell.

We can start today by proving the two propositions stated last lecture:

Proof of Proposition 27. By assumption, because f and g are homotopic, there is a map $H : A \times I \to Y$ such that $H(a, 0) = f(a)$ and $H(a, 1) = g(a)$ for all $a \in A$. If we then consider the space $(X \times I) \amalg_H Y$ (thinking of $A \times I$ as a subspace of $X \times I$), then applying the homotopy extension property allows us to perform a deformation retraction from $(X \times I) \amalg_H Y$ to $((X \times \{0\}) \cup (A \times I)) \amalg_H Y$ – in particular, since we have the identity map on $A \times I$ throughout, the gluing does not get affected. And $X \times \{0\}$ is only being glued to Y at $A \times \{0\}$, so the space we get here is actually $X \amalg_f Y$, because we're only gluing X to Y using $H(\cdot, 0) = f$.

But we may also perform the analogous deformation retraction to $((X \times \{1\}) \cup (A \times I)) \amalg_H Y$ (swapping the roles of 0 and 1). And this time, when we glue $A \times I$ to Y we are only gluing $X \times \{1\}$ to Y through $A \times \{1\}$, using the map $H(\cdot, 1) = g$. So we are deformation retracting to $X \amalg_g Y$ this time. So both $X \amalg_f Y$ and $X \amalg_g Y$ are homotopy equivalent to $(X \times I) \amalg_H Y$, and thus they are homotopy equivalent to each other. □

Proof of Proposition 19. Since A is contractible, there is a homotopy $f_t : A \to A$ such that f_0 is the identity on A and f_1 maps all of A to a single point $a_0 \in A$ (this is also called a **constant map**), which we must think of as a map $f_t : A \to X$. The identity map f_0 extends to all of X (it's still just the identity map), so by the homotopy extension property for the CW pair (X, A) we get a map $\tilde{f}_t : X \to X$ which extends f_t from A to X. In other words, we always have the dotted map making the diagram below commute for any $t \in [0, 1]$:

$$\begin{array}{ccc} X & \overset{\tilde{f}_t}{\dashrightarrow} & X \\ \iota\uparrow & & \iota\uparrow \\ A & \xrightarrow{f_t} & A \end{array}$$

But now $\tilde{f}_t(A)$ must be contained in A for all t (since we are extending the map $f_t : A \to A$, so $\tilde{f}_t$ descends to a map $\overline{f}_t : X/A \to X/A$, and this is well-defined.

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_t} & X \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ X/A & \xrightarrow{\overline{f}_t} & X/A \end{array}$$

(So if $x \in X \backslash A$ and $\tilde{f}_t(x) \in A$, then $\overline{f}_t(x)$ will be mapped to the point $[A]$ corresponding to all of A.) We have that $\overline{f}_0$ is the identity on X/A (since $\tilde{f}_t$ was the identity on X). On the other hand, $\tilde{f}_1(A) = \{a_0\}$, so we can actually construct a "diagonal" map in our diagram here specifically because all of A goes to a single point:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_1} & X \\ \downarrow{\scriptstyle q} & \overset{g}{\nearrow} & \downarrow{\scriptstyle q} \\ X/A & \xrightarrow[\overline{f}_1]{} & X/A \end{array}$$

(We should check explicitly that both triangles here actually commute.) So we have $g \circ q = \tilde{f}_1$, which is homotopic to $\tilde{f}_0 = \text{id}_X$. On the other hand, $q \circ g = \overline{f}_1$, which is homotopic to $\overline{f}_0 = \text{id}_{X/A}$. So the two compositions are homotopic to the corresponding identity maps, which is what we need for homotopy equivalence. (Importantly, this map g only necessarily exists at $t = 1$ because not all of A needs to be mapped to a single point until then.) $\square$

We now have many criteria for proving homotopy equivalence, but we may now want to ask for ways to prove that $X \not\simeq Y$ (for example, if $X = \mathbb{R}^2$ and $Y = \mathbb{R}^2 \backslash \{0\}$). Usually we cannot exhaust all possible maps $X \to Y$ as possibilities, so instead the strategy will be to **find invariants** $I(X)$ that are preserved under homotopy equivalence – such invariants may be numbers, polynomials, groups, or more complicated, as long as we can show that $I(X) = I(Y)$ whenever $X \simeq Y$. (There's also a follow-up question of whether there is a "complete set of invariants" that would allow us to also prove that $X \simeq Y$.) So the three chapters of Hatcher's book we'll go through will give us three examples of such invariants.

We'll start with the **fundamental group**. The idea will basically be that any space X with some base point $x_0 \in X$ is associated with a group $\pi_1(X, x_0)$, and for a large family of manifolds we can actually determine the space from this "fundamental" group. We'll also find that maps between spaces induce group homomorphisms between fundamental groups (which are the same if the maps are homotopic) – category theory abstracts all of this, and we might discuss this a little if there's time.

But for now, we'll start with definitions, and the basic ingredient here will be the **set of loops passing through** x_0.

Definition 35

We can think about loops in one of two ways – they're $f : [0, 1] \to X$ with $f(0) = f(1) = x_0$, and they're also equivalently maps $f : S^1 \to X$ such that $f((1, 0)) = x_0$ – and we'll use both throughout this class. The space of loops in X based at x_0 is denoted $\Omega(X, x_0) = \{f : [0, 1] \to X : f(0) = f(1) = x_0\}$.

Definition 36

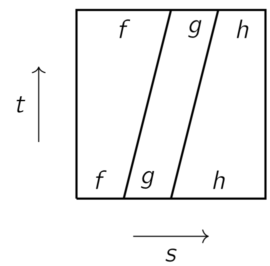
Let $f, g : [0, 1]_s \to X$ be two loops (based at x_0) with $f(0) = f(1) = g(0) = g(1) = x_0$. A **homotopy between based loops** f, g is a map $H : [0, 1]_t \times [0, 1]_s$ such that $H(0, s) = f(s)$, $H(1, s) = g(s)$, and $H(t, 0) = H(t, 1) = x_0$ for all $t \in [0, 1]$.

(This may also be denoted $f \simeq g \text{ rel } \{x_0\}$, and more generally we can have $f \simeq g \text{ rel } A$ if we want to fix a space A throughout the homotopy.) This last condition is basically saying that $H_t : [0,1] \to X$ is always a loop based at x_0 for any t, and it is in fact going to be a necessary additional assumption in some situations.

To define a fundamental group, we'll first recall that a **group** $(G, \cdot)$ is a set with a binary operation $\cdot : G \times G \to G$ which is associative, has an identity element e such that $eg = ge = g$ for all $g \in G$, and has inverses (so for any g there is an element h such that $gh = hg = e$). We may ask whether there's a way to construct such an operation on $\Omega(X, x_0)$. There is indeed a natural map given by **composition**: if f and g are two loops, then we can define $f \cdot g$ to be the loop that "first travels through f" during $[0, \frac{1}{2}]$ and "then travels through g" during $[\frac{1}{2}, 1]$:

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leqslant s \leqslant \frac{1}{2}, \\ g(2s-1) & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

However, this does not satisfy the group axioms that we want: we **don't have associativity** because $(f \cdot g) \cdot h$ has us traveling along h for the second half on $[0,1]$, but $f \cdot (g \cdot h)$ has us traveling along h only for the last quarter. (So even though we travel along the same loops, we do not have the same map $[0,1] \to X$.) To solve this, notice that $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are **homotopic** just by doing a time-change:

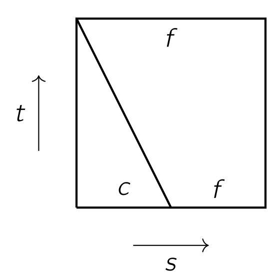


Explicitly, this means that one possible homotopy (there are many) is

$$H(t,s) = \begin{cases} f((4-2t)s) & 0 \leqslant s \leqslant \frac{1}{4-2t}, \\ g(4s) & \frac{1}{4-2t} \leqslant s \leqslant \frac{1}{4-2t} + \frac{1}{4}, \\ h(2+2t)s & \frac{1}{4-2t} + \frac{1}{4} \leqslant s \leqslant 1. \end{cases}$$

So this motivates studying **loops up to homotopy**: define the set $\pi_1(X, x_0) = \Omega(X, x_0)/\sim$, where $f \sim g$ if and only if $f \simeq g \text{ rel } \{x_0\}$.

The product formed by composition in $\pi_1(X, x_0)$ is then associative – it's an exercise to check that we can indeed do the composition and get a homotopic result independent of the elements we choose in the initial equivalence class. We also claim that the constant loop $c : [0,1] \to \{x_0\}$ is the identity element – indeed, we need to check that $c \cdot f \simeq f \simeq f \cdot c$ for any loop f. Indeed (for example for $c \cdot f \simeq f$) we can consider a diagram as follows:



(Explicitly, the computation is that $H(t,s) = \begin{cases} c(s) & 0 \leqslant s \leqslant \frac{1}{2} - \frac{1}{2}t, \\ f((2-t)s) & \frac{1}{2} - \frac{1}{2}t \leqslant s \leqslant 1. \end{cases}$) And to find inverses, we basically traverse our loops in reverse: if $f : [0, 1] \to X$ satisfies $f(0) = f(1) = x_0$, then we can define the loop $\overline{f} : [0, 1] \to X$ by $\overline{f}(s) = f(1 - s)$. And indeed, the idea is that instead of fully traversing f forward and $\overline{f}$ backward, at time t we only traverse a fraction t of the loop forward and then retrace our steps:

$$H(t, s) = \begin{cases} f((2-2t)s) & s \in [0, \frac{1}{2}], \\ \overline{f}((2-2t)s + 2t - 1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

So we do have a group operation on this set $\pi_1(X, x_0)$:

Definition 37

The **fundamental group** of (X, x_0) is the group $(\pi_1(X, x_0), \cdot)$ as defined in the discussion above.

Example 38

Suppose $X = \mathbb{R}^2$ and $x_0 = \{(0, 0)\}$. Then every loop is homotopic to the constant map, because we can construct a homotopy $H(t, s) = (1 - t)f(s) + (0, 0)$ between any based loop and the constant loop. Thus $\pi_1(\mathbb{R}^2, (0, 0))$ is the trivial group with only one element $\{[c]\}$.

We may now ask whether the choice of base point matters in defining $\pi_1(X, x_0)$. If we assume X is path-connected (to avoid any silly examples where loops are restricted to different components), it turns out that the groups will be isomorphic:

Proposition 39

For any two base points x_0, x_1 in a path-connected space X, we have the group isomorphism $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.

Proof. Since X is path-connected, there is a map $h : [0, 1]$ such that $h(0) = x_0$ and $h(1) = x_1$ (this is a path, not a loop). Let $\overline{h}$ be its reverse (so that $\overline{h}(0) = x_1$ and $\overline{h}(1) = x_0$). Notice that this path induces a map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$, because for any loop $f \in \Omega(X, x_1)$, the map $h \cdot f \cdot \overline{h}$ (first traveling along h, then following f, then reversing h) gets us a map in $\Omega(X, x_0)$.

It can be checked (exercise) that this is well-defined (meaning that the result only depends on the homotopy class of h rel $\{x_0, x_1\}$) and that this is a group homomorphism. So we have the map $\beta_h([f]) = [h \cdot f \cdot \overline{h}]$, and we have an explicit inverse $\beta_{\overline{h}} : \pi_1(X, x_0) \to \pi_1(X, x_1)$. We can check that $\beta_h \circ \beta_{\overline{h}}$ is the identity on $\pi_1(X, x_0)$, and $\beta_{\overline{h}} \circ \beta_h$ is the identity on $\pi_1(X, x_1)$, giving us the desired group isomorphism. □

Thus if X is path-connected, we can just denote the fundamental group $\pi_1(X)$ and pick a base point for convenience. And next lecture, we'll explore how to study the fundamental groups of two spaces (X, x_0) and (Y, y_0) given properties of X and Y.

4 October 6, 2022

Last lecture, we proved the homotopy equivalence propositions and introduced the fundamental group $\pi_1(X, x_0)$. We showed that for any path h from x_0 to x_1 we have $\pi_1(X, x_1) \cong \pi_1(X, x_0)$ (through the map β_h mapping loops rooted at x_1 to loops rooted at x_0 described in Proposition 39), so the base point itself can be chosen for convenience. Today, we'll start asking how to relate the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ by looking at continuous maps $X \to Y$.

Theorem 40

Let $\phi : (X, x_0) \to (Y, y_0)$ be a continuous map. Then we have a group homomorphism $\phi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ which sends the equivalence class of $f : [0, 1] \to X$ to the equivalence class of $\phi \circ f : [0, 1] \to Y$.

We need to check that this is actually well-defined and a group homomorphism, but we won't go through the details in lecture here. Instead, we'll mention that this induced map behaves nicely even when we change the base point or compare homotopic maps:

1. Suppose that we have two different base points x_0, x_1 mapping to y_0, y_1 under ϕ. Then we can get the following diagram (with β_h and $\beta_{\phi h}$ again coming from the isomorphism in Proposition 39):

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\phi_*} & \pi_1(Y, y_0) \\ \big\downarrow{\scriptstyle \beta_h} & & \big\downarrow{\scriptstyle \beta_{\phi h}} \\ \pi_1(X, x_1) & \xrightarrow{\phi_*} & \pi_1(Y, y_1) \end{array}$$

   Checking that this is actually a commutative diagram is left as an exercise to us as well.

2. Next, suppose we have two homotopic maps $\phi \simeq \psi$ and want to see if we can relate ϕ_* and ψ_*. One complication is that the corresponding base points $\phi(x_0) = y_0$ and $\psi(x_0) = y_1$ may not be the same, but we want to find a path h that makes the following diagram commute:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\phi_*} & \pi_1(Y, y_0) \\ \big\downarrow{\scriptstyle \psi_*} & \overset{\beta_h}{\nearrow} & \\ \pi_1(Y, y_1) & & \end{array}$$

   In order for such an h to exist, it's important here that ϕ and ψ are homotopic. That means that there is a map $\eta_t : X \to Y$ such that $\eta_0 = \phi$ and $\eta_1 = \psi$, so a natural path between $\eta_0(x_0) = y_0$ and $\eta_1(x_0) = y_1$ is the path $h(s) = \eta_s(x_0)$. (So because x_0's image is shifting from y_0 to y_1 continuously during the homotopy, we should just use that as our path.) Then we can check as an exercise that this is again a commutative diagram.

3. Finally, suppose we have maps $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ which send $x_0 \to y_0 \to z_0$. Then we get a corresponding composition $\pi_1(X, x_0) \xrightarrow{\phi_*} \pi_1(Y, y_0) \xrightarrow{\psi_*} \pi_1(Z, z_0)$, but we can also look at the map $\psi \circ \phi : X \to Z$ and thus get a map $\pi_1(X, x_0) \xrightarrow{(\psi \circ \phi)_*} \pi_1(Z, z_0)$ directly. These maps are the same (by unpacking what happens in each case to an equivalence class of loops $[f]$), so $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Theorem 41

Suppose $X \simeq Y$ are homotopy equivalent, X and Y are path-connected, and we pick (an arbitrary) $x_0 \in X, y_0 \in Y$. Then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

(More generally, we can pick y_0 to be any point in the same path component as $\phi(x_0)$. But when we have fundamental groups rooted at a point x_0, the points in X not in the path component of x_0 don't matter anyway.)

Proof. Since $X \simeq Y$, there are maps $\phi : X \to Y$ and $\psi : Y \to X$ such that $\phi \circ \psi \simeq \text{id}_Y$ and $\psi \circ \phi \simeq \text{id}_X$. Thus by the composition property (3) above,

$$\psi \circ \phi \simeq \text{id}_X \implies \phi_* \circ \psi_* = (\text{id}_X)_* \circ \beta_h = \beta_h$$

(where the second-to-last step is because homotopic maps are different by a base-change β_h (property (2) above), and the last step is because the identity map gives us the identity group homomorphism on the fundamental groups). So we have an isomorphism $\beta_h = \phi_* \circ \psi_*$ and similarly an isomorphism $\beta_{h'} = \phi_* \circ \psi_*$, which implies that ϕ_* and ψ_* are both isomorphisms (since they must both be injective and surjective) betwen the fundamental groups. $\square$

Example 42

If X is contractible, that's equivalent to saying that it is homotopy equivalent to a one-point space, which only contains the trivial loop. Thus the fundamental group of a contractible space (such as $\mathbb{R}^2$) is also trivial.

On the other hand, just because the fundamental group is trivial does not mean the space is contractible – the n-sphere S^n (for $n \geqslant 2$) is not contractible but has trivial fundamental group, because we can shrink any loop to a point (for example by stereographic projection from a point not on the loop, giving us $\mathbb{R}^2$) but S^n is non-contractible (this can be seen for example through homology, which we'll discuss later in the course).

Example 43

Recall that $Y = \mathbb{R}^2 - \{(0,0)\}$ has a projection onto the unit sphere $S^1 : \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ given by a deformation retraction, so $\pi_1(\mathbb{R}^2 - \{(0,0)\}, (1,0))$ is isomorphic to $\pi_1(S^1, (1,0))$.

In the next few lectures, we'll develop some techniques for computing fundamental groups in general (such as the van Kampen theorem and covering spaces). But for today, we'll work towards computing $\pi_1(S^1)$. For any integer n, we define the loop $\omega_n : [0,1] \to S^1$ by setting

$$\omega_n(s) = (\cos(2n\pi s), \sin(2n\pi s)) \in S^1.$$

Geometrically, these loops are basically traveling around the sphere n times counterclockwise at a constant rate, and we give them a group structure by saying that $[\omega_n][\omega_m] = [\omega_{m+n}]$.

Theorem 44

There is a group isomorphism $\Phi : \mathbb{Z} \to \pi_1(S^1, (1,0))$ sending n to ω_n.

In other words, every loop is homotopic to exactly one of the ω_n maps. We'll first show some applications of this result before proving it:

Corollary 45 (Brouwer fixed point theorem)

Any (continuous) map $f : D^2 \to D^2$ (where D^2 is the unit disk $\{(x,y) : x^2 + y^2 \leqslant 1\}$) has some fixed point $x \in D^2$ with $f(x) = x$.

Proof. Suppose otherwise, so that there is some function f with $f(x) \neq x$ for all $x \in D^2$. Then we can construct a retraction $r : D^2 \to S^1$ in the following way: for any x, draw a ray from $f(x)$ towards x, and let $r(x)$ be the intersection of that ray with S^1. We can check that there is always one such intersection point, and in particular if x is on the boundary then $r(x) = x$. But then the composition $S^1 \xrightarrow{\iota} D^2 \xrightarrow{r} S^1$ is the identity on S^1, so this composition should also give us an induced map on the fundamental groups $r_* \circ \iota_* = (\mathrm{id}_{S^1})_* = \mathrm{id}$, since the base point $(1,0)$ is unchanged. But $\pi_1(S^1,(1,0)) \xrightarrow{\iota_*} \pi_1(D^2,(1,0)) \xrightarrow{r_*} \pi_1(S^1,(1,0))$ is a map $\mathbb{Z} \xrightarrow{\iota_*} 0 \xrightarrow{r_*} \mathbb{Z}$ (since D^2 is contractible). This composition must then be both the zero and identity map, which is a contradiction. $\square$

Corollary 46

Let $h : S^1 \to S^1$ be a map such that $h(x) = -h(-x)$ for all x. Then h is not nullhomotopic (in other words, h is not homotopic to the constant map).

The idea is that we can rotate the map h so that $h(1,0) = (1,0)$, so we can think of h as a loop on S^1. Then we wish to show that $[h] \neq [0] \in \pi_1(S^1,(1,0))$. But we'll postpone the proof of this for a bit – the idea is related to the proof that $\pi_1(S^1,(1,0)) \cong \mathbb{Z}$.

Corollary 47 (Borsuk-Ulam)

Let $f : S^2 \to \mathbb{R}^2$ be any map. Then there is some point $x \in S^2$ such that $f(x) = f(-x)$.

(One interpretation of this result is that there are always two antipodal points on Earth with the same temperature and pressure.)

Proof. Suppose otherwise, so that there is some map $f : S^2 \to \mathbb{R}^2$ with $f(x) \neq f(-x)$ for all $x \in S^2$. Then we can define a new map $g : S^2 \to S^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{||f(x) - f(-x)||},$$

and we can look at the image of the equator $S^1 \subset S^2$ under g. Letting $\iota : S^1 \to S^2$ be the inclusion of the equator in S^2, we get a map $h = g \circ \iota : S^1 \to S^1$. But notice that

$$h(-x) = \frac{f(-x) - f(x)}{||f(-x) - f(x)||} = -h(x),$$

and by Corollary 46 h is not homotopic to the constant loop. But $h : S^1 \to S^2 \to S^1$ factors through S^2, and any loop is null-homotopic in S^2, which is a contradiction (again the map $\mathbb{Z} \to 0 \to \mathbb{Z}$ cannot have a nonzero image). □

The two remaining things left to prove – that $\Phi : \mathbb{Z} \to \pi_1(S^1,(1,0))$ is an isomorphism and that we cannot have a nullhomotopic map $h : S^1 \to S^1$ with $h(x) = -h(-x)$ – are both proved using **covering spaces**. To motivate that, recall that S^1 is the unit sphere $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, so we can define a map $q : \mathbb{R} \to S^1$ given by

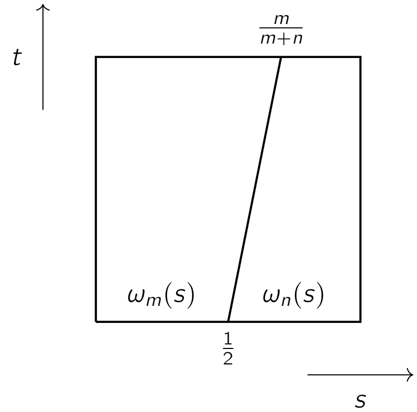
$$q(s) = (\cos(2\pi s), \sin(2\pi s)).$$

We then have $q(n) = (1,0)$ for all $n \in \mathbb{Z}$, and we can imagine that this map loops around the circle once in every unit interval. So this means that any map $[0,1] \to S^1$ can be lifted to a map $[0,1] \to \mathbb{R}$ ($\mathbb{R}$ is called the **universal cover** of S^1), and it turns out such a lift will be unique up to some conditions, which we'll see in subsequent lectures.

5 October 11, 2022

Last lecture, we stated (in Theorem 44) that the fundamental group of S^1 is $\mathbb{Z}$. In other words, we claimed that there is an isomorphism $\Phi : \mathbb{Z} \to \pi_1(S^1,(1,0))$ which sends n to $[\omega_n(s) = (\cos(2n\pi s), \sin(2n\pi s))]$, the loop which goes around S^1 n times counterclockwise.

