The Casimir element for $U_q(\mathfrak{so}_{2n})$

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Abstract

We construct an explicit central element of the quantum groups $U_q(\mathfrak{so}_6)$ and $U_q(\mathfrak{so}_8)$ using a method from [2] and provide progress towards the general case $U_q(\mathfrak{so}_{2n})$ by studying the structure of the dual elements under a q-deformed pairing.

1 Introduction

The **symmetric simple exclusion process** (SSEP) is a continuous-time Markov process, meaning it can be entirely characterized by a generator matrix encoding the jump rates from each state to each other state. This generator matrix can be obtained from the quadratic Casimir element of the Lie algebra $\mathfrak{sl}_2$, which is a distinguished element of the corresponding **universal enveloping algebra** $U(\mathfrak{sl}_2)$.

There are two natural extensions of this construction. One is to generalize the particle system itself by introducing asymmetry; the **asymmetric simple exclusion process** (ASEP) is constructed by a similar process but using the Drinfeld–Jimbo quantum group $U_q(\mathfrak{sl}_2)$. The other is to construct generator matrices using other Lie algebras, since the differences in algebraic structure will lead to different properties for the generator matrices.

This paper explores a combination of the two ideas by analyzing the Drinfeld-Jimbo quantum group $U_q(\mathfrak{so}_{2n})$. Previous work [2] has provided a formula for a q-analog of the Casimir element, and we apply these methods here for the simple cases $n = 3, 4$ and develop ideas for general n. In Section 2, we review the algebraic definitions and relevant constructions for the rest of the paper, and we explain the procedure for applying the formula itself in Section 3. In Section 4, we present the explicit formulas for these central elements and provide some code aiding us in computations. Finally, Section 5 describes some strategies for computing dual elements and suggests a potential method for further research.

2 Background

This section describes the main objects of study for this paper and explains Lemma 3.1 of [2], the central formula of this approach.

Definition 1. *The Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$ is the set of $2n \times 2n$ matrices*

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A = -D^T, B^T = -B, C^T = -C \right\},$$

where $A, B, C, D \in \mathbb{C}^{n \times n}$.

Below, we list a few key algebraic properties associated to this Lie algebra:

- The rank of the Lie algebra $\mathfrak{so}_{2n}$ is n. Specifically, if L_i denotes the linear operator taking a matrix M to its diagonal entry $M_{i,i}$, then the set of roots is $\{L_i - L_j : 1 \leq i \neq j \leq n\}$, and the positive simple roots are $\alpha_i = L_i - L_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n = L_{n-1} + L_n$.

- The *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{so}_{2n}$ is the subalgebra of diagonal matrices spanned (as a vector space) by the matrices $H_n = E_{n-1,n-1} + E_{n,n} + E_{2n-1,2n-1} + E_{2n,2n}$ and, for $1 \leq i \leq n-1$),
$$H_i = E_{i,i} - E_{i+1,i+1} - E_{n+i,n+i} + E_{n+i+1,n+i+1},$$
where $E_{a,b}$ denotes the matrix with a 1 in the (a,b)th entry and 0s elsewhere. In other words, H_i takes its first n diagonal entries from the ith positive simple root and then flips the signs for the next n entries (so that $A = -D^T$).

- The corresponding Dynkin diagram for $\mathfrak{so}_{2n}$ is D_n. In particular, all nonzero off-diagonal entries of the Cartan matrix are -1:
$$a_{ij} = \begin{cases} 2 & i = j, \\ -1 & \{i,j\} = \{n-2, n\} \text{ or } \{k, k+1\}, 1 \leq k \leq n-2, \\ 0 & \text{otherwise.} \end{cases}$$

- In the **fundamental representation** of $\mathfrak{so}_{2n}$, all elements of the Lie algebra act on a $2n$-dimensional vector space. Certain elements correspond to simple roots and will be referred to later in the paper: for all $1 \leq i \leq n-1$, we have
$$E_i = E_{i,i+1} - E_{n+i+1,n+i}, \quad E_n = E_{n-1,2n} - E_{n,2n-1},$$
$$F_i = E_{i+1,i} - E_{n+i,n+i+1}, F_n = E_{2n-1,n} - E_{2n,n-1}.$$
The weights of the fundamental representation are $\pm L_i$ (for $1 \leq i \leq n$).