We will prove this in three steps, showing that Φ is a group homomorphism, that it is surjective, and that it is injective. Step 1 can be verified directly: the composition of the loops ω_m and ω_n is indeed homotopic to the loop ω_{m+n}, as represented in the diagram below (writing down the detailed expression is an exercise for us):



For the second step, we will make use of the map $p : \mathbb{R} \to S^1$ mapping s to $\cos(2\pi s, \sin 2\pi s)$. The point is that any loop $f : [0, 1] \to S^1$ will be lifted to a path $\tilde{f} : [0, 1] \to \mathbb{R}$, because we can figure out more easily what's going on with the homotopy on $\mathbb{R}$. Indeed, any two paths $\tilde{f}, \tilde{g} : [0, 1] \to \mathbb{R}$ with the same start and end point $\tilde{x}_0$ and $\tilde{x}_1$ are homotopic on $\mathbb{R}$ (for example by linear interpolation), so they must descend to homotopic loops on S^1 rel $\{x_0, x_1\}$.

But we'll need to figure out how to ensure that such a lift exists, and it's important that we chose the map p specifically because for any $(a, b) \in \mathbb{R}$ with $b - a \leqslant 1$, we get a **homeomorphism** $p : (a, b) \to p(a, b)$ (in other words, any small enough open interval is mapped to an open interval on S^1.), and we'll use this property to lift the path.

Fact 48 (Covering property of S^1)

For any $x \in S^1$, there is a neighborhood (open set) $U \subseteq S^1$ containing x such that any connected component V of its preimage $p^{-1}(U) \subseteq \mathbb{R}$ yields a homeomorphism $p|_V : V \to U$.

Geometrically, if we imagine a small neighborhood U in S^1, its preimage is a disjoint union of a bunch of intervals, and any of these intervals projects down to U. (This will become a general property when we look at more general **covering spaces** soon.)

Proposition 49

Fix any path $f : [0, 1] \to S^1$, and fix any point $\tilde{x}_0$ such that $p(\tilde{x}_0) = f(0)$. Then there is a unique path $\tilde{f} : [0, 1] \to \mathbb{R}$ such that $\tilde{f}(0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$ (that is, f lifts to $\tilde{f}$).

Proof. For any point $x \in S^1$ there is some neighborhood U_x of x satisfying the covering property. Since S^1 is compact and the set of U_xs covers it, we can cover S^1 by **finitely** many such neighborhoods. Thus we see that $f[0, 1] \subseteq U_0 \cup \cdots \cup U_n$, where each $U_i \in S^1$ satisfies the covering property, and where we assume that $x_0 \in U_0$. Let V_0 be the component of $p^{-1}(U_0)$ which contains $\tilde{x}_0$. Because we have a homeomorphism between U_0 and V_0, we can locally lift f uniquely to $\tilde{f} : [0, \varepsilon) \to V_0$ (by composing with p^{-1}). Now U_0 intersects some other interval U_1, so the same argument shows that we can extend f to $U_0 \cup U_1$. Repeating this eventually lets us extend to $U_0 \cup \cdots \cup U_n$ by patching these extensions together.

And uniqueness follows because locally p is a one-to-one map, so there are no other paths possible besides the one that we constructed. □

Remark 50. *More precisely (to avoid some of the issues with infinitely switching between the U_is), for every point $s \in [0, 1]$ we have a small neighborhood $(a_s, b_s) \in [0, 1]$ so that $f(a_s, b_s) \subseteq U \subseteq S^1$. Then we can use those neighborhoods in our compactness instead.*

But now returning to our goal, recall that what we care about is loops in S^1, and the next step we wanted to perform is to show that $\Phi : \mathbb{Z} \to \pi_1(S^1, (1, 0))$ is **surjective**. To do this, consider any loop $f : [0, 1] \to S^1$ based at $(1, 0)$, and choose $\tilde{x}_0 = 0 \in \mathbb{R}$. The lifting property then tells us that we have a path $\tilde{f} : [0, 1] \to \mathbb{R}$ with $\tilde{f}(0) = 0$ and $p \circ \tilde{f} = f$. Then $\tilde{f}(1)$ should be in the preimage of $p^{-1}(1, 0) = \mathbb{Z}$, so our path ends at some integer on $\mathbb{R}$. There is indeed a path from 0 to n, namely the linear path $\tilde{\omega}_n(s) = ns$. But $\tilde{f}$ and $\tilde{\omega}_n$ both have the same starting and ending points, so they are homotopic rel $\{0, n\}$ on $\mathbb{R}$, and thus (composing with p) they descend to homotopic maps on S^1 rel $\{(1, 0)\}$. But $\tilde{w}_n$ gives us the loop ω_n, and $\tilde{f}$ gives us back our original f. Thus $[f] = [\omega_n] \in \text{im}\Phi$ for some n. This means that the map $\mathbb{Z} \to \Pi_1(S^1, (1, 0))$ sending $n \to [\omega_n(s)]$ is surjective because any element of $\Pi_1(S^1, (1, 0))$ is some class $[\omega_n(s)]$.

On the other hand, we can also prove that $\Phi : \mathbb{Z} \to \pi_1(S^1, (1, 0))$ is **injective**, meaning that we wish to show that ω_n and ω_m are not homotopic rel $\{(1, 0)\}$ for any $m \neq n$. For this, we'll make use of another useful property:

Fact 51 (Homotopy lifting for S^1)

If $f, g : [0, 1] \to S^1$ have the same endpoints $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$, and $f \simeq g$ rel$\{x_0, x_1\}$, let $H : [0, 1]_t \times [0, 1]_s \to S^1$ be such a homotopy (meaning that $H(t, 0) = x_0, H(t, 1) = x_1$, $H(0, s) = f(s)$, and $H(1, s) = g(s)$ for all $s, t \in [0, 1]$). Then if we fix any $\tilde{x}_0 \in p^{-1}(x_0)$, H lifts to a (unique) map $\tilde{H} : [0, 1]_t \times [0, 1]_s \to \mathbb{R}$, such that $p \circ \tilde{H} = H$ and $\tilde{H}(t, 0) = \tilde{x}_0$.

The proof here is very similar to our previous argument – we lift the square piece by piece by compactness instead of lifting the circle. And now we can prove **injectivity**: if we write $\tilde{f}(s) = \tilde{H}(0, s)$ and $\tilde{g}(s) = \tilde{H}(1, s)$, we have $p \circ \tilde{f} = f$ and $p \circ \tilde{g} = g$ (just by restricting $p \circ \tilde{H} = H$ to only part of the domain), so we see that $\tilde{H}$ is a homotopy between the lifts of f and g. Additionally, if we define $\tilde{h} : [0, 1] \to \mathbb{R}$ via $\tilde{h}(t) = \tilde{H}(t, 1)$, we get a path from $\tilde{f}(1)$ to $\tilde{g}(1)$ which descends to $h(t) = H(t, 1)$, the constant path at $(1, 0)$. But by **uniqueness of path lifting**, that means $\tilde{h}$ must be a constant, so $\tilde{f}(1)$ and $\tilde{g}(1)$ must be the same point. But ω_m and ω_n lift to paths from 0 to m and 0 to n, so they can only be homotopic if $m = n$.

Notice that all we've really used is the local homeomorphism covering property – it's not important that we just work with paths into S^1 or even with S^1 and $\mathbb{R}$ in particular. So that's what we'll be generalizing now:

Definition 52

Let $X, \tilde{X}$ be two spaces. A map $p : \tilde{X} \to X$ is a **covering map** if for any $x \in X$, there is a neighborhood U of x such that for any connected component V of $p^{-1}(U)$, the restriction $p|_V : V \to U$ is a homeomorphism. If this holds, then $\tilde{X}$ is a **covering space** of X.

(We require a connected component here because we want to be able to uniquely lift maps into X to maps into $\tilde{X}$.)

Theorem 53 (General lifting property)

Let $p : \tilde{X} \to X$ be a covering map, Y be an arbitrary space, and let $f_t : Y \to X$ be a homotopy of maps such that f_0 lifts to $\tilde{f}_0 : Y \to \tilde{X}$. Then there is a unique lift $\tilde{f}_t : Y \to \tilde{X}$ for all $t \in [0, 1]$, meaning that $p \circ \tilde{f}_t = f_t$.

In particular, if Y is a single point, this gives us the path lifting property, and if Y is the interval $[0, 1]$, we get the homotopy lifting property.

Theorem 54

Let $p : \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. Then the corresponding map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof. Suppose we have two loops $\tilde{f}, \tilde{g} : [0,1] \to \tilde{X}$ with $\tilde{f}(0) = \tilde{f}(1) = \tilde{g}(0) = \tilde{g}(1) = x_0$. Then if $p_*[\tilde{f}] = p_*[\tilde{g}]$, that means that $p \circ \tilde{f} \simeq p \circ \tilde{g}$ rel$\{x_0\}$. But by homotopy lifting this means that $\tilde{f} \simeq \tilde{g}$ rel$\{x_0\}$, so $[\tilde{f}] = [\tilde{g}]$. □

Next, we'll mention a necessary condition for being able to lift a **general** map $f : Y \to X$ to a map $\tilde{f} : Y \to \tilde{X}$. If such a lift exists, then $f = p \circ \tilde{f}$ means that $f_* = p_* \circ \tilde{f}_*$, meaning that we must have the condition $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)$. But it turns out this condition is also sufficient:

Theorem 55

Let $p : \tilde{X} \to X$ be a covering map with $p(\tilde{x}_0) = x_0$. Then if Y is path-connected and locally-path-connected, and $f : Y \to X$ is a map such that $f(y_0) = x_0$ and $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, then f lifts to a map $\tilde{f} : Y \to \tilde{X}$ with $\tilde{f}(x_0) = \tilde{x}_0$.

6 October 13, 2022

Last lecture, we introduced the concept of a covering space and its lifting property. Specifically, a **covering map** is a map $p : \tilde{X} \to X$ where any $x \in X$ has a neighborhood U, such that for any component of the preimage $V \subset p^{-1}(U)$, we have a homeomorphism $p|_V : V \to U$. Such a construction is useful because it allows us to lift paths and homotopies from X to $\tilde{X}$ (Theorem 55): as long as Y is path-connected and locally-path-connected, the relation $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ between the fundamental groups is equivalent to the existence of an extension $\tilde{f} : Y \to \tilde{X}$.

Proof of Theorem 55. For one direction, if a lift $\tilde{f}$ exists, then we have $f = p \circ \tilde{f} \implies f_* = p_* \circ \tilde{f}_*$, which means the image of f_* is contained in the image of p_*.

The other direction is more substantial: suppose $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$, and we want to define the map $\tilde{f}$. We know that $\tilde{f}$ should preserve the base point, so we must define $\tilde{f}(y_0) = \tilde{x}_0$. Now for any $y_1 \in Y$, by path-connectedness, there is some path $h : [0,1] \to Y$ such that $h(0) = y_0$ and $h(1) = y_1$. Then $f \circ h$ is a path in X with $f \circ h(0) = x_0$ and $f \circ h(1) = x_1$, and by the path lifting property $f \circ h$ lifts uniquely to a path $f \tilde{\circ} h : [0,1] \to \tilde{X}$ if we fix the starting point $\tilde{x}_0$. Thus we can define $\tilde{f}(y_1) = f \tilde{\circ} h(1) = \tilde{x}_1$.

To make sure $\tilde{f}$ is well-defined, we must show that it does not depend on the path $h : [0,1] \to Y$ that we chose. Suppose there is some other path $h' : [0,1] \to Y$ also with $h'(0) = y_0$ and $h'(1) = y_1$, so that $f \circ h' : [0,1] \to X$ is a path in X with $f \circ h'(0) = x_0$ and $f \circ h'(1) = x_1$. We must ensure that the lift of this new path $f \tilde{\circ} h'$ also ends at $\tilde{x}_1$. Notice that $\gamma = h' \circ \overline{h}$ is a loop in Y starting and ending at y_0, which means $f \circ \gamma = (f \circ h) \circ (\overline{f \circ h})$ is a loop in X starting and ending at x_0. Since $[\gamma] \in \pi_1(Y, y_0)$, $f_*([\gamma]) \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, x_0))$, meaning that there is some loop $\tilde{\delta}$ on $\tilde{X}$ starting and ending at $\tilde{x}_0$ such that $f_*([\gamma]) = p_*([\delta])$. This means that $f \circ \gamma$ and $p \circ \tilde{\delta}$ have the same homotopy class in X rel $\{x_0\}$. By the homotopy extension property of covering spaces, we thus get a homotopy between $f \tilde{\circ} \gamma$ and $\tilde{\delta}$ rel $\{\tilde{x}_0\}$ (since $\tilde{\delta}$ is a lift of $p \circ \tilde{\delta}$ and lifts are unique). But this means that $f \tilde{\circ} \gamma$ is homotopic to a loop and thus must be a loop itself, and that can only happen if the extension of h' ends at the same point as the extension of h. (Throughout this argument, we have crucially used that lifts are unique in a few spots.)

Finally, we need to make sure that $\tilde{f}$ is actually continuous (since we defined it point-by-point), and this is where locally-path-connectedness matters (which basically means for any point and any open neighborhood of that point, there is a smaller neighborhood which is path-connected). We should read the details of that part of the argument in our textbook. □

The next question is to ask when (and how) such an extension map can be unique:

Proposition 56

Suppose $p : \tilde{X} \to X$ is a covering map such that $f : Y \to X$ has two extensions $f_1, f_2 : Y \to \tilde{X}$, meaning that $p \circ \tilde{f}_1 = p \circ \tilde{f}_2 = f$. If Y is connected (not necessarily path-connected) and there is some point y_0 such that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$, then $\tilde{f}_1 = \tilde{f}_2$ on all of Y.

Proof sketch. Suppose $\tilde{f}_1(y_0) = \tilde{f}_2(y_0) = \tilde{x}_0$. By definition, there is some neighborhood U around $x_0 = p(\tilde{x}_0)$ such that p is a local homeomorphism on that neighborhood. Thus $\tilde{f}_1$ and $\tilde{f}_2$ must coincide locally on all of $f^{-1}(U)$. Thus the set $S = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\}$ is open, and so is the set $\{y \in Y : \{y \in Y : \tilde{f}_1(y) \neq \tilde{f}_2(y)\}$ (because $\tilde{f}_1$ and $\tilde{f}_2$ can be contained in disjoint neighborhoods). Thus $Y - S$ is also open, meaning S is closed, and by connectedness this means S must be the whole space. □

We'll come back to covering spaces next week, but for now we'll cover some other important techniques for computing fundamental groups:

Definition 57

Let G and H be two groups. The **free product** of G and H, denoted $G * H$, is the set of words $\{g_1 g_2 \cdots g_n\}$, where each g_i is an element of G or an element of H, quotienting by the relations that if $g_i g_{i+1}$ are from the same group, then $g_1 \cdots g_i g_{i+1} \cdots g_n = g_1 \cdots g_{i-1}(g_i g_{i+1}) g_{i+1} \cdots g_n$.

In other words, we can simplify our words by using the group multiplication of G and of H separately, but there are no relations between elements of G and H. We can also explain this concept using group presentations: we let

$$G = \langle g_\alpha : \alpha \in I | r_\beta : \beta \in I' \rangle, \quad H = \left\langle g'_\alpha : \alpha \in J \middle| r'_\beta : \beta \in J' \right\rangle,$$

where the g_αs are generators of the groups and r_βs are relations. Then the free product is given by

$$G * H = \left\langle g_\alpha, g'_\alpha : \alpha \in I \cup J \middle| r_\beta, r'_\beta : \beta \in I' \cup J' \right\rangle.$$

Theorem 58 (Von Kampen)

Let X be a topological space, and let A_α be an open cover of X (meaning that each A_α is open and $\bigcup A_\alpha = X$.) Suppose there is some $x_0 \in \bigcap_{\alpha \in I} A_\alpha \subseteq X$, and $A_\alpha \cap A_\beta$ is path-connected for all α, β. Then there is a natural map $\Phi : *_{\alpha \in I} \pi_1(A_\alpha, x_0) \to \pi_1(X)$ which sends any word $(g_1, g_2, \cdots, g_n)$ to $g_1 g_2 \cdots g_n$, which is surjective. Furthermore, for $[\gamma] \in \pi_1(A_\alpha \cap A_\beta, x_0)$, then we can define (by inclusion) $i_\alpha[\gamma] \in \pi_1(A_\alpha, x_0)$ and $i_\beta[\gamma] \in \pi_1(A_\beta, x_0)$. Then if $A_\alpha \cap A_\beta \cap A_\gamma$ is also path-connected for any α, β, γ, the kernel of Φ is generated by elements $i_\alpha[\gamma](i_\beta[\gamma])^{-1}$ for all α, β, γ.

We can read the proof of this result on our own – being able to use it in applications is more important. The algebraic way of saying this is that the fundamental group of X is actually a **pushout** coming from the inclusion maps $A_1 \cap A_2 \to A_1$ and $A_1 \cap A_2 \to A_2$.

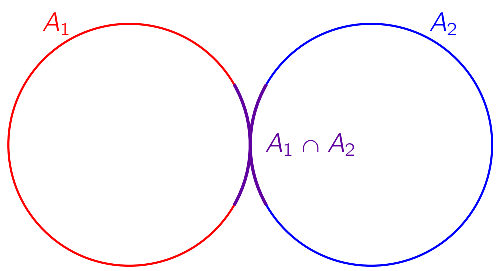
Example 59

Suppose $X = A_1 \cup A_2$, with $x_0 \in A_1 \cap A_2$ and $A_1 \cap A_2$ path-connected. Then if $\pi_1(A_1, x_0)$ has group presentation $\langle q_1, \cdots, q_n | r_1, \cdots, r_u \rangle$, $\pi_1(A_2, x_0)$ has group presentation $\langle f_1, \cdots, f_m | s_1, \cdots, s_v \rangle$, and $\pi_1(A_1 \cap A_2, x_0)$ has group presentation $\langle e_1, \cdots, e_\ell | t_1, \cdots, t_w \rangle$. Then by Von Kampen, we have the group presentation

$$\pi_1(X, x_0) = \langle g_1, \cdots, g_n, f_1, \cdots, f_m | r_1, \cdots, r_u, s_1, \cdots, s_v, i_{1*}(e_1) = i_{2*}(e_1), \cdots, i_{1*}(e_\ell) = i_{2*}(e_\ell) \rangle.$$

Example 60

Consider the wedge $S^1 \vee S^1$; we'll apply Van Kampen's theorem to the diagram below, where the red and purple regions form A_1 and the blue and purple regions form A_2.



By Van Kampen, we know that $\pi_1(X) = \pi_1(A_1) * \pi_1(A_2)/\ker \Phi$. But A_1 and A_2 are homotopy equivalent to circles, and the intersection is contractible and thus provides no new relations. This means the fundamental group of X is $\mathbb{Z} * \mathbb{Z}$.

Example 61

Now suppose $X = S^2$, and let A_1 and A_2 be the upper and lower hemispheres of S^2 extended a bit (so that $A_1 \cap A_2$ is homotopy equivalent to the equator – we call this a **collar** of the equator). But the fundamental group of A_1 and A_2 are each trivial because they are open disks, so we indeed see that $\pi_1(S^2)$ is trivial (we don't even need to consider the quotient).

A similar argument also shows that $\pi_1(S^n)$ is trivial for any $n \geqslant 2$. But we cannot make an analogous argument for S^1 because we need $A_1 \cap A_2$ to be path-connected.

Example 62

Suppose we have a CW complex $X = \cup_n X^n$, where X^n is the n-skeleton of X. We wish to understand $\pi_1(X)$ based explicitly on how cells are attached to form X.

The answer turns out to be yes, but we'll do the argument in parts:

Proposition 63

Attaching an n-cell for $n \geqslant 3$ does not change the fundamental group of X.

Proof. Recall that n-cells are added to X via an attaching map $\phi : \partial D^n \to X^{n-1}$ and use ϕ to glue X^{n-1} to D^n along the boundary. But D^n can be thought of as a mapping cone CS^{n-1}, so $D^n \cong [0, 1] \times S^{n-1}/(\{1\} \times S^{n-1})$. But now we can decompose $X = X^{n-1} \amalg_\phi D^n$ into two parts, where $A^1 = (\frac{1}{3}, 1) \times S^{n-1}/(\{1\} \times S^{n-1})$ is only the "tip" of the mapping cone" and $A^2 = X \amalg_\phi (0, \frac{2}{3}) \times S^{n-1}$. Their intersection is a collar (from $\frac{1}{3}$ to $\frac{2}{3}$) of S^{n-1}, and thus

$$\pi_1(X, x_0) = \pi_1(A_1, x_0) * \pi_1(A_2, x_0)/ \ker \Phi.$$

But the fundamental group of A_1 is trivial (because it is a cone and thus contractible when $n \geqslant 3$), while the fundamental group of A_2 is just $\pi_1(X)$ because we can deformation retract the attached cylinder back to X^{n-1}. Finally, $\pi_1(A_1 \cap A_2, X_0) = \pi_1(S^{n-1})$ is also trivial for $n \geqslant 3$. Thus the fundamental groups of X^{n-1} and of $X^{n-1} \amalg_\phi D^n$ are the same, and the attachment did not change the fundamental group. □

Corollary 64

For any CW complex X with 2-skeleton X^2, we have $\pi_1(X^2) \cong \pi_1(X)$.

We'll understand how to study the attachment of 2-cells and 1-cells later on!

7 October 18, 2022

We'll discuss **covering spaces** in more detail today. Recall that a covering space of X is a space $\tilde{X}$ with a covering map p, such as $p : \mathbb{R} \to S^1$ sending s to $(\cos(2\pi s), \sin(2\pi s))$; we in particular used this space to compute $\pi(S^1)$ because $\mathbb{R}$ is contractible. $\mathbb{R}$ plays a special role for S^1 in that it is a "universal cover:"

Definition 65

Suppose $\tilde{X}$ is a covering space of X which is **simply connected** (meaning that it is path connected and its fundamental group is trivial). Then we call $\tilde{X}$ the **universal cover** of X.

We'll see later that "universal" refers to the universal property that any other covering space of X is also covered by $\tilde{X}$ – in particular, this gives us uniqueness of the universal cover up to homeomorphism, if it exists – but the characterization as a simply connected space is what we'll care about here, because it helps us compute $\pi_1(X)$.

We'll first try to figure out when universal covers do exist, and we'll start with a necessary condition. We'll focus our attention on path-connected and locally-path-connected spaces here – suppose we have a universal cover $\tilde{X}$, so that there is a map $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$, where $\tilde{x}_0$ is any preimage of x_0. Because we have a covering space, we can find a small neighborhood U of $x_0 \in X$ and a small neighborhood V of $\tilde{x}_0 \in \tilde{X}$ so that $p : V \to U$ is a homeomorphism. We thus get the following commutative diagram, where i, j are inclusions:

$$\begin{array}{ccc} (V, \tilde{x}_0) & \xrightarrow{p} & (U, x_0) \\ \downarrow i & & \downarrow j \\ (\tilde{X}, \tilde{x}_0) & \xrightarrow{p} & (X, x_0) \end{array}$$

This gives us a corresponding map between fundamental groups as shown:

$$\begin{array}{ccc} \pi_1(V, \tilde{x}_0) & \xrightarrow{p_*} & \pi_1(U, x_0) \\ \downarrow i_* & & \downarrow j_* \\ \pi_1(\tilde{X}, \tilde{x}_0) & \xrightarrow{p_*} & \pi_1(X, x_0) \end{array}$$

But because $\tilde{X}$ is a universal cover, its fundamental group is trivial, and the p_* map on the top is an isomorphism by definition of U and V because we have a homeomorphism $V \to U$. Thus, j_* **must have trivial image** for any arbitrary base point x_0:

Definition 66

A space X is **semi-locally simply connected** if for any $x \in X$, there is a neighborhood U of x such that $j_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial.