Definition 2. *The* ***quantum group*** $U_q(\mathfrak{so}_{2n})$ *is the algebra generated by* $\{E_i, F_i, q^{H_i} : 1 \leq i \leq n\}$. *These generators satisfy the relations*
$$[E_i, F_i] = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \quad q^{H_i} E_j = q^{(\alpha_i, \alpha_j)} E_j q^{H_i}, \quad q^{H_i} F_j = q^{-(\alpha_i, \alpha_j)} F_j q^{H_i},$$
as well as the Serre relation for all (i,j) *with* $a_{ij} = -1$ *in the Cartan matrix*
$$E_i^2 E_j + E_j E_i^2 = (1+q) E_i E_j E_i, \quad F_i^2 F_j + F_j F_i^2 = (1+q) F_i F_j F_i,$$
and all other pairs of elements commuting. Finally, the coproducts of the generators are
$$\Delta(E_i) = E_i \otimes 1 + q^{H_i} \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes q^{-H_i}, \quad \Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i}.$$

A general definition for the quantum group of a Lie algebra $\mathfrak{g}$ can be found in [1]; this quantum group serves as a deformation of the ordinary universal enveloping algebra $U(\mathfrak{g})$.

Constructing a nondegenerate bilinear form often allows us to study relationships between elements of a vector space, and Chapter 6 of [1] introduces a similar idea. Recall that the **Borel subalgebras** $\mathfrak{b}\pm$ are the Lie subalgebras generated by $\{E_i, H_i\}$ and $\{F_i, H_i\}$, respectively. Let $U_q(\mathfrak{b}\pm)$ denote the corresponding subalgebras of the quantum group generated by the Borel subalgebras (replacing H_i with q^{H_i}), and let $\langle\, ,\, \rangle$ be the bilinear pairing $U_q(\mathfrak{b}-) \times U_q(\mathfrak{b}+) \to \mathbb{Q}(q)$ such that for any linear combinations α, β of the α_is, we have

$$\langle q^{H_\alpha}, q^{H_\beta} \rangle = q^{-(\alpha\cdot\beta)} \text{ and } \langle F_i, E_j \rangle = -\delta_{ij}(q - q^{-1})^{-1},$$

and all other pairings between generators are zero. Furthermore, the pairing can be computed for products via

$$\langle y, xx' \rangle = \langle \Delta(y), x' \otimes x \rangle, \quad \langle yy', x \rangle = \langle y \otimes y', \Delta(x) \rangle,$$

where the coproduct Δ satisfies $\Delta(ab) = \Delta(a)\Delta(b)$ and is explicitly defined for generators in the definition above. (Here, $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle$ is defined to be $\langle x_1, y_1 \rangle \langle x_2, y_2 \rangle$.) From this construction, we can obtain the following result, which is Lemma 3.1 of [2] and is based on results from Chapter 6 of [1]:

Theorem 3. *For each weight μ of the fundamental representation of a Lie algebra $\mathfrak{g}$, let v_μ be a vector in the weight space. Suppose q is not a root of unity, and 2μ is always in the root lattice of $\mathfrak{g}$. Let $e_{\mu\lambda}$ and $f_{\lambda\mu}$ be products of E_is and F_is in $U(\mathfrak{g})$, respectively, such that $e_{\mu\lambda}$ sends v_λ to v_μ and $f_{\lambda\mu}$ sends v_μ to v_λ. If $e^*_{\mu\lambda}$ and $f^*_{\mu\lambda}$ are the corresponding dual elements under the above pairing, and ρ is half the sum of the positive roots of $\mathfrak{g}$, then*

$$\sum_{\mu} q^{(-2\rho,\mu)} q^{H_{-2\mu}} + \sum_{\mu>\lambda} q^{(\mu-\lambda,\mu)} q^{(-2\rho,\mu)} e^*_{\mu\lambda} q^{H_{-\mu-\lambda}} f^*_{\lambda\mu}$$

is a central element of the quantum group $U_q(\mathfrak{g})$.

The main goal of this paper is to apply this result to the quantum group $U_q(\mathfrak{so}_{2n})$.

3 Computing the individual terms

This section describes each of the components of Theorem 3 and efforts towards computing them explicitly. All results from here concern only $\mathfrak{so}_{2n}$.

First of all, we will take q to be a real number different from 1, so q is not a root of unity. In addition, $2\mu = \pm 2L_i$ is always in the root lattice, because all of $\pm L_i \pm L_j$ are roots. Thus, the conditions of the theorem are satisfied.

Our next step is to establish an ordering for the weight spaces to discern when $\mu > \lambda$ holds. Since the E_i and F_i operators serve as raising and lowering operators, and the sum of the positive roots is

$$\sum_{i\neq j}(L_i + L_j) + \sum_{i<j}(L_i - L_j) = (2n-2)L_1 + (2n-4)L_2 + \cdots + L_{n-1},$$

a natural ordering of the weights is

$$L_1 > \cdots > L_{n-1} > L_n = -L_n > -L_{n-1} > \cdots > -L_1.$$

Lemma 4. *The weight μ can be reached from a weight λ by a product of E_is, and the weight λ can be reached from μ by a product of F_is, if and only if $\mu > \lambda$ under the above ordering.*

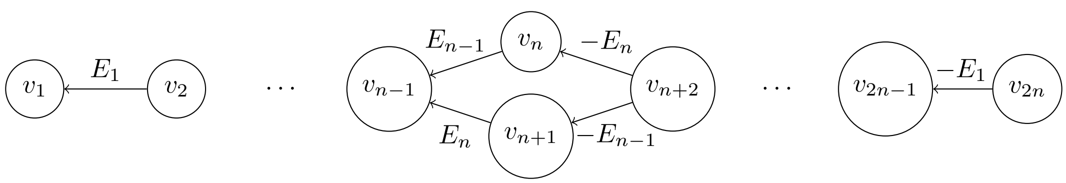
Proof. Define $v_1, \cdots, v_{2n}$ to be the vectors in the weight spaces $L_1, \cdots, L_n, -L_n, \cdots, -L_1$ with only a single nonzero entry of 1. (Specifically, v_i has a 1 in the ith spot for $1 \le i \le n$ and a 1 in the $(3n+1-i)$th spot for $n+1 \le i \le 2n$.) The action of the E_is and F_is can be described explicitly based on their matrix counterparts in the fundamental representation, defined earlier: for all $1 \le i \le n-1$,

$$E_i v_j = \begin{cases} v_i & j = i+1, \\ -v_{2n-i} & j = 2n+1-i, \\ 0 & \text{otherwise,} \end{cases}$$

meaning that the first $(n-1)$ E operators "move us up one weight space," and

$$E_n v_j = \begin{cases} v_{n-1} & j = n+1, \\ -v_n & j = n+2, \\ 0 & \text{otherwise.} \end{cases}$$

Below is a schematic diagram for the weight spaces, which resembles the Dynkin diagram D_n mirrored over itself:



The result can now be directly verified for the E_is by inspection of the diagram (the ordering goes left-to-right, and the E_is always move us left in the diagram). The proof for F_is follows similarly. □

Since E_{n-1} and E_n commute in $\mathfrak{so}_{2n}$, and so do F_{n-1} and F_n, taking the "top path" or the "bottom path" in the diagram above in fact corresponds to applying the same element of $U(\mathfrak{g})$. This means that there is in fact a **unique** product of Es or Fs (up to scalars) that takes any v_i to any v_j whenever j can be reached from i, and this means that we have $\binom{2n}{2} - 1$ total ordered pairs (μ, λ) in the second sum of Theorem 3.

Now that we know how to find the elements $e_{\mu\lambda}$ and $f_{\lambda\mu}$, we can begin to calculate the quantities in the actual sum. Computing $q^{(\mu-\lambda,\mu)}$ and $q^{(-2\rho,\mu)}$ is relatively simple, because μ always consists of a single term of the form $\pm L_i$:

$$q^{(\mu-\lambda,\mu)} = q^{(\mu,\mu)-(\lambda,\mu)} = \begin{cases} q^2 & \lambda = -\mu, \\ 1 & \lambda = \mu, \\ q & \text{otherwise,} \end{cases} \qquad q^{(-2\rho,\mu)} = \begin{cases} q^{2n-2i} & \mu = L_i, \\ q^{2i-2n} & \mu = -L_i. \end{cases}$$

Computing the others requires more casework. To find $q^{H_{-2\mu}}$ and $q^{H_{-\mu-\lambda}}$, we need to write the exponents as linear integer combinations of the H_is. We present the general argument here and also show explicit examples later in the paper.