For example, consider an annular neighborhood of the origin U in $\mathbb{R}^2$. Then $\pi_1(U, x)$ is nontrivial, but any loop is trivial when mapped into $\pi_1(X, x)$. In this case we could have chosen a simpler neighborhood, but there are spaces that are semi-locally simply connected but not simply connected.

Fact 67

There is also a notion of being **locally simply connected**, where for any $x \in X$ there is a neighborhood U of x which is simply connected. In particular, this is a stronger assumption than being semi-locally simply connected (because it means $\pi_1(U, x)$ is always trivial so the map into $\pi_1(X, x)$ will be trivial). And the point is that **all CW complexes are locally simply connected**.

It turns out that this is the only necessary condition:

Theorem 68

Suppose X is path-connected, locally-path-connected, and semi-locally simply connected. Then X admits a universal cover.

To figure out how to construct such a universal cover, recall that such a construction must have the path lifting property (in which any path $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$ has a unique lift $\tilde{\gamma} : [0, 1] \to \tilde{x}$ with $\tilde{\gamma}(0) = \tilde{x}_0$) and also must have the homotopy lifting property (where any $\gamma_t : [0, 1] \to X$ with $\gamma_t(0) = x_0$ and $\gamma_t(1) = x_1$ for all t uniquely lifts to $\tilde{\gamma}_t : [0, 1] \to \tilde{X}$ with $\gamma_t(0) = \tilde{x}_0$ and $\tilde{\gamma}_1(t) = \tilde{x}_1$ for some unique $\tilde{x}_1$ determined by $\tilde{x}_0$). In addition, we must have that any two paths from $\tilde{x}_0$ to $\tilde{x}_1$ are homotopic because $\tilde{X}$ is simply connected. Thus, we are motivated to think of the universal cover as **the space of homotopy classes of paths on X starting at some fixed starting point** x_0, since the path class just depends on the final point $\tilde{\gamma}(1) \in \tilde{X}$ of the lift. Specifically, if a universal cover $\tilde{X}$ exists, then we have the map

$$\Phi : \{[\gamma] : \gamma : [0, 1] \to X, \gamma(0) = x_0\} \to \tilde{X}$$

sending $[\gamma]$ to $\tilde{\gamma}(1)$, and if $\tilde{X}$ is simply connected then this is a bijection. So now we're ready to actually construct the universal cover:

Proof. Define $\tilde{X}$ to be the set of homotopy classes of paths $[\gamma]$ starting at x_0 (meaning we consider all $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$), where the covering map $p : \tilde{X} \to X$ maps $[\gamma]$ to $\gamma(1)$. We need to (1) define a topology on $\tilde{X}$, (2) show that p is actually a covering map, and (3) check that $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial. (It turns out that if we choose a different base point x_0, we'll get a homeomorphic space. But we'll talk about uniqueness later.)

For (1), recall from point-set topology that a **basis** for a topology τ on X is a set of open sets $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ so that for all $U, V \in \mathcal{U}$ and any $x \in U \cap V$, there is some $W \in \mathcal{U}$ such that $x \in W \subseteq U \cap V$. (For example, think about the set of arbitrarily small balls in $\mathbb{R}^n$.) We'll use a basis for the topology on X to construct a basis for the topology on $\tilde{X}$. Specifically, we can define

$$\mathcal{U} = \{U : U \text{ open and path-connected, such that } \pi_1(U) \to \pi_1(X) \text{ is trivial.}\}$$

This forms a basis for X (exercise), and now we can construct a basis for $\tilde{X}$ as follows. For any $U \in \mathcal{U}$, and for any path $\gamma : [0, 1] \to X$ between $\gamma(0) = x_0$ and $\gamma(1) = x_1 \in U$ (that is, any path from x_0 to a point in U), we will define an open set $U_{[\gamma]}$ on $\tilde{X}$ (the "open neighborhood" of γ), which is the **set of compositions** $[\eta \circ \gamma]$, where η is a path in U with $\eta(0) = x_1$. In other words, because U is path-connected we can draw a path from x_1 to another point in U (staying within U), and we can use that path to extend γ. So the set of all extensions gives us a subset of $\tilde{X}$, which will be an element of our basis for $\tilde{X}$.

There are a few important properties of this set $U_{[\gamma]}$:

(a) If $\gamma(1) \in V \subseteq U$ and V is also open and path-connected, then we can construct $V_{[\gamma]}$, which will be a subset of $U_{[\gamma]}$ (because any path within V is also a path within U).

(b) If γ' is another path $[0, 1] \to X$ with $\gamma'(0) = x_0$ and $\gamma'(1) = x_1' \in U$ (not necessarily ending at the same point x_1), **and we assume** $[\gamma'] \in U_{[\gamma]}$, then $\gamma'(1) \in U$, and we actually have $U_{[\gamma']} = U_{[\gamma]}$. (This relies on the fact that the map $\pi_1(U) \to \pi_1(X)$ is trivial.)

We claim that

$$\tilde{\mathcal{U}} = \left\{U_{[\gamma]} : U \in \mathcal{U}, \gamma : [0, 1] \to X \text{ where } \gamma(0) = x_0,\ \gamma(1) \in U\right\}$$

gives us a basis on $\tilde{X}$. To show that this set can actually be a basis which lets us define a topology, we must show that for any $U_{[\gamma_1]}, V_{[\gamma_2]} \in \tilde{U}$ and any point $[\gamma] \in U_{[\gamma_1]} \cap V_{[\gamma_2]}$, there is some $W_{[\gamma_3]}$ such that $[\gamma] \in W_{[\gamma_3]} \subseteq U_{[\gamma_1]} \cap V_{[\gamma_2]}$. We know by (b) that if $[\gamma] \in U_{[\gamma_1]} \cap V_{[\gamma_2]}$, then $\gamma(1) \in U \cap V$, and (by properties of the original basis $\mathcal{U}$ on X) there is some open set $W \in \mathcal{U}$ such that $\gamma(1) \in W \subseteq U \cap V$. But now by (a), because W is a subset of $U \cap V$, $W_{[\gamma]}$ must be a subset of $U_{[\gamma]} \cap V_{[\gamma]}$. And now by the second part of property (b), that also means that $W_{[\gamma]} \subseteq U_{[\gamma_1]} \cap V_{[\gamma_2]}$, as desired. So $\tilde{U}$ is actually a valid basis, and from that we can define a topology on $\tilde{X}$ in the usual way (a set O is open if for any point x in the set, there is some U in the basis so that $x \in U \subseteq O$).

Remark 69. *A more naive approach would be to take the pullback of the topology on X to get a topology on $\tilde{X}$ (which is the weakest topology making $p : \tilde{X} \to X$ continuous). But this doesn't quite work – the pullback of the topology on S^1 to $\mathbb{R}$ collapses all of the different "sheets" and gives us the topology on S^1 again.*

Next, we work on (2), showing that $p : \tilde{X} \to X$ is a covering map. If we pick a small open set $U \in \mathcal{U}$ of X, then any preimage of a point in U is some $\gamma : [0, 1] \to X$ mapping from x_0 to some point in U. Thus we have the "neighborhood of γ" $U_{[\gamma]}$, and we claim that $p : U_{[\gamma]} \to U$ is in fact a one-to-one correspondence sending $[\gamma\eta]$ to $\gamma\eta(1)$ (injective because the fundamental group of U is trivial, and surjective because U is path-connected). Furthermore, p restricted to $U_{[\gamma]}$ is a homeomorphism, because we get a bijection between restrictions of bases $\mathcal{U}|_U \to \tilde{\mathcal{U}}|_{U_{[\gamma]}}$.

So we indeed have a covering space because this covering map p is valid – indeed, for any $U \in \mathcal{U}$, the preimage of U under p is the set of homotopy classes

$$p^{-1}(U) = \bigcup_{\substack{\gamma:[0,1]\to X \\ \gamma(0)=x_0 \\ \gamma(1)\in U}} U_{[\gamma]}.$$

And the key fact is that $U_{[\gamma]} \cap U_{[\gamma']}$ is either empty or the two neighborhoods are the same (since a nontrivial intersection means we have a common point $[\gamma'']$ in both neighborhoods, so $\gamma''(1) \in U$ and $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$). So $p^{-1}(U)$ is a disjoint union of sets, and each is homeomorphic to U, so we do indeed have a covering space.

Finally, we must check (3), which is showing that $\tilde{X}$ is simply connected. We'll do that argument next time! □

8 October 20, 2022

Last time, we discussed the construction of a simply-connected covering space (that is, a universal cover) of a path-connected, locally-path-connected, and semi-locally simply connected space X. Specifically, we define $\tilde{X}$ to be the set of homotopy classes of paths $[\gamma]$ (for $\gamma : [0,1] \to X$ starting at some fixed point x_0), from which we can define a topology and show that $p : \tilde{X} \to X$ is in fact a covering map. Our goal is now to check that this space is simply connected, which we can do by checking first that $\tilde{X}$ is path-connected and then by showing that $\pi_1(\tilde{X})$ is trivial.

Our first step will be to restate the lifting property for $\tilde{X}$ in a better way, because we have an explicit description of the covering space. Given any path $\gamma : [0,1] \to X$ with $\gamma(0) = x_0$, we can describe its lift in $\tilde{X}$ in the following way. We need to choose some $\tilde{x}_0 \in p^{-1}(x_0) \subseteq \tilde{X}$ – in particular, we can choose $\tilde{x}_0$ to be the class of the **constant path** $[x_0]$ (because that constant path ends at x_0, and our covering map sends $[\gamma]$ to $\gamma(1)$). The lifting property then tells us that γ has a unique lift starting at $\tilde{x}_0$. For any $t \in [0,1]$, we can define a new path

$$\gamma_t(s) = \gamma(ts), \quad s \in [0,1],$$

which is the path that travels along only the first t fraction of γ from x_0 to $\gamma(t)$. We can thus define $\tilde{\gamma} : [0,1] \to \tilde{X}$ by letting $\tilde{\gamma}_t = [\gamma_t]$. In other words, the path in $\tilde{X}$ can be thought of as starting with the constant path and then stretching out along γ, and this is always a valid element of $\tilde{X}$ because we're always starting at x_0. Then $\tilde{\gamma}(0) = [\gamma_0] = [x_0]$ and $\tilde{\gamma}(1) = [\gamma_1] = [\gamma]$.

But this shows that $\tilde{X}$ is path-connected – fix $\tilde{x}_0 = [x_0]$ as before to be our chosen base point on the covering space. Then for any $[\gamma] \in \tilde{X}$ (that is, any path on X starting at x_0), we know that γ lifts to $\tilde{\gamma} : [0,1] \to \tilde{X}$, which is a path in the space of (classes of) paths connecting $\tilde{x}_0$ to $[\gamma]$. Thus any point is connected to $\tilde{x}_0$ and thus the whole space is path-connected.

We can now show that $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial. Recall that for a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$, the map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective. So it suffices to show that any loop (of paths) $\tilde{\gamma} : [0,1] \to \tilde{X}$ sends to a loop $\gamma = p \circ \tilde{\gamma}$ (with $\tilde{\gamma}(0) = \tilde{\gamma}(1) = \tilde{x}_0 = [x_0]$ under the covering map, which is nullhomotopic in X. In other words, we wish to show that $[\gamma] = [x_0]$. By assumption, $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_0 = [x_0]$, and because $\tilde{\gamma}$ is a loop, it ends at $\tilde{x}_0$. But by the path-lifting property, γ (now viewed as a path in X) also has a unique lift on $\tilde{X}$ from $\tilde{x}_0$ to $[\gamma]$ on $\tilde{X}$. Thus uniqueness means $[\gamma]$ and $[x_0]$ must be in the same homotopy class of loops, which is what we wanted to show.

We will now return to the question of **uniqueness** of the universal cover:

Theorem 70

Let X be path-connected and locally-path-connected. Suppose we have two connected covering spaces $p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$. Then there exists a homeomorphism $h : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ so that $p_1 = p_2 \circ h$ **if and only if** the images of $p_{1,*}$ and $p_{2,*}$ are the same (in other words, $p_{1,*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2,*}(\pi_1(\tilde{X}_2, \tilde{x}_2)) \subseteq \pi_1(X, x_0)$.

Remembering that $\pi_1(\tilde{X}_1, \tilde{x}_1)$ is trivial for simply-connected covering spaces and thus the image is also trivial in X, we immediately get the following result:

Corollary 71

There is a unique simply-connected covering space (up to homeomorphism).

Proof. For one direction, if the map h exists, then $p_1 = p_2 \circ h$ induces a corresponding diagram between fundamental groups with $p_{1,*} = p_{2,*} \circ h_*$, where h_* is an isomorphism because h is a homeomorphism. Then the images of the fundamental groups will be the same.

The other direction is more challenging. Suppose we know that $\text{im}(p_{1,*}) = \text{im}(p_{2,*})$. Because $(\tilde{X}_2, \tilde{x}_2)$ is a covering space of (X, x_0), the map $(\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$ lifts to a map $h : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ because we satisfy that necessary lifting assumption (Theorem 55). Similarly, switching the roles of the two spaces gives us a map $g : (\tilde{X}_2, \tilde{X}_2) \to (\tilde{X}_1, \tilde{x}_1)$. We now want to show that gh and hg are the identity maps on $\tilde{X}_1$ and $\tilde{X}_2$, respectively. For this, notice that we can lift the covering map $p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$ to its covering space $\tilde{X}_1$ in two ways: we can use the identity map, or we can use $g \circ h$ (because for the latter choice, we have $(p_1 \circ g) \circ h = p_2 \circ h = p_1$). Thus those two maps must be the same lift by uniqueness because they coincide on at least the point $\tilde{x}_1$ (since both the identity at $g \circ h$ send $\tilde{x}_1$ to $\tilde{x}_1$). Similarly we can show that $h \circ g = \text{Id}_{\tilde{X}_2}$, as desired. □

So up to homeomorphism, all path-connected covering spaces of X must correspond to subgroups of the fundamental group of X, and now we want to ask if we can achieve all subgroups. In other words, if N is a subgroup of $\pi_1(X, x_0)$, then we want to know if there is a covering space $p_N : (X_N, \tilde{x}_N) \to (X, x_0)$ with $p_{N,*}(\pi_1(X_N, \tilde{x}_0)) = N$. When X is path-connected, locally-path-connected, and semi-locally path connected, the answer turns out to be **yes** – first, start with the universal cover $(\tilde{X}, \tilde{x}_0)$, and now define an equivalence relation on $\tilde{X}$ (which we can identify with the set of classes of paths $[\gamma]$ starting at $\tilde{x}_0$) by setting $[\gamma_1] \sim_N [\gamma_2]$ if $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_1 \cdot \overline{\gamma}_2]$ (which is a loop based at $\tilde{x}_0$, so we can think of it as an element of the fundamental group) is an element of N. We can check (exercise) that this is actually an equivalence relation, so that we can define $X_N = \tilde{X}/\sim_N$. We then get two natural maps $p' : \tilde{X} \to X_N$ (the quotient map; we can check that this is also a covering map) and $p_N : X_N \to X$, and we can check that p_N gives us the covering space that we desire.

Definition 72

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map. A homeomorphism $h : \tilde{X} \to \tilde{X}$ such that $p = p \circ h$ is called a **Deck transformation**. The group of Deck transformations will be denoted $G(\tilde{X}, p)$.

In other words, h is a homeomorphism such that for any $x \in X$, the **set** $p^{-1}(X) \subseteq \tilde{X}$ is preserved. Notice that if two Deck transformations h_1 and h_2 coincide at **any** point, meaning $h_1(\tilde{x}) = h_2(\tilde{x})$ for some $\tilde{x} \in \tilde{X}$, then h_1 and h_2 coincide on the connected component of $\tilde{X}$ containing $\tilde{x}$.

Definition 73

A covering space $p : \tilde{X} \to X$ is **normal** if for all $x \in X$, $G(\tilde{x}, p)$ acts transitively on the set $p^{-1}(x) \in \tilde{X}$. In other words, for all $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x) \in \tilde{X}$, there is some $h \in G(\tilde{X}, p)$ such that $h(\tilde{x}_1) = \tilde{x}_2$.

Theorem 74

Suppose X is path-connected and locally-path-connected, and we have a path-connected covering space given by $p : \tilde{X} \to X$ (note that X being locally-path-connected also implies that $\tilde{X}$ is locally-path-connected). Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$. Then $\tilde{X}$ is a normal covering space if and only if H is a normal subgroup of $\pi_1(X, x_0)$. Also (independent of that fact), $G(\tilde{X}, p) \cong N(H)/H$, where $N(H)$ is the normalizer of H.

(Here, the **normalizer** of $H \subseteq \pi_1(X, x_0)$ is the subgroup

$$N(H) = \{a \in \pi_1(X, x_0) \text{ such that } a^{-1}Ha = H\},$$

and a subgroup is **normal** if $N(H) = \pi_1(X, x_0)$.)

Proof sketch. For the first fact, suppose that $\tilde{X} \to X$ is a normal covering space; we wish to show that H is normal. So we need to show that for any $[\gamma] \in \pi_1(X, x_0)$, we have $[\gamma]^{-1}H[\gamma] = H$. Notice that $[\gamma]$ is a path $[0, 1] \to X$, so it can be lifted to a path $[\tilde{\gamma}] : [0, 1] \to \tilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ and $\tilde{\gamma}(1) = \tilde{x}_1 \in p^{-1}(x_0)$ (because we started with a loop γ rooted at x_0, we must have $\tilde{\gamma}_1$ end at a point $\tilde{x}_1$ that projects down to x_0). But now we can apply the definition of a normal covering space to find some Deck transformation h that maps $\tilde{x}_0$ to $\tilde{x}_1$. The property $p = p \circ h$ of the Deck transformation then gives us an induced relation on the fundamental groups. But

$$H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_* \circ h_*(\pi_1(\tilde{X}, \tilde{x}_0)) \cong p_*(\pi_1(\tilde{X}, \tilde{x}_1)),$$

and now we can compose with the change-of-basepoint homomorphism $\beta_{\tilde{\gamma}}$ (this is the only place where we use the Deck transformation) to get

$$H = p_* \circ \beta_{\tilde{\gamma}}(\pi_1(\tilde{X}, \tilde{x}_0)).$$

So now if $\tilde{\eta}$ is any loop on $\tilde{X}$ with $\tilde{\eta}(0) = \tilde{\eta}(1) = \tilde{x}_0$, we see that

$$p_* \circ \beta_{\tilde{\gamma}}([\tilde{\eta}])] = p_*([\overline{\tilde{\gamma}}\tilde{\eta}\tilde{\gamma}]) = [p \circ (\overline{\tilde{\gamma}}\tilde{\eta}\tilde{\gamma})]$$

by definition of p_*. But this now simplifies to

$$= [(p \circ \overline{\tilde{\gamma}})(p \circ \tilde{\eta})(p \circ \tilde{\gamma})] = [\overline{\gamma}(p \circ \tilde{\eta})\gamma] = [\gamma]^{-1}p_*([\tilde{\eta}])[\gamma].$$

Since γ was arbitrary and we had an arbitrary element $p_*([\tilde{\eta}]) \in H$, this shows that $H = [\gamma]^{-1}H[\gamma]$, and thus H is normal. □

9 October 25, 2022

We'll start our discussion of **homology** today – it has some connections to the fundamental group but not direct relations, and we'll only cover this at a basic level. (And if we're seeing this for the first time, we should check all of the details at least once.)

We'll start with some motivation for why we want to study homology: when we defined the fundamental group $\pi_1(X, x_0)$, the main ingredient is the set of loops $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1) = x_0$, or equivalently the set of maps $S^1 \to X$ with $(1, 0)$ mapping to x_0, where we have equivalence coming from homotopies $S^1 \times I \to X$ with $H((1, 0), t) = x_0$ for all t. We now want to generalize these objects a bit to remove a few restrictions – topologically we care about more than maps from circles and cylinders. Instead, we'll think about higher-dimensional objects – studying $\gamma : S^n \to X$ will give us the higher **homotopy groups** $\pi_n(X)$ (with some fixed data), but that's not the direction in which we'll go. Instead, we'll allow more general n-dimensional objects, such as the surfaces of arbitrary genus. Additionally, we can also generalize homotopy – instead of going from S^1 to $S^1 \times I$, we'll go more generally from n dimensions to $(n + 1)$ dimensions (for example imagining that a genus-g surface connects our two S^1s rather than just a cylinder / sphere).

We'll start with some **homological algebra**. Let R be a commutative ring with unit 1 – we'll be considering the modules over the ring R. (We can imagine $R = \mathbb{Z}$ for now if we're not too comfortable doing this, and we can just think that we're working with abelian groups.)

Definition 75

A **chain complex** is a pair (C, d), where C is an R-module and $d : C \to C$ is a module morphism (called the **differential**) such that $d^2 = 0$.

Usually the idea is that C is a **graded** module $C = \bigoplus_{i \in \mathbb{Z}} C_i$, where d maps C_i to C_{i-1} (in other words, it has degree -1) – we'll see that d will increase the degree in cohomology, but it decreases it here. We'll then let d_i denote the restriction of d to C_i. In general for the situations that we're seeing, we'll have $C = C_0 \oplus C_1 \oplus \cdots C_n$. We see that this setup means we have a sequence

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots,$$

where $d^2 = 0$ is really saying that $d_i \circ d_{i+1} = 0$, or in other wrods $\text{im } d_{i+1} \subset \ker d_i$.

Definition 76

The **homology** of the chain complex (C, d), denoted $H_*(C, d)$, is $\ker d/\text{im } d$. In other words, for each i, we have the module

$$H_i(C, d) = \ker d_i/\text{im } d_{i+1},$$

and then we define $H_*(C, d) = \bigoplus_i H_i(C, d)$.

Example 77

Suppose our chain complex looks like $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$, where 2 means we have the multiplication-by-2 map. This notation is meant to mean that the only nontrivial parts of the chain complex are $C_0 = \mathbb{Z}$ and $C_1 = \mathbb{Z}$.

This is indeed a chain complex (easy check); to compute the homology, we have

$$H_1(C, d) = \ker d_1/\text{im } d_2 = 0/0 = 0, \quad H_0(C, d) = \ker d_0/\text{im } d_1 = \mathbb{Z}/2\mathbb{Z},$$

which is the field of two elements. All other homology modules / groups $H_i(C, d)$ will be zero because the kernel of d_i must be zero.

What we're interested in studying is often how to relate different chain complexes:

Definition 78

Let $(C, d), (C', d')$ be two chain complexes. A module morphism $f : C \to C'$ is a **chain map** if $d' \circ f = f \circ d$ (that is, the differentials commute with f).

Proposition 79

Any chain map $f : (C, d) \to (C', d')$ induces a canonical module morphism $f_* : H_*(C, d) \to H_*(C', d')$.

Indeed, if we send $[x]$ to $[f(x)]$, we can check that the map is well-defined and actually a module morphism. And there is also a functoriality property (which we can check by computation as well):

Proposition 80

If we have chain maps $(C, d) \xrightarrow{f} (C', d') \xrightarrow{g} (C'', d'')$, then we also get a chain map $g \circ f$, and $(g \circ f)_* = g_* \circ f_*$.

We may ask whether two chain maps $f, g : (C, d) \to (C', d')$ yield identical induced morphisms f_*, g_* – in the case of the fundamental group the answer came from homotopy, and here we have a "chain homotopy" as well:

Definition 81

A module morphism (not necessarily a chain map) $T : (C, d) \to (C', d')$ is a **chain homotopy** between chain maps $f, g : (C, d) \to (C', d')$ if $f - g = d'T + Td$. (We say that f and g are **chain homotopic**.)

Proposition 82

If f and g are chain homotopic, then $f_* = g_*$ as maps $H_*(C, d) \to H_*(C', d')$.

We'll encounter more algebra as we move forward, but this is all the preparation we'll do on that front for now. For now, we'll move on to some other preliminaries:

Definition 83

The **simplex** Δ^n is a subset of $\mathbb{R}^{n+1}$, defined via

$$\Delta^n = \{(x_0, \cdots, x_n) : \sum_{i=0}^{n} x_i = 1, x_i \geqslant 0\}.$$

For example, the 1-simplex is the set of points $(x_0, x_1) \in \mathbb{R}^2$ with $x_0 + x_1 = 1$ in the first quadrant, which is a line segment. A 2-simplex is then a triangle, and a 3-simplex is a tetrahedron (and a 0-simplex is a single point). In general, Δ^n is then a convex space spanned by $(n+1)$ standard basis vectors – we'll let those vectors $v_i = (0, \cdots, 0, 1, 0 \cdots, 0)$ be the **vertices** of our simplex and write the simplex as $[v_0, v_1, \cdots, v_n]$. The **faces** of Δ^n are then spans of subsets:

Definition 84

Let $[v_0, \cdots, v_n]$ be an n-dimensional simplex. Given any nonempty subset $\{v_{i_0}, \cdots, v_{i_m}\}$, we get a subspace $\Delta^m \subseteq \Delta^n$ (the convex space spanned by the subset), which we call an m-dimensional **face** of Δ^n.

For example, picking a subset of size 2 gives us a line segment, and a subset of size 3 gives us a triangle. In making all of these definitions, the order has not mattered (something like $[v_0, v_2, v_1]$ produces the same simplex as $[v_0, v_1, v_2]$ but rearranging the faces), but we do want to describe how they are different in terms of traversing along cyclic paths. This basically comes down to characterizing permutations in a particular way:

Definition 85

A **transposition** is a permutation in which only two elements are switched (so v_i is sent to v_j and vice versa, but all other elements stay fixed).

Fact 86

All permutations are a finite product of transpositions (by induction), and for any permutation, the parity of the number of transpositions required is fixed. (One way to see this is to think about permutations as matrices; requiring an even (resp. odd) number of permutations then corresponds to a determinant of 1 (resp. -1).

Definition 87

A permutation is **even** (resp. **odd**) if the number of transpositions in its representation is always even (resp. odd). The set of orderings of the vertices of a simplex is then divided into two equivalence classes, where equivalence means that we get from one to another by an even permutation. An **orientation** of Δ^n is a choice of one of the two equivalence classes.

For example, if we write $\Delta^2 = [v_0, v_1, v_2]$, we have now denoted an **oriented simplex**, which is ordered in the opposite way as $[v_0, v_2, v_1]$. And for any subsimplex Δ^m of an oriented simplex Δ^n, we get an induced orientation on Δ^m (from the induced ordering).

We're now actually ready to introduce what homology theory looks like for topological spaces – it turns out we really just care about codimension-1 faces, and the point of this orientation is that it gives us some negative signs when we do calculations in chain complexes. There's two ways we can set this up, which give rise to two different homology theories, but it turns out they will be isomorphic:

- **Simplicial homology**, in which we embed simplices into our topological space X (in other words, triangulate the space),
- **Singular homology**, in which we think about continuous maps $\sigma : \Delta^n \to X$.

(There is also **cellular homology** coming from CW complexes, which is what is more useful for actual computations.) For **simplicial homology**, our setup is as follows (this definition will be rewritten next lecture):

Definition 88 (Sketch)

A **simplicial structure** of a topological space X is a collection of embeddings (homeomorphic injection) of simplices $\{\sigma_\alpha^{n_\alpha} : \Delta_\alpha^{n_\alpha} \to X\}_{\alpha \in I}$ (often we'll just write the images as $\Delta_\alpha^{n_\alpha} \subseteq X$) such that $X = \bigcup_{\alpha \in I} \Delta_\alpha^{n_\alpha}$, with the condition that whenever $\Delta_\alpha^{n_\alpha} \cap \Delta_\beta^{n_\beta}$ is nontrivial, there is some $\gamma \in I$ with $\Delta_\gamma^{n_\gamma} = \Delta_\alpha^{n_\alpha} \cap \Delta_\beta^{n_\beta}$ which is a face of both $\Delta_\alpha^{n_\alpha}$ and $\Delta_\beta^{n_\beta}$ (so simplices must intersect along a common face, which is also a simplex in the embedding).

(In particular, this does restrict the set of spaces X we can consider – even for manifolds, there is the triangulation conjecture which was proven false, so not all spaces can be triangulated.) The point of such a simplicial structure will be to define a chain complex on which we can do homology: if $K = \{\Delta_\alpha^{n_\alpha}\}_{\alpha \in I}$ is a simplicial structure on X, then define the chain complex (C^K, d^K) as follows. Fix some commutative ring R with unit 1, and let

$$C^K = \left\{ \sum_{i=1}^n r_i \Delta_i : r_i \in R, \Delta_i \in K \right\}$$

be the set of finite formal sums of all simplices (of any dimension), which is an R-module. We then get a natural decomposition into its various graded parts $C^K = \bigoplus_{m \in \mathbb{Z}} C_m^K$, defining

$$C_m^K = \left\{ \sum_{i=0}^n r_i \Delta_i^m : r_i \in R, \Delta_i^m \in K, \dim \Delta_i^m = m \right\}.$$

Then the differentials will basically take m-dimensional simplices to $(m-1)$-dimensional simplices, but we'll see that in more detail next time.

10 October 27, 2022

We introduced **simplices** last time, with the goal of creating simplicial structures on topological spaces for the purpose of computing homology. We'll begin by slightly restating the simplicial structure definition to account for a few extra conditions:

Definition 89

A **simplicial structure** is a collection of maps $K = \{\sigma_\alpha^{n_\alpha} \to X\}_{\alpha \in I}$ with the following conditions:

1. For any α, σ_α restricted to the interior of $\Delta_\alpha^{n_\alpha}$ is an embedding (injective map).
2. If our simplicial structure contains $\sigma_\alpha : \Delta_\alpha^{n_\alpha} \to X$ and $\Delta^m \subseteq \Delta_\alpha^{n_\alpha}$ is a face of the simplex, then $\sigma_\alpha|_{\Delta^m} : \Delta^m \to X$ is also one of the embeddings in K.
3. For any two maps $\sigma_\alpha : \Delta_\alpha^{n_\alpha} \to X$ and $\sigma_\beta : \Delta_\beta^{n_\beta} \to X$, if $\sigma_\alpha(\Delta_\alpha^{n_\alpha}) \cap \sigma_\beta(\Delta_\beta^{n_\beta}) \neq \varnothing$, then we have a "common face" (meaning that $\sigma_\alpha^{-1}(\sigma_\alpha(\Delta_\alpha^{n_\alpha}) \cap \sigma_\beta(\Delta_\beta^{n_\beta})) = \tilde{\Delta}_\alpha \subseteq \Delta_\alpha^{n_\alpha}$, and similarly $\sigma_\beta^{-1}(\sigma_\alpha(\Delta_\alpha^{n_\alpha}) \cap \sigma_\beta(\Delta_\beta^{n_\beta})) = \tilde{\Delta}_\beta \subseteq \Delta_\beta^{n_\beta}$).
4. $X = \bigcup_{\alpha \in I} \sigma_\alpha(\Delta_\alpha^{n_\alpha})$; that is, X is covered by the simplices.
5. A subset $U \subseteq X$ is open if and only if $U \cap \sigma_\alpha(\mathrm{int}(\Delta_\alpha^{n_\alpha}))$ is open for any α.
6. The interior of any two simplices do not intersect in X.

In other words, we want to glue simplices along the common boundary, restricting the topology formed by those gluings (so that we can't make the topology arbitarily stronger).

Remark 90. *We often also call a simplicial structure on X a* ***triangulation****. A simplicial structure has more requirements than a CW complex because of point (3) – gluings can only happen along entire faces in simplicial structures. So simplicial structures give rise to CW complexes but not necessarily the other way around. Expanding on what was mentioned last time, all smooth manifolds do admit simplicial structures, and all manifolds of dimension at most* 3 *do as well, but there are n-dimensional topological manifolds which do not for any* $n \geqslant 4$*.*

We can now describe simplicial homology (the most restrictive homology theory) in more detail than we did last time. We start by defining a chain complex in terms of the simplicial structure K of X – here, we'll work with abelian groups to make the construction simpler and define

$$C^k(X) = \left\{ \sum_{i=1}^{n} r_i \sigma_i : r_i \in \mathbb{Z}, \sigma_i : \Delta_\alpha^{n_\alpha} \to X \text{ simplex in } K \right\}$$

to be the set of $\mathbb{Z}$-finite sums of simplices (more generally $\mathbb{Z}$ can be replaced with any commutative ring, but we'll just think about this in terms of abelian groups for simplicity). We can place a grading on this ring, where the mth graded part C_m^K is restricted only to sums over the m-dimensional simplices. To define the differential map d^K for this chain complex, we just need to define $d_m^K : C_m^K \to C_{m-1}^K$ for each m; we know the generators of C_m^K are the m-dimensional simplices, so we just need to define the differential of any simplex and extend by linearity: we'll basically sum over all faces in an alternating fashion:

$$d_m([v_0, v_1, \cdots, v_m]) = [v_1, v_2, v_3, \cdots, v_m] - [v_0, v_2, v_3, \cdots, v_m] + [v_0, v_1, v_3, \cdots, v_m] - \cdots + (-1)^m [v_0, v_1, \cdots, v_{m-1}].$$

In other words, when we remove the ith vertex, we get $(-1)^i$ times the $(m-1)$-dimensional face without i. This choice of signs is to ensure that we actually have $d^2 = 0$, and it's also motivated by the induced orientation that we discussed last time. Checking that we do have $d^2 = 0$ will be left as an exercise, but as an example notice that

$$\begin{aligned} d^2[v_0, v_1, v_2] &= d([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) \\ &= (v_1 - v_2) - (v_0 - v_2) + (v_0 - v_1) \\ &= 0. \end{aligned}$$

Intuitively, the differential gives us the boundary of our simplex, and the boundary of a boundary is zero. So (C^K, d^K) is a chain complex, and we can define the **simplicial homology** $H_*^K(X) = H_*(C^K, d^K)$.

Remark 91. *If we want to make this definition, we have two main problems: X may not have a simplicial structure, and we need to check that this is well-defined regardless of the choice of our chain complex. Additionally, if (X, K_X) and (Y, K_Y) are two spaces equipped with simplicial structures, then we may want to relate X and Y with a continuous map f. But f does not generally preserve simplicial structure, so it is hard for us to describe how f gives a map $H_*^K(X) \to H_*^K(Y)$.*

In classical simplicial homology there is a way to show well-definedness and also deal with continuous maps (specifically using barycentric subdivision) but we won't explain that here because it takes a bit of work. Instead, we'll now show some of the more advanced techniques so that we don't have to think about these kinds of questions. We're mentioning simplicial homology mostly because it came first in mathematical development and as motivation for more powerful theories.

In **singular homology**, the point is that we no longer require $\sigma_\alpha : \Delta^{n_\alpha} \to X$ to be injective in the interior, so that we can work with more general objects:

Definition 92

A **singular simplex** is a continuous map $\sigma : \Delta^n \to X$.

We'll again just work with $\mathbb{Z}$ coefficients here and construct a chain complex. This time, we have

$$C(X) = \left\{ \sum_{i=0}^{n} r_i \sigma_i : r_i \in \mathbb{Z}, \sigma_i : \Delta^{n_i} \to X \text{ singular simplices} \right\};$$

notice that we're allowing arbitrary continuous maps into X now, so this space is now much bigger than the $C(X)$ that we had before. And now we define the differential in a similar way: we can decompose $C(X) = \bigoplus_{m \geqslant 0} C_m(X)$, where $C_m(X)$ is generated by the m-dimensional singular simplices, and then we can define $d : C_m(X) \to C_{m-1}(X)$ by defining it on the generators, so that for any $\sigma : [v_0, v_1, \cdots, v_m] \to X$ we have

$$d\sigma = \sigma([v_1, \cdots, v_m]) - \sigma([v_0, v_2, \cdots, v_m]) + \cdots + (-1)^m \sigma([v_0, v_1, \cdots, v_{m-1}]).$$

This is again restricting to faces, so the argument for showing that $d^2 = 0$ is identical. So we have singular homology $H_*^s(X) = H_*(C(X), d)$ defined in the usual way for a chain complex.

This time, we don't have issues with specifying a simplicial set anymore, but because our chain complex is generally infinitely generated it's hard to answer questions like "is homology finite-dimensional?". In general, simplicial homology is more computable if we have a finite simplicial structure (because it's induced by simplices rather than continuous maps or other properties), but it has worse functoriality properties and it's hard to see why it's well-defined. We'll mention just a few cases where singular homology can be actually computed:

Example 93

If X is a single point $\{p\}$, then $C_m(X)$ only has a single unique generator which sends all of Δ^m to p.

Our goal is then to calculate the homology of the chain complex $\cdots \to C_m \xrightarrow{d_m} C_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} C_0$ – the actual calculation is left as an exercise for us, but we can check that each C_k is freely generated (as $\mathbb{Z}$) – in particular, we're not modding out by relations from permutations – and then we get

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, \\ 0 & n \neq 0. \end{cases}$$

(And now if we replace $\mathbb{Z}$ with an arbitrary commutative ring, all of the $\mathbb{Z}$s become Rs, including our final answer.)

Example 94

Now suppose X is a path-connected space; we will still be able to compute $H_0(X)$.

We need to understand the chain complex $C_1(X) \xrightarrow{d_1} C_0(X) \xrightarrow{d_0} 0$ and calculate $\ker d_0/\text{im } d_1$, but all of $C_0(X)$ maps to zero so the kernel of d_0 is all of $C_0(X)$. But remember that zero-dimensional simplices are maps from a point into X, so $C_0(X)$ can be regarded as $\{\sum_{i=0}^n r_i x_i : r_i \in \mathbb{Z}, x_i \in X\}$. On the other hand, $C_1(X)$ is the set of maps from a line segment into X, which can be thought of as the formal sum of **paths** in X. Furthermore, if $\sigma : [0, 1] \to X$ is an element of $C_1(X)$, then $d\sigma = \sigma(1) - \sigma(0)$. So modding out by the kernel of d_1 means $\sigma(0) = \sigma(1)$ for any path σ in $C_0(X)$, but X is path connected so any two points are equal under this relation! Thus $H_0(X)$ is isomorphic to the group generated by any one point, which is $\mathbb{Z}$.

Theorem 95

For any space X that admits a simplicial structure, we have $H_*^K(X) \cong H_*^S(X)$.

It turns out that (when we introduce it later) cellular homology will also be isomorphic to these homology theories. This is not a coincidence – there turns out to be an set of axioms that guarantee isomorphism, which we may explore more if we have time. But the point is that we can pick whichever is easiest to use to make arguments or perform computations.

11 November 1, 2022

We introduced singular homology last time, in which we study the set of singular simplexes (that is, the set of continuous maps $\sigma : \Delta^n \to X$). We get the chain complex $C(X)$ of linear combinations $\sum_{i=1}^n r_i \sigma_i$ (where $r_i \in \mathbb{Z}$ and σ_i are simplexes), and we get the differential map $d : C(X) \to C(X)$ in which $d_i : C_i(X) \to C_{i-1}(X)$ restricts each simplex to its faces with alternating signs and orientations as we've previously described:

$$d(\sigma([v_0, \cdots, v_n])) = \sigma([v_1, v_2, \cdots, v_n]) - \sigma([v_0, v_2, \cdots, v_n]) + \cdots$$

$$+(-1)^i \sigma([v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n]) + \cdots + (-1)^n \sigma([v_0, v_1, \cdots, v_{n-1}]).$$

Last time we shows that the homology groups $H_i(X)$ of a point are $\mathbb{Z}$ for $i = 0$ and 0 otherwise, and we also showed that whenever X is path-connected, $H_0(X) = \mathbb{Z}$. Today, we'll understand how continuous maps $f : X \to Y$ relate to singular homology. For any such map f, we get a natural map $f_\# : C(X) \to C(Y)$ which sends any singular simplex $\sigma : \Delta^n \to X$ to $f \circ \sigma : \Delta^n \to Y$.

We can check that **(1)** this is in fact a chain map (meaning that it commutes with the differential, so $d_Y \circ f_\# = f_\# \circ d_X$), **(2)** if f is the identity map $X \to X$, then $f_\#$ is the identity map $C(X) \to C(X)$, and **(3)** if we have maps $f : X \to Y$ and $g : Y \to Z$, then $g_\# \circ f_\# = (g \circ f)_\#$. In particular, **(1)** says that because $f_\#$ is a chain map, we get a natural map on homology $f_* : H_*(X) \to H_*(Y)$, **(2)** tells us that the induced map id_* on homology is also the identity map $H_*(X) \to H_*(X)$, and **(3)** tells us that $g_* \circ f_* = (g \circ f)_*$ at the homology level too. And what's nice is that with singular homology, we get this induced map in a very natural way.

The next natural question to ask is whether the maps $f_\# : g_\# : C(X) \to C(Y)$ or $f_*, g_* : H_*(X) \to H_*(Y)$ can be related if we have homotopic maps $f, g : X \to Y$. Recall that $f \simeq g$ can be restated as having a continuous map $H : X \times I \to Y$, and any singular simplex $\sigma : \Delta^n \to X$ gives us a composition of maps $\Delta^n \times [0, 1] \to Y$ by first applying $\sigma \times \mathrm{id}_I$ and then the homotopy H. This composition restricted to $\{0\}$ is $H|_{X\times\{0\}} : \Delta^n \to Y$, which is $f \circ \sigma$ (the image of σ under $f_\#$), and this composition restricted to $\{1\}$ is $H|_{X\times\{1\}} : \Delta^n \to Y$, which is $g \circ \sigma$ (the image of σ under $g_\#$). So with that, we can now define a chain homotopy $T : C_n(X) \to C_{n+1}(Y)$ making use of H: for any $\sigma : [v_0, \cdots, v_n] \to X$, denote $v_i \times \{0\}$ by the original v_i and denote $v_i \times \{1\}$ by v_i'. Then we basically "triangulate" $[v_0, v_1, v_2] \times I$ by defining

$$T(\sigma) = H \circ (\sigma \times \mathrm{id})|_I[v_0, v_0', v_1', \cdots, v_n'] - H \circ (\sigma \times \mathrm{id})|_I[v_0, v_1, v_1', \cdots, v_n']$$

$$+ \cdots + (-1)^i H \circ (\sigma \times \mathrm{id})|_I[v_0, v_1, \cdots v_i, v_i', \cdots, v_n'] + \cdots + (-1)^n H \circ (\sigma \times \mathrm{id})|_I[v_0, v_1, \cdots, v_n, v_n'] + \cdots,$$

a $(n+1)$-simplex into Y. We can check by computation that

$$g_\# - f_\# = d_Y T + T d_X,$$

so by Proposition 82 we indeed have identical maps $f_*, g_* : H_*(X) \to H_*(Y)$. And similarly, if $X \simeq Y$, then $H_*(X) \to H_*(Y)$ by constructing inverse maps on homology, so for example for any contractible space we get the homology of a single point ($\mathbb{Z}$ for $n = 0$ and 0 otherwise).

Example 96

Consider the case where $\Delta^1 = [v_0, v_1]$. By the construction above, we have $T([v_0, v_1]) = [v_0, v_0', v_1'] - [v_0, v_1, v_1']$, and we can check the chain map condition here explicitly, using that $g_\#([v_0, v_1]) = [v_0', v_1']$ and $f_\#([v_0, v_1]) = [v_0, v_1]$.

Indeed, we have

$$d_Y T([v_0, v_1]) = [v_0', v_1'] - [v_0, v_1'] + [v_0, v_0'] - [v_1, v_1'] + [v_0, v_1'] - [v_0, v_1]$$

and

$$T d_X([v_0, v_1]) = T([v_1] - [v_0]) = [v_1, v_1'] - [v_0, v_0'],$$

so adding these together preserves only the terms $[v_0', v_1'] - [v_0, v_1]$, which indeed corresponds to $g_*([v_0, v_1]) - f_*([v_0, v_1])$.

We'll now discuss **relative homology**: let $A \subseteq X$ be a subspace, and we define the homology $H_*(X, A)$ in the following way. We have $C_k(A) \subseteq C_k(X)$ for each k, so we can form the quotient complex $C(X)/C(A)$ (or equivalently making formal sums zero if they consist of terms inside $C(A)$), which is a direct sum $\sum_{k\in\mathbb{Z}} C_k(X/A) = \sum_{k\in\mathbb{Z}} C_k(X)/C_k(A)$. Additionally, the differential $d_K : C_k(X) \to C_{k-1}(X)$ also maps $C_k(A) \to C_{k-1}(A)$, so we get a natural differential on the quotient complex X/A.

Definition 97

The **relative homology** $H_*(X, A)$ is the homology of the quotient complex $H_*(X/A)$.

This relative homology turns out to be useful for doing certain computations:

Proposition 98

If $U \subseteq X$ is an open set with $A \subseteq U$ a deformation retraction of U, then $H^n(X, A) \cong H_n(X/A)$ for all $n > 0$.

For example, if $X = D^2$ and $A = \partial X = S^1$, then we can let U be an open collar of the boundary. Since U deformation retracts onto A, we have $H_n(X/A) \cong H_n(X, A)$ for all $n > 0$, meaning that $H_n(X, A) \cong H_n(S^2)$ for all $n > 0$ (and the zero-dimensional case is easy because we have a connected space, so $H_0(S^2) \cong \mathbb{Z}$). Similarly, whenever we have $X = D^n$ and $A = \partial D^n$, then $X/A \cong S^n$ so $H_k(S^n) \cong H_k(X, A)$ for all $k > 0$ and $H_0(S^n) \cong \mathbb{Z}$.