- To compute $q^{H_{-2\mu}}$ when $\mu = \pm L_i$, we first find that (with some abuse of notation)

$$\boxed{H_{-2L_n}} = H_{-(L_{n-1}-L_n)+(L_{n-1}+L_n)} = \boxed{H_{n-1} - H_n},$$

and then use a telescoping sum: since

$$H_{-2L_i} - H_{-2L_n} = \sum_{j=i}^{n-1}(-2L_j + 2L_{j+1}) = -2\sum_{j=i}^{n-1} H_j,$$

we can rearrange and substitute to obtain

$$\boxed{H_{-2L_i} = H_{n-1} - H_n - 2\sum_{j=i}^{n-1} H_j}$$

for any $1 \le i \le n-1$. This gives us the answer for $\mu = +L_i$, and we simply negate the expressions for the case $\mu = -L_i$. (For example, $n = 3$, $\mu = L_2$ yields $H_{-2\mu} = -H_2 - H_3$, so $\mu = -L_2$ yields $H_{-2\mu} = H_2 + H_3$.)

- A similar telescoping sum works when $\mu > \lambda$ and we want to find $q^{H_{-\mu-\lambda}}$. (Such exponents always look like $H_{\pm L_i \pm L_j}$ for $i \neq j$.) First, note that when $i < j$,

$$\boxed{H_{L_i - L_j}} = \sum_{k=i}^{j-1} H_{L_k - L_{k+1}} = \boxed{\sum_{k=i}^{j-1} H_k},$$

  and the $i > j$ case follows by negating both sides. This then allows us to find any $H_{\pm L_i \pm L_j}$ by subtracting or adding H_{2L_i} and H_{2L_j} appropriately, both of which have been computed in the previous case.

We now turn our attention to computing $e_{\mu\lambda}$, $f_{\lambda\mu}$, and their dual elements. Since all $e_{\mu\lambda}$ and $f_{\lambda\mu}$s are products of E_is and F_is, respectively, the dual elements $e^*_{\mu\lambda}$ and $f^*_{\lambda\mu}$ will be products of F_is and E_is. The next result serves to characterize these elements, and it also explains more explicitly how the above bilinear pairing is calculated.

Lemma 5. *The value of $\langle F_{x_1} F_{x_2} \cdots F_{x_m}, E_{y_1} E_{y_2} \cdots E_{y_n} \rangle$ is only nonzero if $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_n)$ are permutations of each other (in particular, $n = m$), in which case it evaluates to $(q - q^{-1})^{-n}$ times an element of the ring $\mathbb{Z}[q, q^{-1}]$.*

Proof. We proceed by induction on n. The base case $n = 1$ can be verified using the rule $\langle yy', x\rangle = \langle y \otimes y', \Delta(x)\rangle$ and the fact that 1 and q^{H_i} both pair to 0 with any F_i or product of F_is. Then $\langle F_i, E_j \rangle$ is always equal to $-\delta_{ij}(q - q^{-1})^{-1}$, which proves the remainder of the claim.

For the inductive step, using the coproduct relation, the above pairing evaluates to

$$\left\langle \prod_{i=1}^{n} \left(1 \otimes F_{x_i} + \boxed{F_{x_i} \otimes q^{-H_{x_i}}}\right), E_{y_2} \cdots E_{y_n} \otimes E_{y_1} \right\rangle.$$

By the inductive hypothesis, the only way this pairing is nonzero is if $(n-1)$ of the terms on the left are of the boxed type (so that there are exactly $(n-1)$ Fs to pair with the $(n-1)$ Es in $E_{y_2} \cdots E_{y_n}$). In addition, to yield a nonzero result, the remaining term (of the form $(1 \otimes F_{x_i})$) must have the same index as E_{y_1}. Thus, we've proved the first claim, and this expression evaluates to

$$\sum_{i:\, x_i = y_1} \langle F_{x_1} \cdots \hat{F_{x_i}} \cdots F_{x_n}, E_{y_2} \cdots E_{y_n} \rangle \langle q^{-H_{x_1}} \cdots F_{x_i} \cdots q^{-H_{x_n}}, E_{y_1} \rangle,$$

where the hat means that F_{x_i} is omitted. By the inductive hypothesis again, the first pairing here is always $(-(q-q^{-1})^{-1})^{n-1}$ times an element of $\mathbb{Z}[q,q^{-1}]$, and the second pairing is (moving the F to the front)

$$\langle q^{-H_{x_1}}\cdots F_{x_i}\cdots q^{-H_{x_n}}, E_{y_1}\rangle = q^{(\alpha_{x_i},\alpha_{x_1}+\cdots+\alpha_{x_{i-1}})}\langle F_{x_i}q^{-H_{x_1}}\cdots q^{-\hat{H}_{x_i}}\cdots q^{-H_{x_n}}, E_{y_1}\rangle$$

by the commutativity relations of the quantum group. Now using the coproduct relation again on this final pairing and substituting shows that it is always $-(q-q^{-1})^{-1}$, so we have the right power of $(q-q^{-1})$, as desired. □

Now, computing the dual elements can be done as follows. For a given set of indices $(x_1,\cdots,x_n)$, all elements of the form $e' = E_{\sigma(x_1)}\cdots E_{\sigma(x_n)}$ (for a permutation of the indices σ) have nonzero pairing only with elements of the form $f' = F_{\tau(x_1)}\cdots F_{\tau(x_n)}$ (for a permutation τ). However, some of these elements may be identical or linearly dependent (due to the relations of the quantum group). Thus, pick $\{e_1,\cdots,e_m\}$ and $\{f_1,\cdots,f_m\}$ to be (linear) bases of the spaces of possible e's and f's, and let M be the matrix such that $M_{ij} = \langle e_i, f_j\rangle$. Results from Chapter 6 of [1] show that this pairing is nondegenerate, so M must be an invertible matrix. The rows of M^{-1} then tell us the correct $\mathbb{Z}[q,q^{-1}]$-combinations to take for the dual element.

Example 6. *Take $n=4$. If we wish to find the dual element for E_2E_3, first note that E_2E_3 and E_3E_2 both have nonzero pairing with F_2F_3 and F_3F_2, and none of these elements have a nonzero pairing with anything else. Taking the bases to be $\{E_2E_3, E_3E_2\}$, $\{F_2F_3, F_3F_2\}$, we evaluate the pairings to find*

$$M = (q-q^{-1})^{-2}\begin{bmatrix}1 & 1/q\\ 1/q & 1\end{bmatrix} \implies M^{-1} = (q-q^{-1})^2\frac{1}{q-q^{-1}}\begin{bmatrix}q & -1\\ -1 & q\end{bmatrix}.$$

Thus, reading off the first row, E_2E_3 has dual element $(q-q^{-1})(qF_2F_3 - F_3F_2)$.

4 The special cases $n=3$ and $n=4$

This section gives the explicit form of the distinguished central element from Theorem 3 for the cases $n=3$ and $n=4$ in terms of the generators E_i, F_i, q^{H_i}. (We do not present $n=2$ here because $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is a degenerate case. While $\mathfrak{so}_6 \cong \mathfrak{sl}_4$, the central element presented here is still new.)

Computing the bases $\{e_1,\cdots,e_m\}$ described above, as well as many of the dual elements, was done using Python code. We have included the code in Appendix A at the end of this report, as the same algorithm can be applied to other Lie algebras beyond $\mathfrak{so}_{2n}$.

To make the expression more readable, we adopt some shortcuts: set $r = q - \frac{1}{q}$, and let $E_{x_1x_2\cdots x_n}$ denote $E_{x_1}E_{x_2}\cdots E_{x_n}$ (and similar for Fs). (For example, $r^2E_{21}F_3$ would denote $(q-\frac{1}{q})^2E_2E_1F_3$.)

Theorem 7. *The following element of the quantum group $U_q(\mathfrak{so}_6)$ is central:*