But we want a way to actually compute the relative homology of $C(X, A) = C(X)/C(A)$, and it turns out we can make use of an **exact sequence**:

Definition 99

A sequence of group homomorphisms between abelian groups $\cdots \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \cdots$ is **exact** if for any n, we have $\ker(f_{n-1}) = \text{im}(f_n)$.

(Remember that in chain complexes, we have a sequence of such maps in which each kernel contains the previous image, but exactness is stronger – a chain complex is exact if and only if $H_n(C, d) = 0$ for all n.) For example, $0 \to A \xrightarrow{f} B$ is exact if and only if f is injective (since $\ker f = 0$), $A \xrightarrow{g} B \to 0$ is exact if and only if g is surjective (since $\ker g = B$), and $0 \to A \xrightarrow{h} B \to 0$ is exact if and only if f is an isomorphism. So the inclusion $\iota : A \to X$ induces a map $\iota_\# : C(A) \to C(X)$, and we also get a quotient map $q : C(X) \to C(X)/C(A)$. We can check that ι_* is injective, q is surjective, and $\ker(q) = \text{im}(\iota_*)$, so we have an exact sequence (called a **short exact sequence** because it is of the form $0 \to A \to B \to C \to 0$)

$$0 \to C(A) \to C(X) \to C(X)/C(A) \to 0.$$

We can now make use of an important theorem in homological algebra:

Theorem 100 (Zigzag lemma)

Suppose $0 \to (A, d_A) \to (B, d_B) \to (C, d_C) \to 0$ is a short exact sequence of graded chain complexes. In other words, we have chain maps $f : A \to B$ and $g : B \to C$ which preserve gradings of A, B, C (meaning $f(A_n) \subseteq B_n$ and $g(B_n) \subseteq C_n$), such that f is injective, g is surjective, and $\ker(g) = \text{im}(f)$. Then we can construct a long exact sequence with the connecting maps ∂_*

$$\cdots \to H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{f_*} H_{n-1}(B) \xrightarrow{g_*} H_{n-1}(C) \xrightarrow{\partial_*} H_{n-2}(A) \to \cdots.$$

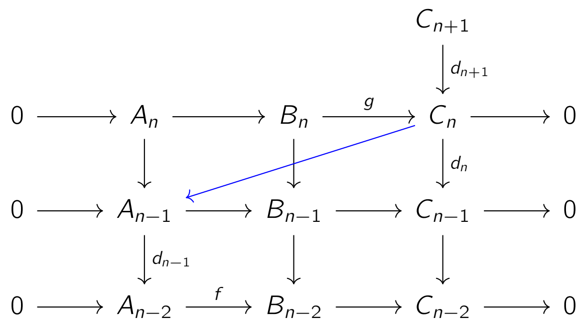
Applying this to our particular question, we find the following property for relative homology:

Corollary 101

If $A \subseteq X$ is a subspace, then there is a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to H_{n-1}(X, A) \to \cdots.$$

The essential step to showing the zigzag lemma is constructing the connecting map $\partial_* : H_n(C) \to H_{n-1}(A)$, and we can visualize this in the diagram below (we will do some "diagram chasing"):



To construct the blue map $H_n(C) \to H_{n-1}(A)$, we need to start with an element of $\ker(d_n)/\text{im}(d_{n+1})$. We can pick some arbitrary representative $\alpha \in \ker(d_n) \subset C_n$ (sent to zero under d_n), and because g is surjective, we can find some $\beta \in B_n$ such that $g(\beta) = \alpha$. But then $d_n(\beta) \in B_{n-1}$ must be sent to zero because of the commutative square formed by $B_n, C_n, B_{n-1}, C_{n-1}$, so $d_n(\beta) \in \ker(g)$. By exactness, this means it is also in the image of f, so there is some element γ in A_{n-1} which maps to $d_n(\beta)$ – we wish to define $\partial_*([\alpha]) = [\gamma]$.

Next, we check that γ is actually in $\ker(d_{n-1})$. For the latter, by the commutative square formed by $A_{n-1}, B_{n-1}, A_{n-2}, B_{n-2}$, we know that $f(d_{n-1}\gamma) = d_{n-1}(d_n(\beta)) = 0$ (because B is a chain complex), but $f : A_{n-2} \to B_{n-2}$ is injective we must have $d_{n-1}\gamma = 0$. Thus γ does represent a homology class in $H_{n-1}(A)$.

But there is still work to do – we need to check that our choices of α, β, γ do not affect the final homology class $H_{n-1}(A)$, and we also have to check that ∂_* actually fits into the long exact sequence. But that's left as an exercise for us, and what's important is that this is a very powerful result for computation:

Example 102

Let $X = D^n$, $A = \partial X = S^{n-1}$. The short exact sequence $0 \to C(A) \to C(X) \to C(X, A) \to 0$ yields a long exact sequence including

$$\cdots \to H_k(S^{n-1}) \to H_k(D^n) \to H_k(D^n, S^{n-1}) \to H_{k-1}(S^{n-1}) \to H_{k-1}(D^n) \to H_{k-1}(D^n, S^{n-1}) \to \cdots.$$

But for all $k > 1$ we have $H_k(D^n) = H_k(\text{point}) = 0$, so in fact the blue part of the long exact sequence yields an isomorphism $H_k(D^n, S^{n-1}) \cong H_{k-1}(S^{n-1})$ for all $k > 1$.

12 November 3, 2022

Last lecture, we introduced relative homology, defining a quotient chain complex $C(X, A) = C(X)/C(A)$ and defining the relative homology $H_*(C(X, A))$ in terms of the induced differential map of the quotient. We mentioned that there is a short exact sequence of chain complexes $0 \to C(A) \to C(X) \to C(X, A) \to 0$, which then gives rise to a long exact sequence of the homology groups (with ∂_* "connecting maps" constructed by the zigzag lemma). We care about this quotient chain complex, because it turns out the relative homology is actually related to the homology of the quotient space – $H^n(X, A) = H_n(X/A)$ for all $n > 0$ if A is a deformation retraction of some open set $U \subseteq X$.

Remark 103. *The case $n = 0$ is generally a bit trickier – we don't expect $H_0(X, A) \cong H_0(X/A)$, because in the case where $A = X$ (so the deformation retraction condition is clearly satisfied) the right-hand side is $\mathbb{Z}$, but $H_0(X, X) = 0$ because $C(A) = C(X)$ so the relative homology chain complex is just identically trivial.*

So if we want to include the case $n = 0$ in our characterization, we'll make use of the **reduced homology**, where we extend the original (singular) chain complex $C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ to a complex

$$C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} \mathbb{Z} \to 0.$$

So most of the maps are defined as usual, but we now need to define d_0 in a way that still makes this a chain complex. Remembering that $C_0(X)$ is spanned by the set of maps from a single point into X, we can basically write $C_0(X) = \{\sum_i r_i x_i : r_i \in \mathbb{Z}, x_i \in X\}$ as a formal sum of points. Since $C_1(X)$ is generated by paths (maps from $[0, 1]$ into X) γ which are sent under $d_1 : C_1(X) \to C_0(X)$ to $\gamma(1) - \gamma(0)$, we want to make sure $d_0(\gamma(1) - \gamma(0)) = 0$ so that the chain complex does satisfy $d^2 = 0$. So this motivates us to define

$$d_0\left(\sum_{i=0}^{n} r_i x_i\right) = \sum_{i=0}^{n} r_i,$$

which works because we get a $1 - 1$ contribution from any image $\gamma(1) - \gamma(0)$ of a path. Keeping all other d_is the same, this **extended chain complex** is often denoted $(\tilde{C}(X), \tilde{d})$, and we define the reduced homology

$$\tilde{H}_n(X) = H_n(\tilde{C}(X), \tilde{d}) = \ker(\tilde{d}_n)/\mathrm{im}(\tilde{d}_{n+1}).$$

By the way we've defined this chain complex, $\tilde{H}_n(X)$ and $H_n(X)$ will agree for all $n > 0$, and the only difference comes in the $n = 0$ term: we have $\boxed{H_n(X) \cong \tilde{H}_n(X) \oplus \mathbb{Z}}$. So now we can rephrase our previous result for relative homology:

Theorem 104

Let $A \subseteq X$ be a subspace such that there is some open set $U \subseteq X$ with $A \subseteq U$ a deformation retraction. Then $\tilde{H}_*(X/A) \cong H_*(X, A)$.

Example 105

The n-cell $X = D^n$ is homotopy equivalent to a point (it is contractible). Then the reduced homology $\tilde{H}_n(X)$ is zero for any n (since we have one less copy of $\mathbb{Z}$ in the reduced homology compared to the ordinary one).

Example 106

This reduced homology being completely trivial actually helps us do some other calculations. For example, take $A = \partial X = S^{n-1}$ as discussed last time. Then $H_n(D^n, S^{n-1}) \cong \tilde{H}_n(D^n/S^{n-1}) \cong \tilde{H}_n(S^n)$, since quotienting the boundary of an n-ball gives us an n-sphere.

Applying a long exact sequence to the **reduced** homology, we have

$$\tilde{H}_k(S^{n-1}) \to \tilde{H}_k(D^n) \to H_k(D^n, S^{n-1}) \to \tilde{H}_{k-1}(S^{n-1}) \to \tilde{H}_{k-1}(D^n) \to \cdots.$$

But the two red terms are trivial, so the two terms between them must be isomorphic (because exactness proves injectivity and surjectivity), and we see that $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$. So now we can inductively compute the reduced homology of S^n. First of all, $S^0 = \{p, q\}$ has $H_0(S^0) = \mathbb{Z}^2$ and all other groups zero, so we also have $\tilde{H}_k(S^0) = \mathbb{Z}$ (one rank lower) and all other groups zero. This then shows that $\tilde{H}_k(S^n)$ is $\mathbb{Z}$ for $k = n$ and 0 for all other groups, and thus the actual homology is

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 107. *In doing this induction, there's two ways we can deal with the cases where $k < n$. First of all, we can compute $\tilde{H}_0(S^n)$ for each n, noticing that it's trivial because we have a path-connected space. Or alternatively, we can notice that the zigzag lemma for reduced homology can be extended to negative homology groups $-k$, with $C_{-1} = \mathbb{Z}$ (this is the space we added when extending the chain complex) and $C_{-2} = C_{-3} = \cdots = 0$.*

We'll now turn to applications: much like for fundamental groups, we can take advantage of the nontrivial homology groups $H_n(S^n)$. The details are left as exercises to us, but the argument is very similar to before:

Example 108

There is no retraction $\gamma : D^n \to S^{n-1} = \partial D^n$, because the induced map on the homology groups would need to be trivial. Also, for any continuous map $f : D^n \to D^n$, f admits a fixed point.

Example 109

For $n \neq m$, $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic – intuitively, we're saying that the dimension of $\mathbb{R}^n$ and $\mathbb{R}^m$ are actually different.

Indeed, if $f : \mathbb{R}^n \to \mathbb{R}^m$ were a homeomorphism, pick $x \in \mathbb{R}^n$ and $y = f(x)$, so that we have a homeomorphism from $\mathbb{R}^n \backslash \{x\}$ to $\mathbb{R}^m \backslash \{y\}$ (just by restricting f). Thus the homology of $\mathbb{R}^n \backslash \{x\}$ should be isomorphic to the homology of $\mathbb{R}^m \backslash \{y\}$, but $\tilde{H}_{n-1}(\mathbb{R}^n - \{x\}) = \mathbb{Z}$ because $\mathbb{R}^n - \{x\}$ retracts to S^{n-1}, while $\tilde{H}_{n-1}(\mathbb{R}^m - \{y\}) = \tilde{H}_{n-1}(S^{m-1}) = 0$ because $n \neq m$, a contradiction.

We'll now move to the discussion of **multiplicity** – we may have first encountered this when thinking about roots of polynomials (for example, $(x-1)^2 = 0$ has a root $x = 1$ of multiplicity 2), and it comes up in algebraic geometry because the map $f(z) = z^2$ in the complex plane is a 2-to-1 covering except at $z = 0$. So that tells us that the multiplicity of f at $z = 0$ is 2 (since at this singular point we get two sheets collapsing).

Our question is therefore to ask how to describe multiplicity of general continuous maps $f : X \to Y$ in algebraic topology. We'll restrict to the case where X and Y are both manifolds (meaning that for any point $x \in X$, there is an open neighborhood $U \subset X$ of x such that U is homeomorphic to some Euclidean space). To define this multiplicity $m_f(x)$, we'll need a result in homology theory:

Theorem 110 (Excision theorem)

Suppose we have subspaces $Z \subseteq A \subseteq X$ such that $\overline{Z} \subset \text{int}(A)$ (so the closure of Z is contained in the interior of A). (This is a very weak condition.) Then $H_*(X, A) \cong H_*(X - Z, A - Z)$.

(We should read the proof of this on our own – it's in our textbook.) With this, we can consider a map $f : X \to Y$ between n-dimensional manifolds and consider points $x \in X, y = f(x) \in Y$. We then take $U \subseteq X$ to be some open neighborhood of X homeomorphic to $\mathbb{R}^n$; by excision for the triple where $A = X \backslash \{x_0\}$ and $Z \backslash X - U$ (so throwing away everything except the local behavior), we see that

$$H_*(X, X - \{x\}) \cong H_*(U, U - \{x\}).$$

But U is homeomorphic to $\mathbb{R}^n$ for some n, so we can treat it as D^n so that $U - \{x\}$ is effectively S^{n-1}. So this right-hand side is basically $H_*(D^n, S^{n-1}) \cong \tilde{H}_*(S^n)$; in particular, we see that

$$H_n(X, X - \{x\}) \cong \mathbb{Z}.$$

This means that for a map $f : X \to Y$ and whenever $x = f^{-1}(y)$ for $y \in Y$ (not just $f(x) = y$), we induce a map $f_* : H_n(X, X - \{x\}) \to H_n(Y, Y - \{y\})$, which is a map $\mathbb{Z} \to \mathbb{Z}$. We then want to define the multiplicity at the point x to be $m_f(x) = f_*(1)$. Intuitively, what's happening near x is that m_f tells us how many times the input S^n wraps around the target S^n. But notice that we've switched from $x = f^{-1}(y)$ to $f(x) = y$ in the condition above – the point is that if the preimage $f^{-1}(y)$ contains more points than just x, then $f(X - \{x\}) \subsetneq Y - \{y\}$ so we can't say that we have a map $(X, X - \{x\}) \to (Y, Y - \{y\})$. But multiplicity is a local property anyway, so we can use excision to only look at a local neighborhood.

Definition 111

A map $f : X \to Y$ is **proper** if for any compact set $C \subseteq Y$, $f^{-1}(C) \subseteq X$ is also compact.

Compactness is useful here because for any proper $f : X \to Y$, plus some additional constraints, the preimage $f^{-1}(y)$ is actually **finite** for any $y \in Y$, so for any $x \in f^{-1}(y)$ we can pick some open neighborhood $U \subseteq X$ such that $U \cap f^{-1}(y) = \{x\}$. This lets us make a proper definition:

Definition 112

Let $f : X \to Y$ be a continuous map with finite preimages, let $x \in f^{-1}(y)$, and suppose we have U as above. Then f induces a map $f_* : H_n(U, U - \{x\}) \to H_n(Y, Y - \{y\})$, which is a map $\mathbb{Z} \to \mathbb{Z}$. The **multiplicity** is then $m_f(x) = f_*(1)$.

We'll now consider the special case where we have a map $f : S^n \to S^n$, in which the nth homology of S^n is $\mathbb{Z}$ so we have $f_* : \mathbb{Z} \to \mathbb{Z}$ without needing to look locally at any point.

Definition 113

The **degree** of a map $f : S^n \to S^n$ is the value of $f_*(1)$ in $\mathbb{Z}$.

This "global multiplicity" is in fact related to the multiplicity:

Theorem 114

Let $f : S^n \to S^n$ be a continuous map, and suppose $y \in S^n$ with $f^{-1}(y)$ finite. Then

$$\deg(f) = \sum_{x \in f^{-1}(y)} m_f(x)$$

(in particular, we claim this is independent of y).

Example 115

If $f : S^n \to S^n$ is the identity map, then f_* is the identity map as well so $\deg(f) = 1$. Meanwhile, if $f : S^n \to S^n$ is not surjective, then f is homotopic to a constant map (if x is missing from the image we can use a stereographic projection from x to map to $\mathbb{R}^n$), so $\deg(f) = 0$ because $H_n(S^n) \to H_n(\{x\}) \to H_n(S^n)$ is always the zero map. (On the other hand, there are indeed surjective maps with degree zero.)

Example 116

The map $f : \mathbb{C} \to \mathbb{C}$ sending $z \to z^n$ indeed has $m_f(0) = n$, so this multiplicity we've defined coincides with the algebraic one at least in this simple case.

Fact 117

Notice that two maps $f, g : S^n \to S^n$ of different degree must not be homotopic to each other, so deg is a homotopy invariant. But the converse is also true, and this is a result of Hopf: if $f, g : S^n \to S^n$ are two maps with $\deg(f) = \deg(g)$, then we actually have $f \simeq g$. (But this is a very hard theorem to prove – we can take a look at the fourth chapter of Hatcher for the details.)

This degree is actually what will lead us to **cellular homology**, a more computable homology theory for CW complexes. We'll discuss that next time – basically the degree will give us the coefficients in the differential maps when defining the chain complex.

13 November 10, 2022

Last time, we described an long exact sequence for pairs $A \subset X$ relating the homology groups of X, A, and relative homology (X, A). We mentioned that if there is an open set $U \subseteq X$ with $A \subseteq U$ a deformation retraction, then we have $\tilde{H}_*(X/A) \cong H_*(X, A)$. For example, we saw that $\tilde{H}_k(S^n) = H_k(D^n, S^{n-1})$ is $\mathbb{Z}$ if $k = n$ and 0 otherwise; we then used this to define the **degree** of a map $S^n \to S^n$ to be the image of 1 in the map $f_* : H_n(S^n) \to H_n(S^n)$ (a map $\mathbb{Z} \to \mathbb{Z}$), and we mentioned that the degree can be calculated locally if preimages are finite.

Today, we'll discuss how to use degree to define cellular homology, another homology theory which applies to CW complexes. If X is a CW complex (in which we have cells that are glued along boundaries to lower-dimensional skeletons) and we have some fixed CW structure $\{\psi_\alpha : e_\alpha^{n_\alpha} \to X^{n_\alpha - 1}\}$ for X, our chain complex $C(X)$ will be generated by the set of cells $\{e_\alpha\}$ in that structure; in other words, we have

$$C(X) = \left\{ \sum_{i=1}^{n} r_i e_i^{n_i} : r_i \in \mathbb{Z}, e_i^{n_i} \text{ cell} \right\},$$

graded by the dimension of the cell (so $C_k(X)$ is the $\mathbb{Z}$-span of the k-cells e_i^k). We now need to define a differential, which is a map $d_k : C_k(X) \to C_{k-1}(X)$ satisfying $d_{k-1} \circ d_k = 0$, meaning that for each cell $e_\alpha^{n_\alpha}$ with $n_\alpha = k$, the map takes the form

$$d_k(e_\alpha^{n_\alpha}) = \sum_{\beta \in I, n_\beta = k-1} d_k^{\alpha\beta} e_\beta^{n_\beta}.$$

(In words, the differential of a k-cell is some linear combination of $(k-1)$-cells.) Our job is then to specify all coefficients $d_k^{\alpha\beta}$ by thinking about how cells are glued to their boundary. We know that there is a map from any k-cell $\psi_\alpha : \partial e_\alpha^{n_\alpha} \to X^{k-1}$, and the skeleton X^{k-1} in particular contains $e_\beta^{n_\beta}$, so we can consider the quotient

$$X^{k-1}/(X^{k-1} \backslash \text{int}(e_\beta^{n_\beta}))$$

where we identify everything outside the interior of the $(k-1)$-cell β, which actually just gives us a sphere S^{k-1}. So composing ψ_α with this quotient map, we get a map $\tilde{\psi}_{\alpha\beta}$ in which we start with $\partial e_\alpha^{n_\alpha}$, which is a sphere S^{k-1}, and end up in the quotient sphere, which is also S^{k-1}. We will thus define

$$d_k^{\alpha\beta} = \deg(\tilde{\psi}_{\alpha\beta})$$

to be the degree of the map $S^{k-1} \to S^{k-1}$ (which is some integer), and that gives us a complete characterization of the map. Intuitively, this number tells us how many times we wrap around the $(k-1)$-cell β.

We might be concerned that there might be infinitely many nonzero coefficients $d_k^{\alpha\beta}$, since elements of the chain complex should only contain finite linear combinations. But this is where the CW complex definition comes in: $\psi_\alpha : \partial e_\alpha^{n_\alpha} \to X^{k-1}$ is a map from a compact set, so the image must also be compact in X^{k-1} (by how our topology is defined). And any compact set only intersects finitely many cells, even if the CW complex has infinitely many.

Additionally, we also have to check that this is a valid differential map – we need that $d^2 = 0$. We won't do this in too much detail, instead just explaining this at an intuitive level: if the image of ψ_α is contained in a single β-cell, meaning $\text{im}(\psi_\alpha) \subseteq \text{int}(e_\beta^{n_\beta})$, then because $e_\beta^{n_\beta}$ is contractible, ψ is nullhomotopic and thus $d_k^{\alpha\beta} = 0$ for any β, meaning the image itself is zero (we don't even need to look at d^2). On the other hand, if the image of ψ_α wraps around each of the two adjacent $(k-1)$-cells $e_\beta^{n_\beta}$ and $e_\gamma^{n_\gamma}$ once, and those two $(k-1)$-cells overlap on the X^{k-2}-skeleton. But the coefficients from $\alpha \to \beta$ and $\alpha \to \gamma$ are both 1, and then the coefficients from β and γ to their common intersection will cancel out (because we wrap around in opposite directions). That's why we should expect any coefficient to be zero.

So we do have a chain complex $C(X)$ along with a valid differential map d, and thus we can define the CW homology

$$H^{\text{CW}}(X) = H_*(C(X), d).$$

Theorem 118

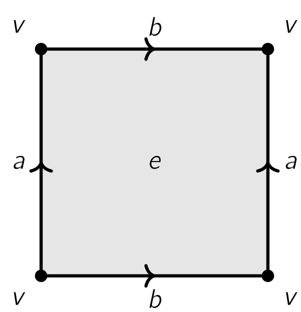
The simplicial, singular, and cellular homology theories $H_*^k(X), H_*^S(X)$, and $H_*^{\text{CW}}(X)$ are all isomorphic to each other.

In particular, for both simplicial and cellular homology we even have to answer the question of well-definedness – the homology is independent of the choice of simplicial or CW structure we put on X. And this is powerful for computation because CW structures have fewer restrictions than simplicial ones.

Example 119

We'll compute the cellular homology for surfaces, specifically thinking about the (one-holed) torus S.