$$
\begin{aligned}
q^{-4-2H_1-H_2-H_3} + q^{-2-H_2-H_3} + q^{H_2-H_3} + q^{H_3-H_2} + q^{2+H_2+H_3} + q^{4+2H_1+H_2+H_3} + \tfrac{r^2}{q^3} F_1 q^{-H_1-H_2-H_3} E_1 \\
+\tfrac{r^2}{q} F_2 q^{-H_3} E_2 - \tfrac{r^2}{q} F_3 q^{-H_2} E_3 + r^2 q F_2 q^{H_3} E_2 - r^2 q F_3 q^{H_2} E_3 + r^2 q^3 F_1 q^{H_1+H_2+H_3} E_1 \\
+\tfrac{r^2}{q^3}(qF_{12} - F_{21}) q^{-H_1-H_3}(qE_{21} - E_{12}) - \tfrac{r^2}{q^3}(qF_{13} - F_{31}) q^{-H_1-H_2}(qE_{31} - E_{13}) \\
+r^2 q(qF_{21} - F_{12}) q^{H_1+H_3}(qE_{12} - E_{21}) - r^2 q(qF_{31} - F_{13}) q^{H_1+H_2}(qE_{13} - E_{31}) \\
-\tfrac{r^2}{q^3}(q^2 F_{123} - qF_{213} - qF_{312} + F_{231}) q^{-H_1}(q^2 E_{231} - qE_{312} - qE_{213} + E_{123}) \\
-\tfrac{r^2}{q}(q^2 F_{231} - qF_{312} - qF_{213} + F_{123}) q^{H_1}(q^2 E_{123} - qE_{213} - qE_{312} + E_{231}) \\
-\tfrac{r^4}{q^2}((q^2+1) F_{1231} - qF_{1312} - qF_{2131})((q^2+1) E_{1231} - qE_{1312} - qE_{2131}) \\
-r^4 F_2 F_3 E_2 E_3.
\end{aligned}
$$

This element acts as $q^6 + q^2 + 2 + q^{-2} + q^{-6}$ times the identity matrix in the fundamental representation of $\mathfrak{so}_6$.

The result for $n = 4$ looks similar but is much longer, and we have left the full expression of certain terms, labeled $\boxed{A_i}$, to Appendix B. Those elements have been boxed for easy reference. In addition, for this element, the terms from the sum over μ are included first, and then all subsequent sums are included in order of μ and then λ (for example, the first term is $(\mu, \lambda) = (L_1, L_2)$, and the last term is $(\mu, \lambda) = (-L_2, L_1)$) for easy reference.

Theorem 8. *The following element of the quantum group $U_q(\mathfrak{so}_8)$ is central:*

$$
\begin{aligned}
q^{-6-2H_1-2H_2-H_3-H_4} + q^{-4-2H_2-H_3-H_4} + q^{-2-H_3-H_4} + q^{H_3-H_4} \\
+q^{H_4-H_3} + q^{2+H_3+H_4} + q^{4+2H_2+H_3+H_4} + q^{6+2H_1+2H_2+H_3+H_4} \\
+\tfrac{r^2}{q^5} F_1 q^{-H_1-2H_2-H_3-H_4} E_1 + \tfrac{r^2}{q^5}(qF_{12} - F_{21}) q^{-H_1-H_2-H_3-H_4}(qE_{21} - E_{12}) \\
+\tfrac{r^2}{q^5}(q^2 F_{123} - qF_{132} - qF_{213} + F_{321}) q^{-H_1-H_2-H_4}(q^2 E_{321} - qE_{213} - qE_{132} + E_{123}) \\
-\tfrac{r^2}{q^5}(q^2 F_{124} - qF_{142} - qF_{241} + F_{421} q^{-H_1-H_2-H_3}(q^2 E_{421} - qE_{241} - qE_{142} + E_{124}) \\
-\tfrac{r^2}{q^5}\boxed{A_1} q^{-H_1-H_2}\boxed{A_4} - \tfrac{r^2}{q^5}\boxed{A_5} q^{-H_1}\boxed{A_8} - \tfrac{r^4}{q^4}\boxed{A_9}\boxed{A_{10}} \\
+\tfrac{r^2}{q^3} F_2 q^{-H_2-H_3-H_4} E_2 + \tfrac{r^2}{q^3}(qF_{23} - F_{32}) q^{-H_2-H_4}(qE_{32} - E_{23}) - \tfrac{r^2}{q^3}(qF_{24} - F_{42}) q^{-H_2-H_3}(qE_{42} - E_{24}) \\
-\tfrac{r^2}{q^3}(q^2 F_{234} - qF_{324} - qF_{423} + F_{432}) q^{-H_2}(q^2 E_{432} - qE_{324} - qE_{423} + E_{234}) \\
-\tfrac{r^4}{q^2}((q^2+1) F_{2342} - qF_{3242} - qF_{2423})((q^2+1) E_{2342} - qE_{3242} - qE_{2423}) \\
-\tfrac{r^2}{q^3}\boxed{A_7} q^{H_1}\boxed{A_6} + \tfrac{r^2}{q} F_3 q^{-H_4} E_3 - \tfrac{r^2}{q} F_4 q^{-H_3} E_4 \\
-r^4 F_3 F_4 E_4 E_3 - \tfrac{r^2}{q}(q^2 F_{432} - qF_{324} - qF_{423} + F_{234}) q^{H_2}(q^2 E_{234} - qE_{324} - qE_{423} + E_{432}) \\
-\tfrac{r^2}{q}\boxed{A_3} q^{H_1+H_2}\boxed{A_2} - r^2 q F_4 q^{H_3} E_4 - r^2 q(qF_{42} - F_{24}) q^{H_2+H_3}(qE_{24} - E_{42}) \\
-r^2 q(q^2 F_{421} - qF_{241} - qF_{142} + F_{124}) q^{H_1+H_2+H_3}(q^2 E_{124} - qE_{142} - qE_{241} + E_{421}) \\
+r^2 q F_3 q^{H_4} E_3 + r^2 q(qF_{32} - F_{32}) q^{H_2+H_4}(qE_{23} - E_{32}) \\
+r^2 q(q^2 F_{321} - qF_{213} - qF_{132} + F_{123}) q^{H_1+H_2+H_4}(q^2 E_{123} - qE_{132} - qE_{213} + E_{321}) \\
+r^2 q^3 F_2 q^{H_2+H_3+H_4} E_2 + r^2 q^3 (qF_{21} - F_{12}) q^{H_1+H_2+H_3+H_4}(qE_{12} - E_{21}) + r^2 q^5 F_1 q^{H_1+2H_2+H_3+H_4} E_1.
\end{aligned}
$$