We can place a simplicial structure on S and compute the simplicial homology, but we'll do so for cellular homology here. We can cut along the usual two circles for a torus and unfold to get the following picture:



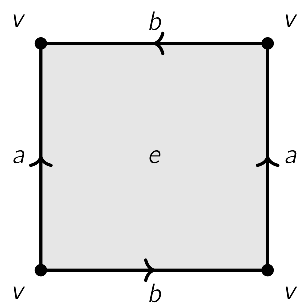
We have one 0-cell v, two 1-cells a and b, and one 2-cell e. Then $C_2(S) = \mathbb{Z}\langle e \rangle$, $C_1(S) = \mathbb{Z}\langle a, b \rangle$, and $C_0(S) = \mathbb{Z}\langle v \rangle$, and we need to define the differential map. Orienting e counterclockwise, we see that $\partial e = a - b - a + b = 0$. Similarly, $\partial a = \partial b = v - v = 0$. Thus all of the differential maps are zero, and we see that

$$H_k(S) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can similarly consider a chain complex for a two-holed torus, in which we again have a single 0-cell, four 1-cells (two for each hole), and one 2-cell, Again there will be no differentials at all, so the homology will be $\mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}$. This can in fact be generalized to general genus g.

Example 120

Next, we can compute the cellular homology for a Klein bottle, in which the gluing looks slightly different:



This time everything is the same as for the torus, except $\delta e = a - b - a - b = -2b$, so that our chain complex looks like

$$C_2(S) = \langle e \rangle \xrightarrow{e \mapsto -2b} C_1(S) = \langle a, b \rangle \xrightarrow{0} C_0(S) = \langle v \rangle \to 0.$$

Then we have $H_2(S) = 0$ because there's no kernel in the map $e \mapsto -2b$, and $H_1(S) = \langle a, b \rangle / (-2b) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Finally, $H_0(S) = \mathbb{Z}$ and we've computed all of our homology groups.

Example 121

Now, we'll find the cellular homology for S^n for $n \geqslant 2$.

We know that one possible CW complex structure for S^n consists of one 0-cell plus one n-cell, so we have $C_n(S) \to 0 \to \cdots \to 0 \to C_0(S)$. So if $n \geqslant 2$ all such maps are zero, and this shows quickly that the homology groups $H_k(S^n)$ are $\mathbb{Z}$ for $n = 0, k$ and 0 otherwise.

We'll mention one more important tool for homology computations now, in which we glue two spaces together to make a new one:

Theorem 122 (Mayer-Vietoris)

Let X be a topological space with subspaces $A, B \subseteq X$, such that $X = \text{int}(A) \cup \text{int}(B)$. Then there is a long exact sequence

$$H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to H_{n-1}(A \cap B) \to \cdots,$$

where we can notice that $H_n(A \cup B) = H_n(X)$. In fact, we can describe all of the maps here explicitly. Additionally, there is also a version with reduced homology which also holds.

In other words, knowing homology of $A \cap B$, A, and B tells us homology of $A \cup B$. The idea of the proof is that we have a standard process for generating a long exact sequence given a short exact sequence of chain complexes (the zigzag lemma), so we can make use of the short exact sequence $0 \to C(A \cap B) \to C(A) \oplus C(B) \to C(A \cup B) \to 0$ in which the maps $C(A \cap B) \to C(A) \oplus C(B)$ and $C(A) \oplus C(B) \to C(A \cup B)$ can be explicitly given: we have natural inclusions $i : C(A \cap B) \to C(A)$ and $j : C(A \cap B) \to C(B)$, and we also have natural inclusions $k : A \to A \cup B$ and

$\ell : B \to A \cup B$. We then get maps on the chain level $i_\sharp, j_\sharp, k_\sharp, \ell_\sharp$, and we claim the maps that give us a short exact sequence of chain complexes are

$$0 \to C(A \cap B) \xrightarrow{i_\sharp \oplus j_\sharp} C(A) \oplus C(B) \xrightarrow{k_\sharp - l_\sharp} C(A \cup B) \to 0.$$

So through the proof of the zigzag lemma, we can figure out the connecting map $H_n(A \cup B) \to H_{n-1}(A \cap B)$ by hand. But it turns out the answer is that elements of $H_n(A \cup B)$ are represented by n-dimensional "objects" α^n, which can be decomposed into a part in A α_A^n and a part B α_B^n, so that $\alpha^n = \alpha_A^n \cup \alpha_B^n$ intersecting along a common boundary $\partial \alpha_A^n = \partial \alpha_B^n$. Then we are basically sending α^n to this common boundary, which will be an object in $H_{n-1}(A \cap B)$.

Example 123

Consider the case where X is a two-holed torus, and we pick A and B so that A contains one of the holes and B contains the other one. (So imagine cutting the two-holed torus in half, and letting each half plus a little bit of the other be one of the two spaces we're considering.)

Then the homology of $A \cap B$ follows the long exact sequence in Mayer-Vetoris for reduced homology

$$\tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_1(B) \to \tilde{H}_1(A \cup B) \to \tilde{H}_0(A \cap B)$$

Since the intersection of A and B is a connected annular region, $\tilde{H}_0(A \cap B)$ and $\tilde{H}_1(A \cap B) \cong \tilde{H}_1(S)$. Similarly, we have the sequence

$$\boxed{\tilde{H}_2(A \cap B) \to \tilde{H}_2(A) \oplus \tilde{H}_2(B) \to \tilde{H}_2(A \cup B) \to \tilde{H}_1(A \cap B) \to \tilde{H}_1(A) \oplus \tilde{H}_2(B)},$$

in which we know that $\tilde{H}_2(A \cap B) = \tilde{H}_2(S^2) = 0$, but we don't know how to compute $\tilde{H}_2(A)$ or $\tilde{H}_2(B)$. for that, notice that both of them are basically a punctured torus, so we can think about the Mayer-Vietoris sequence for a torus covered by A and D^2. That gives us

$$\tilde{H}_2(A \cap D^2) \to \tilde{H}_2(A) \oplus \tilde{H}_2(D^2) \to \tilde{H}_2(A \cup D^2) \xrightarrow{\partial *} \tilde{H}_1(A \cap D^2),$$

and these spaces are $0 \to \tilde{H}_2(A) \to \mathbb{Z} \xrightarrow{\text{isomorphism}} \mathbb{Z}$, so in fact $\tilde{H}_2(A)$ must be trivial. Similarly, we can find that $\tilde{H}_1(A) = \mathbb{Z}^2$ by looking further down that sequence. So plugging back in to our boxed sequence, we find that it is $0 \to 0 \to \tilde{H}_2(A \cup B) \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \oplus \mathbb{Z}^2)$, and this shows that we must have $\tilde{H}_2(A \cup B) \cong \mathbb{Z}$. Similarly, we can find that $\tilde{H}_1(A \cup B) \cong \mathbb{Z}^4$, and we indeed see that we get back the same answer with Mayer-Vietoris as with our previous method above.

14 November 15, 2022

We'll finish our discussion of homology theory today with a few remarks. The first thing we'll talk about is **orientation** – we'll focus on manifolds here, in which we have a topological space X with an open cover $\{U_\alpha\}_{\alpha \in I}$ such that there are homeomorphisms $\rho_\alpha : \mathbb{R}^n \to U_\alpha$ for each α. (There are also some other conditions that are required, but this is the most important one.) We see that our manifold X is smooth (this also gives definitions for C^1, C^2, and so on), if the transition map $\rho_\beta^{-1} \circ \rho_\alpha$ from $\rho_\alpha^{-1}(U_\alpha \cap U_\beta)$ to $\rho_\beta^{-1}(U_\alpha \cap U_\beta)$ is a smooth map $\mathbb{R}^n \to \mathbb{R}^n$.

Our goal is then to define an orientation for manifolds. For smooth manifolds this is easy, because the differential $d(\rho_\beta^{-1} \circ \rho_\alpha)_x$ at any $x \in \rho_\alpha^{-1}(U_\alpha \cap U_\beta)$ is a map $\mathbb{R}^n \to \mathbb{R}^n$. Since $\rho_\beta^{-1} \circ \rho_\alpha$ is a diffeomorphism, $d(\rho_\beta^{-1} \circ \rho_\alpha)_x$ is nondegenerate for all x (meaning that the corresponding $n \times n$ matrix is of full rank).

Definition 124

A manifold X is **orientable** if we can pick an open cover $\{U_\alpha\}_{\alpha \in I}$ so that $\det(d(\rho_\beta^{-1} \circ \rho_\alpha)_x) > 0$ for all $\alpha, \beta \in I$ and $x \in \rho_\alpha^{-1}(U_\alpha \cap U_\beta)$.

Intuitively, the point is that we have local **coordinate charts** at every open set in our cover, and we want the canonical choice of $\mathbb{R}^n$ to agree in orientation when we transition from one open set to another. But this only works if the transition map is differentiable, and for an arbitrary topological manifold we cannot use this definition. So instead we want to ask what happens in general. In particular, the map $\rho_\beta^{-1} \circ \rho_\alpha : \rho_\alpha^{-1}(U_\alpha \cap U_\beta) \to \rho_\beta^{-1}(U_\alpha \cap U_\beta)$ will just be continuous, not differentiable, and in such a case we will **use homology to define orientation instead**.

Definition 125

Let $f : U \to V$ be a homeomorphism between open subsets U, V of $\mathbb{R}^n$. Then for any $x \in U$, we have a map $H_*(U, U - \{x\}) \to H_*(V, V - \{f(x)\})$ (this is well-defined because $f(x)$ is only mapped to by x), which is a map $\mathbb{Z} \to \mathbb{Z}$ by excision. Then 1 is sent to either 1 or -1 because f is a homeomorphism; we say that f is **orientable** if 1 is sent to 1.

Definition 126

A topological manifold is **orientable** if there is an open cover $\{U_\alpha\}$ such that for any $\alpha, \beta \in I$, $\rho_\beta^{-1} \circ \rho_\alpha$ is orientable.

In most cases we can use the differential definition, but there are some manifolds that are not "smoothable," so this homology definition is more general.

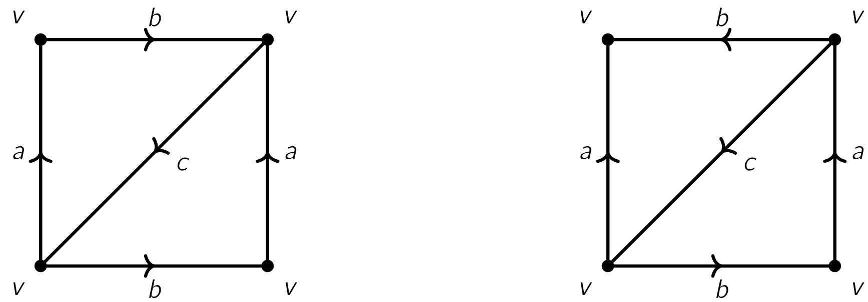
Next, we'll see how all of this is related to simplicial complexes – recall that we can define orientation in that setting, but not all topological manifolds can be triangulated. Suppose X is a topological manifold which does admit this simplicial structure. Notice that if we have an n-dimensional topological manifold, then we must basically take a bunch of n-dimensional simplices and glue them along $(n-1)$-dimensional faces, which is much more constrained than the usual construction.

But recall that for a single simplex, we define an orientation by choosing the parity of ordering of its vertices (we get the same orientation if we perform an even permutation on that ordering). And for any codimension-1 (meaning dimension $n-1$) face of an n-simplex $\Delta^n = [x_0, \cdots, x_n]$, we get an **induced orientation** on the n vertices $x_0, \cdots, x_{i-1}, x_{i+1}, x_n$ – specifically, we need to use $(-1)^i[x_0, x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$. (That's what appears in the definition of the differential map when we constructed homology that way.) Repeating this process repeatedly gives us orientation on all sub-simplices.

So turning back to our n-dimensional topological manifold X admitting a simplicial structure, it turns out that X is orientable if and only if each n-simplex in the given structure can be oriented so that the induced orientations on common codimension-1 faces are **opposite to each other**.

Example 127

Recall that both the torus and Klein bottle are obtained by gluing opposite edges of a square together, and we can form triangulations in both cases by drawing one of the diagonal lines of the square.



The torus is shown on the left, and the Klein bottle is shown on the right, with their corresponding triangulations. Consider the top left triangle Δ_1 and place an orientation on it as given by the arrows. If we call the bottom left vertex $v_0^{(1)}$, the top left vertex $v_1^{(1)}$, and the top right vertex $v_2^{(1)}$, then $\Delta_1 = [v_0^{(0)}, v_1^{(1)}, v_2^{(2)}]$, so that $a = [v_0^{(1)}, v_1^{(1)}]$, $b = [v_1^{(1)}, v_2^{(1)}]$, and $c = -[v_0^{(1)}, v_2^{(1)}]$. We can then check that we can choose one of the two orientations on the bottom right triangle Δ_2, so that the orientations for a, b, c (which are all identified with a, b, c in Δ_1) are all opposite. But we can check that kind of process will not work for either of the two choices in the Klein bottle, which is not orientable.

So now if we look at the top of the chain complex for an orientable n-dimensional manifold X, then we already know some elements of the kernel $d_n : C_n(X) \to C_{n-1}(X)$: the sum $\sum \Delta^n$ will be in $\ker(d_n)$ because the opposite faces will all cancel out. So there is some nontrivial element in the top homology $H_n(X) = \ker(d_n)/\text{im}(d_{n+1})$. This turns out to be a general fact:

Theorem 128

Let X be a closed, connected n-dimensional manifold. Then $H_n(X) \cong \mathbb{Z}$ if X is orientable and $H_n(X) \cong 0$ otherwise.

In particular, the actual result holds whether or not X admits a simplicial structure. And it will turn out that certain duality results only hold if we have a nontrivial top homology, which we will see later.

Given the result above, it turns out we can also define degree for a general orientable manifold:

Definition 129

If $f : X \to Y$ is a continuous map between connected, closed, oriented manifolds, then there is a map $f_* : H_n(X) \to H_n(Y)$ which is a map $\mathbb{Z} \to \mathbb{Z}$. Then the **degree** of f is the image of 1 under f_*.

We're now going to turn to some "abstract nonsense:" we've discussed topological spaces and continuous maps between them (along with homotopy and some other properties), and there's also some similar structure that comes up when we have groups and homomorphisms between them, or abelian groups and homomorphisms between them. These are all very similar "packages," and we can even map between them (for example, π_1 maps from topological spaces to groups, and H_* maps from topological spaces to abelian groups) in a way that also sends continuous maps to group homomorphisms. The point is that we can make all of this more abstract using **category theory**, a way to describe this phenomenon more generally.

Definition 130

A **category** $\mathcal{C}$ consists of a collection of objects $\text{ob}(\mathcal{C})$ (such as the collection of groups, or topological spaces, or abelian groups), as well as a collection of morphisms $\text{Mor}(X, Y)$ for any objects $X, Y \in \mathcal{C}$, such that the following properties hold:

- For any $X \in \mathcal{C}$, there is a distinguished element $\text{id}_X \in \text{Mor}(X, X)$.
- There is a map $\circ : \text{Mor}(X, Y) \times \text{Mor}(Y, Z) \to \text{Mor}(X, Z)$ sending (f, g) to $g \circ f$, such that $\text{id}_Y \circ f = f \circ \text{id}_X = f$.
- For all morphisms f, g, h, we have $(f \circ g) \circ h = f \circ (g \circ h)$.

Remark 131. *There's some problem in set theory where we can't actually say that we have a "set of all topological spaces" without a Russell's paradox coming up – that's why we use the word "collection" instead.*

For example, **Top** is the category of topological spaces, where the set of morphisms from X to Y is the set of all continuous maps $X \to Y$. We can make similar definitions for the category of sets **Set**, the category of groups **Grp**, and the category of abelian groups **AbGrp**. And then all of the conditions are basically generalizing the structures of composition and identity and so on that we're used to.

One particular example that's relevant for us is the category **Pairs**, which has objects of the form (X, A) where $A \subseteq X$ and has morphisms $f : (X, A) \to (Y, B)$ in which $X \to Y$ is continuous and $f(A) \subseteq B$. We can also define the category **Based**, which is a subcategory of **Pairs** in which we must have (X, x_0) and where a morphism is a map $f : (X, X_0) \to (Y, y_0)$ in which $f(x_0) = y_0$. Finally, we can think about a category **Homotopy**, in which the objects are topological spaces and the morphisms are a refinement of those in **Top**: they are the set of continuous maps up to homotopy. Since homotopy composes, composition is still respected, and we just have a smaller set of morphisms between any two topological spaces.

Our construction of fundamental groups and homology groups can both be thought of as relations between categories, in which we change from one to another:

Definition 132

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of maps $F : \text{ob}(\mathcal{C}) \to \text{ob}(\mathcal{D})$ and $F : \text{Mor}(X, Y) \to \text{Mor}(F(X), F(Y))$ for all $X, Y \in \mathcal{C}$, such that $F(\text{id}_X) = \text{id}_{F(X)}$ and $F(f \circ g) = F(f) \circ F(g)$ (this is called the **functoriality** property).

Example 133

Our construction of fundamental groups π_1 : **Based** $\to$ **Group** is a functor. Indeed, we map each (X, x_0) to a group $\pi_1(X, x_0)$, and we go from a continuous map of based topological spaces to a group homomorphism of their fundamental groups.

Example 134

For any $k \in \mathbb{Z}$, we have a functor H_k : **Top** $\to$ **AbGrp** sending X to $H_k(X)$. Similarly, reduced homology $\tilde{H}_k$ is also a functor **Top** $\to$ **AbGrp**, and relative homology is a functor **Pairs** $\to$ **AbGrp** sending (X, A) to $H_k(X, A)$.

Example 135
We have a functor **Top** → **Homotopy** sending X to X itself and f to its class $[f]$, but we know that two continuous maps that are in the same homotopy class have the same homology. Thus H_k : **Top** → **AbGrp** factors through **Homotopy** via the ("forgetful") functor **Top** → **Homotopy**.

Remark 136. *Noticing that we can compose functors, we may ask whether there is a category of all categories with morphisms given by functors. But it turns out we run into set-theoretic issues there.*

Recall that we stated that the different H_k homology constructions give us the same map, but we didn't show fundamentally why that was the case. It turns out that there's some fundamental axioms for homology called the **Eilenberg-Steenrod axioms**, such that if a homology theory satisfies those axioms, it will agree with the constructions we've already made. We say that we have a **homology theory** if we have a sequence of functors H_k : **Pair** → **AbGrp** (where we can treat a single topological space X as $(X, \varnothing)$) together with a **natural transformation** (we won't make the definition here, but it's basically a map between functors) $\partial_* : H_k(X, A) \to H_{k-1}(A)$ (previously ∂_* was the connecting map). Then we want the homology theory to satisfy these axioms:

1. Homotopy: under the functor, $f : (X, A) \to (Y, B)$ is sent to $f_* : H_k(X, A) \to H_k(Y, B)$. Then if $f \simeq g$, we require that $f_* = g_*$.
2. Excision: if $Z \subseteq A \subseteq X$ with $\text{cl}(Z) \subseteq \text{int}(A)$, then $H_k(X, A) \cong H_k(X - Z, A - Z)$.
3. If X is a single point, then $H_k(X, \varnothing) = \mathbb{Z}$ if k is trivial and 0 otherwise.
4. Union: if $X = \amalg_{\alpha \in I} X_\alpha$, then $H_k(X) = \bigoplus_{\alpha \in I} H_k(X_\alpha)$. (Notice that direct sum and product agree when we only have finitely many terms, but not in general.)
5. There is a long exact sequence $H_n(A) \to H_n(X) \to H_{n-1}(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$.

So with (1) and (3) we can compute all contractible spaces, (5) allows us to compute pairs, (2) lets us compute quotients, particularly S^n, which allows us to pass to cellular homology along with (4).

15 November 17, 2022

We'll start our last topic of the course, cohomology, today – it'll be similar to homology but with some different constructions and some different properties. We can use any of the homology theories we've developed to construct cohomology – we'll use singular homology here. Recall that $C_n(X)$ is the set of sums $\{\sum_{i=1}^n r_i \sigma_i\}$, where $r_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ are continuous maps. We'll now construct the corresponding dual space

$$C^n(X) = \text{Hom}(C_n(X), \mathbb{Z}),$$

the set of group homomorphisms $C_n(X) \to \mathbb{Z}$ (equivalently, the $\mathbb{Z}$-valued linear functions on $C_n(X)$). (Note that we have to be careful trying to identify C_n with C^n because of infinite-dimensional considerations.) To make this into a chain complex, we need to construct a new differential $\delta^n : C^n(X) \to C^{n+1}(X)$ (going in the opposite direction as before, increasing the grading). We already have an old differential $d_{n+1} : C_{n+1}(X) \to C_n(X)$ from homology, and we can define δ^n in the following way: any $f \in C^n(X)$ is a homomorphism $C_n(X) \to \mathbb{Z}$, and if we pre-compose it by d_{n+1}, we get a homomorphism $f \circ d_{n+1} : C_{n+1}(X) \to \mathbb{Z}$. That will then be an element of $C^{n+1}(X)$, and we call that element $\delta^n f$. (We then call this the "dual" of the map d_{n+1} – this construction is following the general idea that a map

$f : A \to B$ leads us to a map $f^* : B^* \to A^*$ for a general abelian group A.) This can be represented in the following diagram:

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{\text{dual}} & \text{Hom}(C_{n+1}, \mathbb{Z}) \\ \big\downarrow{\scriptstyle d_{n+1}} & & {\scriptstyle \delta^n}\big\uparrow \\ C_n & \xrightarrow{\text{dual}} & \text{Hom}(C_n, \mathbb{Z}) \end{array}$$

Then $\delta^{n+1} \circ \delta^n = 0$ as a map $C^n(X) \to C^{n+2}(X)$ just from the definition: for any $f \in C^n(X)$, we have

$$\delta^{n+1} \circ \delta^n(f) = \delta^{n+1}(f \circ d_{n+1}) = f \circ d_{n+1} \circ d_{n+2},$$

which is always the zero map and thus $\delta^{n+1} \circ \delta^n$ sends anything to zero. So given any topological space X with a singular chain complex $C_n(X)$, we can define $C^n(X) = \text{Hom}(C_n(X), \mathbb{Z})$ and define a differential $\delta^n(f) = f \circ d_{n+1}$, and (C^n, δ^n) will be a valid chain complex and we can define a homology theory for it, which we call cohomology:

Definition 137

The ***n*th cohomology of *X***, denoted $H^n(X)$, is

$$H^n(X) = \ker(\delta^n)/\text{im}(\delta^{n-1}).$$

Since we're looking at the dual spaces to the ones from homology, there are a few expected properties that we now have:

- A continuous map $f : X \to Y$ now yields a map in cohomology in the opposite order – it induces a map $f^\sharp : C^*(Y) \to C^*(X)$ and thus a map $f^* : H^*(Y) \to H^*(X)$.
- If $f \sim g : X \to Y$ are homotopic maps, then we induce equal maps $f^* = g^* : H^*(Y) \to H^*(X)$.
- The composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ gives us an induced map $(g \circ f)^* = f^* \circ g^* : H^*(Z) \to H^*(X)$. So this is functoriality in the opposite order.
- We can define a cohomology for pairs similar to the homology for pairs: if $A \subseteq X$, then we can define $C^n(X, A) = \{f \in C_n(X) : f|_A = 0 \in C^n(A)\}$. We can still define a δ differential on this chain complex, which gives us cohomology $H^n(X, A)$.
- Just like in homology, we can write a long exact sequence coming from the short exact sequence of chain complexes, where the sequence of arrows is reversed:

$$\cdots \leftarrow H^n(A) \leftarrow H^n(X) \leftarrow H^n(X, A) \leftarrow H^{n-1}(A) \leftarrow \cdots.$$

- Excision still holds: if $Z \subseteq A \subseteq X$ so that $\overline{Z} \subseteq \text{int}(A)$, then $H^*(X, A) \cong H^*(X - Z, A - Z)$.
- Similarly, we still have a Mayer-Vietoris sequence but with all arrows reversed: if $A, B \subseteq X$ with $X = \text{int}(A) \cup \text{int}(B)$, then we have a long exact sequence

$$\cdots \leftarrow H^n(A \cap B) \leftarrow H^n(A) \oplus H^n(B) \leftarrow H^n(X) \leftarrow H^{n-1}(A \cap B) \leftarrow \cdots.$$

Beyond reversing arrows, though, the reason for cohomology theory is that there are some unique structures that don't show up in homology. Specifically, there is a **cup product** $\cup : H^k(X) \times H^\ell(X) \to H^{k+\ell}(X)$ (which we will talk about next time), which makes cohomology into a **ring** (not just an abelian group). Additionally, cohomology has various applications, which we'll show now:

Fact 138

On a smooth manifold, there is a de Rham cohomology – there are covariant derivatives on differential forms, where the derivative of a k-form is a $(k+1)$-form and the square of that derivative is zero. So we can in fact do analysis on manifolds, which connects to many branches of mathematics. And like with homology, the different cohomology theories are all equivalent – there is also a set of axioms for cohomology just like the ones we saw last time for homology.

Fact 139

Characteristic classes are certain cohomology classes that obstruct or classify certain structures (like existence of bundles over a manifold). For example, if M is a closed and smooth n-manifold, then the Whitney embedding theorem says that M embeds into $\mathbb{R}^{2n}$, and $2n$ is the minimal bounding constant. Specifically, $\mathbb{R}P^n$ does not embed into $\mathbb{R}^{2n-1}$, and we can see this using a certain cohomology construction that obstructions the embedding. We can read Milnor's book 'Characteristic Classes" for more.

Before we dive more into properties of cohomology, we want to first find a relation between homology and cohomology. We know that $H^n(X) = \ker(\delta^n)/\text{im}(\delta^{n-1})$ and $H_n(X) = \ker(d_n)/\text{im}(d^{n+1})$, so given an element class $[\alpha] \in H^n(X)$ and an element $[x] \in H_n(X)$, α can be thought of as some element of $C^n(X)$ with $\delta^n\alpha = 0$ (an actual element in the cochain complex sent to zero under the differential), and similarly x can be thought of as some element of $C_n(X)$ with $d_n x = 0$. But by definition, C^n is the dual of C_n – α is a map from $C_n(X) \to \mathbb{Z}$. Thus we can evaluate α at x, and we can check that this evaluation gives rise to a **well-defined** map

$$e : H^n(X) \times H_n(X) \to \mathbb{Z}, \quad ([\alpha], [x]) \to \alpha(x)$$

which is $\mathbb{Z}$-bilinear. Thus we get a natural map $e : H^n(X) \to \text{Hom}(H_n(X), \mathbb{Z})$ sending each element to the evaluation map we wrote above. On the chain level this is an identity map, but on the homology level we won't necessarily have an isomorphism:

Theorem 140 (Universal coefficients theorem)

There is a split exact sequence

$$0 \to \text{Ext}(H_{n-1}(X), \mathbb{Z}) \to H^n(X) \to \text{Hom}(H_n(X), \mathbb{Z}) \to 0,$$

where being **split** means that there is an isomorphism that represents the middle term $H^n(X)$ as a direct sum $\text{Ext}(H_{n-1}(X); \mathbb{Z}) \oplus \text{Hom}(H_n(X), \mathbb{Z})$, and where Ext is explained below.

So what this means is that we can calculate cohomology from homology as long as we know what the Ext map is. We won't define what the Ext functor is in full generality, but we will explain how to compute it:

- If A, B, C are abelian groups, then $\text{Ext}(A \oplus B, C) = \text{Ext}(A, C) \oplus \text{Ext}(B, C)$.
- $\text{Ext}(A, B) = 0$ if A is free.
- $\text{Ext}(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA = \text{coker}(A \xrightarrow{n} A)$ (that is, the cokernel of the multiplication-by-n map). We will sometimes write $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}_n$.

However, note that Ext is not symmetric: for example,

$$\text{Ext}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0, \quad \text{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}.$$

The idea is that if we have a torsion part of $H_n(X)$, then it will not contribute to $\text{Hom}(H_n(X), \mathbb{Z})$, and if we have a free part of $H_{n-1}(X)$, it will not contribute to $\text{Ext}(H_{n-1}(X), \mathbb{Z})$.

Here we actually just have $\text{Ext}(H_{n-1}(X), \mathbb{Z}) = \text{Tor}(H_{n-1}(X))$, but if we were doing cohomology with **different coefficients**, not necessarily $\mathbb{Z}$, then Ext could be more complicated. For a more general abelian group G, we can define the singular chain and cochain complex

$$C_n(X; G) = \left\{ \sum_{i=1}^{n} g_i \circ \sigma_i : g_l \in G, \sigma_i : \Delta^n \to X \right\}$$

and $C^n(X; G) = \text{Hom}(C_n(X), G)$. We then get corresponding differential maps $d_n : C_n(X; G) \to C_{n-1}(X; G)$ and $\delta^n : C^n(X; G) \to C^{n+1}(X; G)$ as before – all the proofs and constructions are the same – so we get homology $H_n(X; G)$ and cohomology $H^n(X; G)$ with the same definition (kernel of one map modded out by the image of the next). And with this, we can state a more general universal coefficients theorem as well – we'll write $H_n(X)$ for $H_n(X; \mathbb{Z})$ and specify the group G otherwise.

Theorem 141 (Universal coefficient theorem, general version)
We have the split exact sequence

$$0 \to \text{Ext}(H_{n-1}(X), G) \to H^n(X; G) \to \text{Hom}(H_n(X); G) \to \mathbb{Z}$$

where $H_{n-1}(X)$ and $H_n(X)$ are using integer coefficients. We also have the split exact sequence

$$0 \to H_n(X) \otimes_{\mathbb{Z}} G \to H_n(X; G) \to \text{Tor}(H_{n-1}(X), G) \to 0.$$

Similarly, we'll mention a few properties of the Tor functor so that we can do computations for it:

- For any abelian groups A, B, C, $\text{Tor}(A \oplus B, C) = \text{Tor}(A, C) \oplus \text{Tor}(B, C)$.
- $\text{Tor}(A, B) = \text{Tor}(B, A)$ (so Tor is symmetric, unlike Ext),
- $\text{Tor}(A, B) = 0$ if either A or B is free.
- $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{n} A)$.

Example 142
Suppose X is a closed connected manifold of dimension n. Recall that the top dimension homology is either $\mathbb{Z}$ or 0 depending on whether X is orientable or not, and if X admits a simplicial structure and is orientable, then $H_n(X)$ is generated by $\sum_{\Delta^n} \Delta^n$, the sum of all n-simplices (since each codimension-1 face is in two simplices with opposite orientations which cancel out under the differential d). But if we instead work with $\mathbb{Z}/2\mathbb{Z}$-coefficients, no matter which orientation we give, each codimension-1 face will be counted twice, and $2 = 0$. So $H_n(X; \mathbb{Z}_2)$ is always nonzero.

In particular, if X is not orientable, $H_n(X) = 0$, but $H_n(X; \mathbb{Z}_2)$ is nontrivial. Thus by the universal coefficients theorem, looking at the second split exact sequence, we know that $\text{Tor}(H_{n-1}(X), \mathbb{Z}_2) \cong H_n(X; \mathbb{Z}_2)$ because $H_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2$

is zero. Thus $\text{Tor}(\mathbb{Z}_2, H_{n-1}(X)) \neq 0$, meaning that multiplication by 2 has a nontrivial kernel in $H_{n-1}(X)$. That can be restated in the following way:

Proposition 143

Let X be a closed n-dimensional manifold which is not orientable. Then $H_{n-1}(X)$ has 2-torsion.

(For example, if X is the Klein bottle, $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ (from our past calculations), and if X is $\mathbb{R}P^2$, then $H_1(X) \cong \mathbb{Z}_2$.) Now to look at another kind of example, we can apply the first split exact sequence for $H^1(X)$ (since we know $H_0(X)$): applying it with $G = \mathbb{Z}$, we see that

$$H^1(X) \cong \text{Ext}(H_0(X); \mathbb{Z}) \oplus \text{Hom}(H_1(X); \mathbb{Z}).$$

But $H_0(X)$ is a free abelian group, so by our rules above Ext of it is zero. So $\boxed{H^1(X) \cong \text{Hom}(H_1(X); \mathbb{Z})}$, and in fact $\text{Hom}(H_1(X); \mathbb{Z})$ is a free abelian group. That tells us the following statement:

Proposition 144

For any topological space X, $H^1(X)$ has no torsion.

In the first example above, we saw that whenever X is non-orientable, $H_{n-1}(X)$ has 2-torsion. We'll see later on in the class that whenever X is orientable, **Poincaré duality** tells us that $H_{n-1}(X) \cong H^1(X)$, so $H_{n-1}(X)$ must be free. So there will be more systematic approaches for studying the structure of these different homology and cohomology groups, and we'll study those in the rest of the course.

16 November 29, 2022

We'll discuss the **cup product** construction today: our goal is to construct a map $\cup : H^m(X) \times H^n(X) \to H^{m+n}(X)$, making cohomology into a ring with unit. We will define it by first constructing a product on the cochain level $\cup : C^m(X) \times C^n(X) \to C^{m+n}(X)$, so first we will review what all of that means. Recall that if we're working with singular homology, then $C_k(X)$ consists of formal sums $\sum_{i=1}^{\ell} r_i \sigma_i$ where $r_i \in \mathbb{Z}$, $\sigma_i : \Delta^k \to X$ is a singular simplex, and we have the differential map $d_k : C_k(X) \to C_{k-1}(X)$ defined by an alternating sum over codimension 1 simplices. We then define $C^k(X) = \text{Hom}(C_k, X)$ to be the dual space of $C_k(X)$, where we define the dual map δ_{k-1} by saying that for any $f : C_{k-1}(X) \to \mathbb{Z}$, the map $\delta_{k-1}(f) \to C_k(X)$ is defined via

$$\delta_{k-1}(f)(\sigma^k) = f(d_k \sigma^k).$$

So now suppose we have two maps $\psi : C_m(x) \to \mathbb{R}$ and $\phi : C_n(X) \to \mathbb{R}$, and we want to define $\psi \cup \phi : C_{m+n}(X) \to \mathbb{R}$. Remembering that the simplex Δ^k has a standard embedding $\{(x_0, \cdots, x_k) : \sum_{i=0}^{k} x_i = 1, x_i \geqslant 0\}$, giving us a canonical ordering of its vertices $v_0 = (1, 0, \cdots, 0)$, $v_1 = (0, 1, \cdots, 0), \cdots, v_k = (0, 0, \cdots, 1)$. A map $C_{m+n} \to \mathbb{Z}$ would then need to be defined on all simplices $[v_0, v_1, \cdots, v_{m+n}]$, and we define

$$(\psi \cup \phi)([v_0, v_1, \cdots, v_{m+n}]) = \psi([v_0, \cdots, v_m])\phi([v_m, v_{m+1}, \cdots, v_{m+n}]).$$

This then defines a map $C_{m+n} \to \mathbb{Z}$. (More generally, if we're working with homology and cohomology with coefficients, we will need a ring for the base coefficients so that we can actually multiply those two values together.) We can explore a few properties of this map we've just defined:

- Applying the differential to our cup product, $\delta(\psi \cup \phi) = (\delta\psi) \cup \phi + (-1)^m \psi \cup (\delta\phi)$. (where $\psi \in C^m(X)$).
- Since a class in cohomology $H^m(X)$ is a class $[\psi]$ where $\delta\psi = 0$, we can define the cup product on cohomology $\cup : H^m(X) \times H^n(X) \to H^{m+n}(X)$ by taking the cup product on the representative ψs:

$$[\psi] \cup [\phi] \mapsto [\psi \cup \phi].$$

  We must check that $\psi \cup \phi$ is in fact in the kernel of the differential map if ψ, ϕ are, and we must also check that this definition is well-defined (yielding the same result within a given class). But by the previous point, $\delta(\psi \cup \phi) = (\delta\psi) \cup \phi + (-1)^m \psi \cup (\delta\phi)$, and by assumption both terms on the right-hand side are zero so the left-hand side is zero as well. Thus $\psi \cup \phi$ does represent a cohomology class. For well-definedness, we must show for example that if $[\psi'] = [\psi]$, then $[\psi \cup \phi] = [\psi' \cup \phi] \in H^{m+n}(X)$ (and then we just need to do the same with the second argument, but it's the same reasoning). If $[\psi'] = [\psi]$, that means $\delta\psi' = \delta\psi = 0$ and $\psi - \psi' \in \text{im}(\delta)$. Thus $\psi - \psi' = \delta\eta$ for some $\eta \in C^{m-1}(X)$. But then

$$\psi \cup \phi - \psi' \cup \phi = (\psi - \psi') \cup \phi = (\delta\eta) \cup \phi,$$

  and now we can apply the first bullet point again: because $\delta\phi = 0$, this is also

$$= (\delta\eta) \cup \phi + (-1)^{m-1} \eta \cup (\delta\phi) = \delta(\eta \cup \phi),$$

  which is zero in cohomology. Thus if ψ and ψ' are in the same class, then $[\psi \cup \phi] = [\psi' \cup \phi]$ as desired.

Remark 145. *We may ask why this is more natural in cohomology than in homology – one reason comes from the case where X is a smooth manifold and we can look at de Rham cohomology, studying the differential forms. Then the wedge product actually coincides with the cup product.*

- Next, suppose $f : X \to Y$ is a continuous map. Then $f^* : H^*(Y) \to H^*(X)$ preserves the cup product – that is, $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$. In other words, f^* is actually a **ring homomorphism**.
- This cup product has a unit when X is path-connected. Indeed, since the cup product maps $H^m(X) \times H^n(X) \to H^{m+n}(X)$, the unit must be something in $H^0(X)$. We know that $H^0(X) \cong \mathbb{Z}$ (for example by the universal coefficients theorem because $H_0(X) \cong \mathbb{Z}$), and $1 \in H^0(X)$ will then be a unit. More explicitly, $C_0(X)$ consists of finite integer linear combination of points in X, so $C^0(X)$ is a map $C_0(X) \to \mathbb{Z}$, which we can think of basically as a map $f : X \to \mathbb{Z}$. So taking the constant map c sending all of X to 1 (which represents the generator in cohomology – exercise), we can check that $c \cup \psi = \psi \in C^m(X)$.

It turns out that this ring structure almost forms a commutative ring:

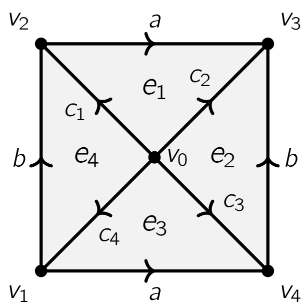
Theorem 146
If $\alpha \in H^m(X)$ and $\beta \in H^n(X)$, then $\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha$.

This is very similar to how wedge products work for differential forms, since we need to switch mn "pairs" and pick up a negative sign each time. This result is true on the homology level, and if we try to pass it to the chain level we should expect a chain homotopy. (But we should read through the proof on our own.)

Example 147

To make our computation easier, we'll go back to simplicial structures and look only at simple surfaces via triangulation. Consider the simplicial structure of the torus X given by identifying opposite edges of a square in the same orientation:

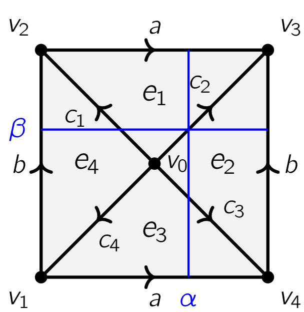
Here, c_1, c_2, c_3, c_4 represent the outgoing edges from the central vertex, and e_1, e_2, e_3, e_4 represent the faces.



We'll orient everything so that the central vertex is always the "first vertex," and for each of the four faces we orient the other two vertices in terms of the orientation of the edge mentioned. We then find that $de_1 = c_1 + a - c_2$, $de_2 = c_3 + b - c_2$, $de_3 = c_4 + a - c_3$, $de_4 = c_4 + b - c_1$, and the differentials for each edge are just the differences of the end and start vertices. Recall that $H_2(X) \cong \mathbb{Z}$, but explicitly it turns out to be generated by $e_1 - e_2 - e_3 + e_4$. Then $H_1(X) \cong \mathbb{Z}^2$, generated by a and b, and $H_0(X)$ is generated by any of the vertices, say v_0. To find the cohomology, we can then use the universal coefficients theorem:

$$H^n(X) \cong \text{Hom}(H_n(X), \mathbb{Z}) \oplus \text{Ext}(H_{n-1}(X), \mathbb{Z}),$$

but all homology groups are free so there is no torsion, meaning we have isomorphism between homology and cohomology. Thus $H^n(X) \cong \mathbb{Z}$ if $n = 0, 2$, $\mathbb{Z}^2$ if $n = 1$, and 0 otherwise. Our goal is then to find the **dual basis** for $H^1(X)$ – that is, we want $H^1(X)$ generated by $[\alpha]$ and $[\beta]$ such that $\alpha(a) = 1, \alpha(b) = 0, \beta(a) = 0, \beta(b) =$, and then we want to understand $\alpha \cup \beta$. To do this, we must first find maps $C_1(X) \to \mathbb{Z}$ such that $\delta\alpha = \delta\beta = 0$. There are six one-dimensional simplexes $(a, b, c_1, c_2, c_3, c_4)$, but we can think about the two blue lines in the diagram below which are really loops on the torus:



Motivated by where these points intersect the lines, we define $\alpha(a) = \alpha(c_2) = \alpha(c_3) = 1$ and $\alpha(b) = \alpha(c_1) = \alpha(c_4) = 0$, and similarly we define $\beta(b) = \beta(c_1) = \beta(c_2) = 1$, and $\beta(a) = \beta(c_3) = \beta(c_4) = 0$. We can check that $\delta\alpha = \delta\beta = 0$ as map $C_2(X) \to \mathbb{Z}$, but that just needs to be done by checking that $\delta\alpha(e_i) = 0$ and $\delta\beta(e_i) = 0$ for each i. Indeed,

$$\delta\alpha(e_1) = \alpha(de_1) = \alpha(c_1 + a - c_2) = 0 + 1 - 1 = 0,$$

and similarly all of the other checks work out, so α, β do represent classes in cohomology. And indeed α, β evaluate correctly on a and b (more precisely we evaluate in cohomology, with $[\alpha]([a]) = \alpha(a) = 1$ and so on), so we have found a dual basis for $H^1(X)$. We can now compute the cup product, but $\alpha \cup \beta$ should lie in $H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z}) \cong \mathbb{Z}$, so we should find a generator in $H^2(X)$. Indeed, the generator of $H_2(X)$ is $[e_1 - e_2 - e_3 + e_4]$, so we can choose $[\gamma] \in H^2(X)$ such that $[\gamma]([e_1 - e_2 - e_3 + e_4]) = 1$. We now just need to compare $[\alpha \cup \beta]$ and $[\gamma]$. By definition of how we've oriented our simplices,

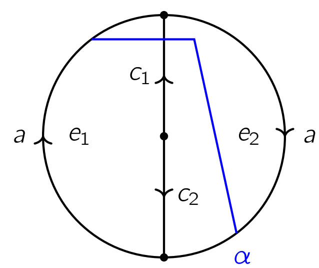
$$(\alpha \cup \beta)(e_1) = \alpha(c_1)\beta(a) = 0$$

and similarly $(\alpha \cup \beta)(e_2) = \alpha(c_3)\beta(b) = 1$, $(\alpha \cup \beta)(e_3) = \alpha(c_4)\beta(a) = 0$, and $(\alpha \cup \beta)(e_4) = \alpha(c_4)\beta(b) = 0$. This means that $(\alpha \cup \beta)(e_1 - e_2 - e_3 + e_4) = -1$, and thus we must have $\boxed{[\alpha \cup \beta] = -[\gamma]}$ in cohomology.

This type of computation works in general, just dealing with more simplices as the surface gets more complicated. And it works for non-orientable surfaces too:

Example 148

Next consider $X = \mathbb{R}P^2$, which is obtained by taking S^2 and identifying antipodal points $x \sim -x$. We get a simplicial structure by taking the upper half hemisphere and identifying opposite points on the equator, which we can write as shown below:



This time we can compute in $\mathbb{Z}_2$-coefficients to get something nontrivial with the differential map $H_2(X) \to H_1(X)$: we find that $H^n(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $n = 0, 1, 2$ and 0 otherwise and the same for cohomology. And we can similarly find $\alpha \in C^1(X; \mathbb{Z}_2)$ which generates $H^1(X; \mathbb{Z}_2)$, specifically evaluating to 1 on a and c_1 and 0 on c_2, and we find that $[\alpha \cup \alpha] = [\alpha] \cup [\alpha]$ is a generator of $H^2(X; \mathbb{Z}_2)$, which we write α^2. This means that $H^*(\mathbb{R}P^2; \mathbb{Z}_2)$ can be thought of as a polynomial ring $\mathbb{Z}_2[\alpha]/(\alpha^3)$, and that turns out to be a general fact: replacing $\mathbb{R}P^2$ with $\mathbb{R}P^n$, we get the polynomial ring $\mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ instead. (And we do need $\mathbb{Z}_2$ coefficients here, because $\mathbb{R}P^n$ is orientable if and only if n is odd.)

17 December 1, 2022

We'll discuss the cup product some more today: recall that we define $\cup$ first on the complex level (for singular homology) by setting

$$(\psi \cup \phi)([v_0, \cdots, v_{m+n}]) = \psi([v_0, \cdots, v_m]) \cdot \phi([v_m, \cdots, v_{m+n}])$$

for any $\psi \in C^m(X)$ and $\phi \in C^n(X)$. We showed last time that this actually induces a cup product on cohomology, so we have a map $\cup : H^m(X) \times H^n(X) \to H^{m+n}(X)$. It turns out we can also generalize this to relative cohomology – if $A \subseteq X$, recall that $C^n(X, A)$ is defined to be $C^n(X)/C_n(A)$ for each n, so $C^n(X, A)$ can be interpreted as the set of functions $f : C_n(X) \to \mathbb{Z}$ that vanish when restricted to $C_n(A)$. Thus, we can define a cup product $C^m(X, A) \times C^n(X, A) \to C^{m+n}(X, A)$, and in fact even if A and B are two different subspaces, we can define a cup product $C^m(X, A) \cup C^n(X, B) \to C^{m+n}(X)$. We are taking a function ψ vanishing on $C_m(A)$ and a function ϕ vanishing on $C_n(B)$, so $\psi \cup \phi$ vanishes on $C_{m+n}(A)$ and $C_{m+n}(B)$. We can compare that to the space of functions $C_{m+n}(X) \to \mathbb{Z}$, which are the functions which vanish on $C_{m+n}(A \cup B)$. We then have an inclusion $C^{m+n}(X, A \cup B)$ into this space $\{f : C_{m+n}(X) \to \mathbb{Z} : f|_{C_{m+n}(A)} = 0, f|_{C_{m+n}(B)} = 0\}$ which is not necessarily an isomorphism (since there might be some simplices partially contained in A and partially contained in B), but under some weak conditions it turns out that this inclusion induces isomorphism **in cohomology**:

Proposition 149

If $A, B \subseteq X$ are open subspaces, then we have a cup product on relative homology $\cup : H^m(X, A) \times H^n(X, B) \to H^{m+n}(X, A \cup B)$.