This element acts as $q^8 + q^4 + q^2 + 2 + q^{-2} + q^{-4} + q^{-8}$ times the identity matrix in the fundamental representation of $\mathfrak{so}_8$.

5 Progress on the dual elements

We now turn our attention to the general case $U_q(\mathfrak{so}_{2n})$. Since we have already written out the expressions for all terms that are powers of q or products of q^{H_i}, it remains to find an explicit expression for the dual elements. Because we know the structure of the weight spaces and of the raising and lowering operators, we only need to find the dual elements for certain products of E_is and F_is. In particular, $e_{\mu\lambda}$ and $f_{\lambda\mu}$ only include each index at most twice, and when the indices appear twice, the element must be of the form

$$\pm(E_i \cdots E_{n-2})E_{n-1}E_n(E_{n-2}\cdots E_j)$$

(or Es replaced with Fs, respectively), where the parenthetical expressions can each also have zero terms.

All explicit progress presented so far has been accomplished using brute force: to calculate the dual elements that have a set of indices $(x_1, \cdots, x_n)$, we have first found some basis of elements with those indices and explicitly calculated M and M^{-1}. However, this quickly becomes computationally infeasible, and we demonstrate that there may be a cleaner approach to find the answers in general. First, we explain how to compute the dual element for an element like $E_3E_4E_5\cdots E_{10}$, where the indices are consecutive but non-repeating:

Proposition 9. *Suppose each index only shows up once in an element of $e_{\mu\lambda}$ or $f_{\lambda\mu}$. Then the matrix M^{-1} can be inductively computed by tensoring the inverse matrix from a smaller set of indices repeatedly with $\begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix}$.*

Proof. We assume we are working with $e_{\mu\lambda}$s without loss of generality. The base cases are when there are 1 or 2 generators, which are computed directly: the dual element of E_i is $-(q-q^{-1})F_i$, and the dual element of E_iE_j is either $-(q-q^{-1})F_iF_j$ if E_i and E_j commute, or it is of the form in Example 6 above.

For the inductive step, let the set of indices that appear be $(x_1, \cdots, x_m)$, ordered in nondecreasing order. If $x_1 \le x_2 - 2$, then the only indices must be $(n-2, n)$, covered in the base case. (There is no other case in which two E_is on the diagram of Lemma 4 are adjacent.) Otherwise, $x_1 = x_2 - 1$, which means either the indices are $(n-2, n-1, n)$, which is also easily computed directly, or that x_1 commutes with everything except x_2 and the Serre relation is satisfied for indices (x_1, x_2). Notably, this means $E_{x_1}E_{x_2}$ and $E_{x_2}E_{x_1}$ are linearly independent.