We will next discuss the **Kunneth formula**, which helps us understand the product of two topological spaces and how homology and cohomology behave under that operation. For this, it will be convenient to work with CW complexes, since the product of an m-cell e_1^m of X and an n-cell e_2^n of Y is an $(m+n)$-cell $e_1^m \times e_2^n$ of $X \times Y$ with boundary

$$\partial(e_1^m \times e_2^n) = ((\partial e_1^m) \times e_2^n) \cup (e_1^m \times (\partial e_2^m)).$$

Recall that in cellular homology, $C_m(X)$ is generated by m-cells of X and $C_n(Y)$ is generated by n-cells of Y, so $C_k(X \times Y)$ is generated by cells of the form $e_1^m \times e_2^n$ for any $m, n \geqslant 0$ such that $m + n = k$. Then it turns out

$$C_k(X \times Y) = \bigoplus_{m+n=k, m,n \geqslant 0} C_m(X) \otimes C_n(Y),$$

which we will write in short as $C_*(X \times Y) = C_*(X) \otimes C_*(Y)$. And the differentials for the product can also be computed in terms of the differentials of the individual spaces: whenever $a \in C_m(X)$, we have

$$d_{X \times Y}(a \otimes b) = (d_X a) \otimes b + (-1)^m a \otimes (d_Y b).$$

The Kunneth formula then gives a formula for the homology of the tensor product chain complex in terms of the individual chain complexes:

Theorem 150

There is a split exact sequence

$$0 \to H_*(X) \otimes H_*(Y) \to H_*(X \times Y) \to \mathrm{Tor}(H_*(X), H_*(Y)) \to 0.$$

In the special case in which $H_*(X)$ or $H_*(Y)$ is free, we know that the Tor term goes to zero, so $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$.

A similar thing happens for cohomology as well, and we also get similar results if we work with general coefficients:

Theorem 151 (Kunneth formula, special case)

Suppose X and Y are CW complexes and R is a commutative ring. If $H^*(Y; R)$ is a finitely generated free R-module, then we have a module isomorphism $\mu : H^*(X \times Y; R) \cong H^*(X; R) \otimes H^*(Y; R)$.

We may then ask how the cup product or relative cohomology interact with this homomorphism – both of them do turn out to work. There is a natural cup product $H^*(X \times Y; R)$, and we can define the cup product for our tensor product in the following way: for any $a, c \in H^*(X)$ and $b, d \in H^*(Y)$, we can set

$$(a \otimes b) \cup (c \otimes d) = (-1)^{mn}(a \cup c) \otimes (b \cup d),$$

where $b \in H^m(X)$ and $c \in H^n(Y)$.

Theorem 152 (Kunneth formula with cup product)

The isomorphism μ in Theorem 151 preserves the cup product as defined above.

Very similarly, if (X, A) and (Y, B) are CW pairs and R is a commutative ring, then we have an analogous result:

Theorem 153 (Kunneth formula, relative version)

Let $(X, A), (Y, B)$ be CW pairs, R be a commutative ring, and suppose $H^*(Y, B; R)$ is a free finitely-generated R-module. Then there is an isomorphism $\mu : H^*(X, A, R) \otimes H^*(Y, B; R) \to H^*(X \times Y, (A \times Y) \cup (X \times B); R)$ which preserves the cup product.

Here, the reason for $(A \times Y) \cup (B \times X)$ is that this is the space on which all functions must vanish. And in particular, this theorem holds whenever R is a field and we have finite CW complexes, because modules over R would be vector spaces and there is no torsion.

Example 154

We'll compute the cohomology ring of the projective spaces $\mathbb{R}P^n = P^n$ (for simplicity) over $\mathbb{Z}_2$ – we'll prove that $H^*(\mathbb{R}P^n : \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/x^{n+1}$ where x has degree 1 (that is, it's the generator of $H^1(\mathbb{R}P^n, \mathbb{Z}_2)$.

Omit the $\mathbb{Z}_2$ in notation for brevity. We did this for $\mathbb{R}P^2$ last time using an explicit simplicial structure, but here we will use cell structures and apply the Kunneth formula to get a more general result. (And something similar would work with complex projective spaces too.) Recall that $P^n = (\mathbb{R}^{n+1} - \{0\})/\sim$, where we identify x and λx for any nonzero constant λ. (So projective space is the set of lines through the origin.) Alternatively, we can think of P^n as $S^n/\sim$, where $\sim$ now identifies antipodal points. In particular, P^1 is just S^1, so nothing interesting happens there, and P^2 can be thought of as the upper hemisphere of S^2 with its boundary identified via the antipodal map, or equivalently gluing D^2 to S^1 via the quotient map $S^1 \to P^1$. More generally, we can look at the upper hemisphere of S^n, whose boundary is S^{n-1}, and we get a cell structure for P^n in which there is one cell in each dimension: the m-skeleton is just P^m, and we glue via the natural quotient map $S^m \to P^m$. Thus if $C_i(P^n)$ is generated by the cell e_i, our chain complex is

$$C^{\mathrm{CW}}(P^n) = \bigoplus_{i=0}^{n} \mathbb{Z}_2\langle e^i \rangle.$$

To figure out the differential map, we're basically adding up the local degrees, but the quotient map is 2-to-1 and we're working in $\mathbb{Z}_2$ so in $\mathbb{Z}_2$-coefficients **all differentials** $d : C_*(P^n) \to C_*(P^n)$ **vanish**. Then because δ is the dual of d, $\delta = 0$ as well, and thus

$$H_i(P^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \leqslant i \leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

and for similar reasons we also have

$$H^i(P^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & 0 \leqslant i \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $P^n - P^{n-1}$ is the interior of our single n-cell and thus homeomorphic to $\mathbb{R}^n$.) We now want to study the cup product, and if we want to know what happens to $\cup : H^i(P^n) \times H^j(P^n) \to H^{i+j}(P^n)$ for any $n \geqslant i + j$, we can just look at $H^i(P^{i+j}) \times H^j(P^{i+j}) \to H^{i+j}(P^{i+j})$, since the inclusion $P^{i+j} \to P^n$ induces an isomorphism (all differentials are zero so the chain complexes literally just include). So we can reduce to thinking about the case where $i + j = n$. The trick is to think about the relative homology

$$H^i(P^n, P^n - P^j) \times H^j(P^n, P^n - P^i) \to H^n(P^n, (P^n - P^j) \cup (P^n - P^i)),$$

because if we start with an $(i + j)$-dimensional disk which we think of as a product of an i-dimensional disk and a j-dimensional disk, quotienting by the antipodal map gives us the spaces listed. Checking that we then have an isomorphism $H^i(P^n, P^n - P^j) \to H^i(P^n)$ and $H^j(P^n, P^n - P^i) \to H^j(P^n)$ is then an exercise, and then $H^n(P^n, (P^n - P^j) \cup (P^n - P^i))$ is exactly $H^n(P^n, P^n - \{p\})$ (where p is the only point of intersection between the two disks – this is where we use $i + j = n$). So the point is that we get the following diagram of maps:

$$\begin{array}{ccc} H^i(P^n) \times H^j(P^n) & \xrightarrow{\quad\cup\quad} & H^n(P^n) \\ \cong\big\uparrow & & \cong\big\uparrow \\ H^i(P^n, P^n - P^j) \times H^j(P^n - P^i) & \xrightarrow{\cup} & H^n(P^n, P^n - \{p\}) \end{array}$$

By the excision theorem, we now want to look at $P^{n-1} \subseteq P^n$ – removing it from P^n yields just an open ball homeomorphic to $\mathbb{R}^n$, and the subset P^i of P^n correspondingly has $\mathbb{R}^i$ removed. Thus we get the additional blue part of the diagram by the excision theorem:

$$\begin{array}{ccc} H^i(P^n) \times H^j(P^n) & \xrightarrow{\quad\cup\quad} & H^n(P^n) \\ \cong\big\uparrow & & \cong\big\uparrow \\ H^i(P^n, P^n - P^j) \times H^j(P^n - P^i) & \xrightarrow{\cup} & H^n(P^n, P^n - \{p\}) \\ \big\downarrow\text{excision} & & \big\downarrow\text{excision} \\ H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \times H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) & \longrightarrow & H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \end{array}$$

But then $H^i(\mathbb{R}^n; \mathbb{R}^n - \mathbb{R}^j) \cong H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\})$ and similarly $H^j(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i) \cong H^i(\mathbb{R}^j, \mathbb{R}^j - \{0\})$ by "removing the dimensions that are unnecessary," and finally thinking of $\mathbb{R}^i, \mathbb{R}^j, \mathbb{R}^n$ instead as i, j, n-dimensional cubes and retract everything but 0 to the boundary in all three cases, we get a map

$$H^i(I^i, \partial I^i) \times H^j(I^j, \partial I^j) \to H^n(I^n, \partial I^n),$$

and now we can identify $(I^n, \partial I^n) = (I^i \times I^j, (\partial I^i \times I^j) \cup (I^i \times \partial I^j))$. But now at this last stage, we finally an isomorphism by the Kunneth formula which maps generators to generators, so tracing this back up to $H^i(P^n) \times H^j(P^n) \to H^n(P^n)$ gives us the desired result.

It turns out that the cohomology ring structure $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]/x^{n+1}$ tells us something powerful:

Corollary 155

Suppose $\mathbb{R}^n$ admits a division algebra structure (a not-necessarily abelian group structure where every nonzero element is invertible). Then n is a power of 2.

Proof. The multiplication $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ also gives us a continuous map between spaces $h : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$, from which we get a cohomology ring structure $h^* : H^*(\mathbb{R}P^{n-1}) \to H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1})$ preserving the cup product. By the Kunneth formula, $H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1})$ has the structure $\mathbb{Z}_2[x_1, x_2]/(x_1^n, x_2^n)$; in particular, $H^1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1})$ is $\mathbb{Z}_1\langle x_1, x_2\rangle$. If we say that $H^1(\mathbb{R}P^n)$ (on the left) is generated as $\mathbb{Z}_2\langle x\rangle$, then the map h^* must send x to $k_1x_1 + k_2x_2$, where $k_1, k_2 \in \{0, 1\}$. But because we have a division algebra, if we keep the first argument in $h : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}p^{n-1}$ fixed then we get an isomorphism, which lets us deduce that we must have $h(x) = x_1 + x_2$. But then $H^*(\mathbb{R}P^{n-1}) = \mathbb{Z}_2[x]/x^n$, so $(x_1 + x_2)^n = (h^*(x))^n = h^*(x^n) = 0$ in $\mathbb{Z}_2[x_1, x_2]/(x_1^n, x_2^n)$. The binomial expansion then tells us that $\sum_{k=1}^{n-1} \binom{n}{k} x_1^k x_2^{n-k} + x_1^n + x_2^n = 0$, which is true if and only if all coefficients $\binom{n}{k}$ for $1 \leqslant k \leqslant n-1$ are even. And this is only true if n is a power of 2. □

It turns out that there is a full classification of the allowed division algebras (for example, the only associative ones are $\mathbb{R}, \mathbb{C}$, and the quaternions, and then we also have some more interesting constructions like the octonions), but the point is that we can prove some interesting things with cohomology.

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Our last topic for this class is **Poincaré duality** – we've understood some relations between homology and cohomology in terms of the Ext and Tor functors through the universal coefficient theorem, and now we'll see another useful tool along those lines. Our discussion will be restricted to the case where M is a connected, closed manifold (meaning that connectedness and path-connectedness are the same thing, and that things look like $\mathbb{R}^n$ locally). By compactness, it is a basic fact that $H_*(M^n)$ is always finitely generated.

We previously introduced a cup product $\cup : C^m(X) \times C^n(X) \to C^{m+n}(X)$, and we will now introduce a new operator called the **cap product**, which maps $\cap : C_m(X) \times C^n(X) \to C_{m-n}(X)$ for $m \geqslant n$ (notice that we have a chain in one case and a cochain in the other). So we take in some simplex $\sigma : [x_0, \cdots, x_m] \to X$ and some map $\Psi : C_n(X) \to \mathbb{Z}$, and we will produce a simplex $\sigma \cap \psi \in C_{m-n}(X)$, defined by setting

$$\sigma \cap \psi = \psi(\sigma|_{[x_0, \cdots, x_n]})\sigma|_{[x_n, x_{n+1}, \cdots, x_m]}.$$

(So we get a number times an $(m-n)$-dimensional singular simplex.) Using the definitions of chain and cochain complexes, we can check that this cap product gives us an operation on homology and cohomology as well, so there is a map $\cap : H_m(X) \times H^n(X) \to H_{m-n}(X)$. Furthermore, the cup and cap product have an interesting relation:

$$\sigma \cap (\psi \cup \phi) = (\sigma \cap \psi) \cap \phi$$

Finally, there is some naturality here: for any continuous map $f : X \to Y$, we may think about how to relate $H_m(X) \times H^n(X) \to H_{m-n}(X)$ and $H_m(Y) \times H^n(Y) \to H_{m-n}(Y)$. We do have maps f_* from the homology groups $H_m(X) \to H_m(Y)$ and $H_{m-n}(X) \to H_{m-n}(Y)$, and we have a map f^* backwards as well (from $H_n(Y) \to H^n(X)$). The point is then that if we have some class $[x] \in H_m(X)$ and some $[\psi] \in H^n(Y)$, we indeed have analogous results of the cap products:

$$f_*([x] \cap f^*[\psi]) = (f_*[x]) \cap [\psi].$$

(We can prove this on the level of chain complexes and then have it descend to homology.)

Example 156

Consider the example of the torus from a previous lecture (Example 147) and use the same diagram. We found the homology groups ($\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}$ for $n = 0, 1, 2$, 0 otherwise) and the cohomomology groups (the same). We then found the generators of homology and cohomology: $H_1(X)$ is generated by $[a], [b]$, $H_2(X)$ is generated by $[e_1 - e_2 - e_3 + e_4]$, and then (with corresponding dual bases) $H^1(X)$ is generated by $[\alpha]$ and $[\beta]$, and $H^2(X)$ is generated by $[\gamma]$. We then found that $[\alpha] \cup [\beta] = -[\gamma]$.

We'll now try to compute the cap product – the case $C_1(X) \times C^1(X) \to C_0(X)$ is left as an exercise, and we'll demonstrate $C_2(X) \times C^1(X) \to C_1(X)$ here. For example, we see that

$$e_1 \cap \alpha = [v_0, v_2, v_3] \cap \alpha = \alpha(c_1)[v_2, v_3] = 0, \quad e_2 \cap \beta = \beta(e_1)[v_2, v_3] = a.$$

Similarly, we have

$$e_2 \cap \alpha = \alpha(c_3)[v_4, v_3] = b, \quad e_2 \cap \beta = \beta(c_3)[v_4, v_3] = 0,$$

and also that $e_3 \cap \alpha, e_3 \cap \beta, e_4 \cap \alpha, e_4 \cap \beta$ are all zero. But $H_2(X)$ is generated by $e_1 - e_2 - e_3 + e_4$, so to compute $H_2(X) \times H^1(X) \to H_1(X)$, we see that

$$[e_1 - e_2 - e_3 + e_4] \cap [\alpha] = [-b], \quad [e_1 - e_2 + e_3 + e_4] \cap [\beta] = [a].$$

And notice that $H^1(X)$ is generated by $[\alpha]$ and $[\beta]$, while $H_1(X)$ is generated by $[a]$ and $[b]$. It turns out this is a general phenomenon – for any closed oriented n-dimensional manifold, we know that the top homology group $H_n(M)$ is a copy of $\mathbb{Z}$, and we can write its generator as $[M]$. (If we had a simplicial structure on M, it would be the sum of all oriented top-dimensional simplices.) We call M the **fundamental class**)of M, and we then define a map $D : H^k(M) \to H_{n-k}(M)$ via

$$D([\psi]) = [M] \cap [\psi] \in H_{n-k}(M).$$

Theorem 157 (Poincaré duality)
If M is closed and oriented, then the map $D : H^k(M) \to H_{n-k}(M)$ is an isomorphism.

Remark 158. *If M is an oriented manifold, then for any commutative ring we get the isomorphism $H^k(M; R) \cong H_{n-k}(M; R)$ as well – the proof is independent of the coefficient ring. On the other hand, if M is unorientable, we no longer get duality results with $\mathbb{Z}$ coefficients – for example, $H_n(M) \cong H^n(M) = 0$ but $H_0(X) \cong \mathbb{Z}$ for a connected unoriented manifold. However, if R is a commutative ring* **in characteristic 2**, *we still have $H_n(X) = R$, and thus we still have an isomorphism $H^k(M; R) \cong H_{n-k}(M; R)$.*

Example 159
Suppose Y^n is a connected, closed, oriented n-dimensional manifold, and suppose $R = \mathbb{Q}$ so we don't need to think about quotients. By the universal coefficient theorem (and using that the Ext functor is trivial) we then have $H^k(X; \mathbb{Q}) \cong \text{Hom}(H_k(X; \mathbb{Q}); \mathbb{Q})$.

So for any nonzero cohomology class $[\psi] \in H^k(X; \mathbb{Q})$, we have a nonzero Poincar'e dual $D([\psi]) \in H_{n-k}(X; \mathbb{Q})$ (because D is an isomorphism), and then there is an element $\phi \in H^{n-k}(X, \mathbb{Q})$ such that

$$[\phi](D[\psi]) \neq 0.$$

But by definition, this is saying that

$$0 \neq [\phi]([M] \cap [\psi]) = [M] \cap ([\psi] \cup [\phi]),$$

and thus we must have $[\psi] \cup [\phi] \neq 0$. In words, any cohomology class (in $H^k(M; \mathbb{Q})$) has a cohomology class of complementary dimension (in $H^{n-k}(M; \mathbb{Q})$), such that their cup product is nonzero (in $H^n(M; \mathbb{Q}) \cong \mathbb{Q}$). So the cup product is nondegenerate as long as we work with rational coefficients, and the nondegeneracy result works when we quotient out the torsions too.

Example 160

Poincaré duality also has applications to the signature of manifolds of dimension a multiple of 4. We'll work with a special case where X is still connected, closed, and oriented, and we'll keep using $\mathbb{Q}$-coefficients, but now say that Y is a manifold of dimension $2n$ for some n. Then the cup product $\cup : H^n(X;\mathbb{Q}) \times H^n(X;\mathbb{Q}) \to H^{2n}(X;\mathbb{Q}) \cong \mathbb{Q}$ is a bilinear form on the middle dimension n.

Notice that $[\psi] \cup [\phi] = (-1)^{n^2}[\phi] \cup [\psi]$, so this bilinear form is symmetric if n is even and antisymmetric if n is odd. In the former case, we can then represent the bilinear form as a symmetric $m \times m$ rational matrix, where $m = \dim H^n(X;\mathbb{Q})$. We can then define the **signature** $\sigma(X)$ of X to be the signature of that $m \times m$ matrix, which is the number of positive eigenvalues minus the number of negative eigenvalues.

Poincaré duality then tells us that the map $\cup : H^n(X;\mathbb{Q}) \times H^n(X;\mathbb{Q}) \to \mathbb{Q}$ becomes a map $I : H_n(X;\mathbb{Q}) \times H_n(X;\mathbb{Q}) \to H_0(X;\mathbb{Q}) \cong \mathbb{Q}$ (often called the **intersection form**). And there is a famous result along these lines:

Fact 161 (Freedman's theorem)

Let X be a closed, connected, oriented, **smooth** 4-dimensional manifold with $\pi_1(X)$ trivial (so X is simply connected). Then X is classified by its intersection form I, meaning that two such spaces are homeomorphic if they have the same intersection form.

(For another result on $\sigma(X)$, we can look up the **Atiyah-Singer index theorem**, which is a different deep result.) For now, we'll turn back to the intersection form and think about its geometric interpretation. Assume that everything is smooth for simplicity. Suppose X is a $(2n)$-dimensional manifold and that we're using $\mathbb{Q}$-coefficients, so our map is $I : H_n(X) \times H_n(X) \to H_0(X) \cong \mathbb{Q}$. Suppose now that $M \subseteq X$ is an n-dimensional closed, connected, orientable submanifold – then $H^n(M) \cong \mathbb{Q}$, so there is a fundamental class $[M] \in H_n(M)$. The natural inclusion $\iota : M \to X$ then gives us a map $\iota_* : H_n(M) \to H_n(X)$, and in particular we may ask about $\iota_*(M)$, which we will also write as $[M] \in H_n(X)$. So any such manifold M gives us a class in $H_n(X)$.

Then given two such n-dimensional closed, connected, orientable submanifolds M and N, we can use facts from differential topology, because we're assuming things are smooth. In the special case $n = 1$, we have two curves inside a 2-dimensional surface, and we can perturb them slightly so that the curves generically intersect transversally at various points. Then $M \cap N$ will be a set of finitely many points (because of compactness), and the generators of $H_0(X)$ are points themselves. So it's reasonable that $[M] \times [N]$ is just going to be a signed count of the points in $M \cap N$:

$$\boxed{I([M],[N]) = \sum_{x \in M \cap N} \operatorname{sgn}(x)}.$$

More generally, if M and N are n-dimensional and X is $2n$-dimensional, near any point $x \in M \cap N$, we can find a neighborhood D of x in $M \cap N$ so that M and N are "intersecting transversally" – that is, we can choose $D \cong \mathbb{R}^{2n}$ so that $M \cap D \cong \mathbb{R}^n$, $N \cap D \cong \mathbb{R}^n$, and $x \in D$ maps to the origin. We can then use the orientations to define $\operatorname{sgn}(x)$: specifically, **we define** $\operatorname{sgn}(x) = 1$ if the orientations of M and N together (by putting together the oriented bases of M and N at x) coincide with that of X, and -1 otherwise.

Remark 162. *More formally, having a local orientation is equivalent to choosing an element of each of the relative homologies $H^n(M, M - \{x\})$, $H^n(N, N - \{x\})$, and $H^{2n}(X, X - \{x\})$ with $\mathbb{Z}$-coefficients (each of which is isomorphic to $\mathbb{Z}$ by excision) – the sign of x then corresponds to whether the cup product $\cup : H^n(M, M - x) \times H^n(N, N - x) \to H^{2n}(X, X - \{x\})$ gives us $+1$ or -1.*