Suppose we have a basis $e = \{e_1, \cdots, e_k\}$ for the set of indices $(x_2, \cdots, x_m)$. Then

$$e' = \{E_{x_1}e_1, \cdots, E_{x_1}e_k, e_1E_{x_1}, \cdots, e_kE_{x_1}\}$$

is a valid basis for the set of indices $(x_1, \cdots, x_n)$. This is because E_{x_1} does not commute only with E_{x_2}, so it can either be moved to the beginning or end of any element with these indices. This basis is linearly independent, and we can verify this by computing that the new matrix of pairings is the tensor product is

$$M' = M \otimes -(q-q^{-1})^{-1} \begin{bmatrix} 1 & 1/q \\ 1/q & 1 \end{bmatrix},$$

so the inverse matrix is just

$$(M')^{-1} = M^{-1} \otimes -(q-q^{-1}) \frac{1}{q-q^{-1}} \begin{bmatrix} q & -1 \\ 1 & q \end{bmatrix} = \boxed{M^{-1} \otimes \begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix}},$$

as desired. ∎

Corollary 10. *In $U_q(\mathfrak{so}_{2n})$, the dimension of the space of E_is with indices a permutation of $(a, a+1, \cdots, a+m)$ (with $1 \le a$ and $a+m \le n$) is always a power of 2.*

Proof. Tensoring with a 2×2 matrix multiplies the matrix dimension by 2, and all base cases in the above example either have dimension 1 (a single index or $(n-1, n)$), 2 (any other pair (i, j)), or 4 (the indices $(n-2, n-1, n)$), which are all powers of 2. ∎

Example 11. *Again, take $n = 4$. We know that the dual element for E_2E_3 can be computed from the matrix*

$$M^{-1} = (q - q^{-1}) \begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix}$$

using the basis $\{E_2E_3, E_3E_2\}$ and similar for Fs. Following the above proposition, a basis for the indices $(1, 2, 3)$ is then $\{E_1E_2E_3, E_1E_3E_2, E_2E_3E_1, E_3E_2E_1\}$ (and similar for F's), which corresponds to a matrix of

$$(M')^{-1} = (q - q^{-1}) \begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix} \otimes \begin{bmatrix} q & -1 \\ -1 & q \end{bmatrix} = (q - q^{-1}) \begin{bmatrix} q^2 & -q & -q & 1 \\ -q & q^2 & 1 & -q \\ -q & 1 & q^2 & -q \\ 1 & -q & -q & q^2 \end{bmatrix}.$$

Thus, the dual element of $E_1E_2E_3$ in $\mathfrak{so}_8$ is

$$(q - q^{-1})(q^2E_1E_2E_3 - qE_1E_3E_2 - qE_2E_3E_1 + E_3E_2E_1).$$

The above procedure allows us to construct all elements that are restricted to the left or right half of the diagram from Lemma 4. However, it does not directly generalize to the case where repeated indices may be present, because there are not always an equal number of ways to "build up our basis." For instance, one basis of e's for the index set $(2, 2, 3, 4)$ for $n = 4$ is

$$\{E_2E_2E_3E_4, E_3E_2E_2E_4, E_3E_4E_2E_2, E_4E_2E_2E_3, E_2E_4E_3E_2\}.$$

From here, we can add E_1 to the element $E_2E_4E_3E_2$ to add three new elements to our basis (depending only on the relative position of E_1 and E_2), but we cannot do so to $E_2E_2E_3E_4$ because the three resulting elements are linearly dependent by our Serre relation. Nevertheless, the result of Corollary 10, combined with the fact that the dimensions for $(1, 1, 2, 2, 3, 4)$ and $(1, 2, 2, 3, 4)$ in $n = 4$ are both multiples of the dimension for $(2, 2, 3, 4)$ – the three dimensions are $15, 20$, and 5, respectively – indicates that there may be more structure in the construction of the basis when adding a new index to a general set $(x_1, \cdots, x_m)$.

Conjecture 12. *Suppose the index $x_1 - 1$ is being added to a set of indices $S = (x_1, \cdots, x_m)$ of dimension d, where $x_1 \le \cdots \le x_m$. Further assume that $x_1 \le n-2$, so that E_{x_1-1} and E_{x_1} do not commute, but E_{x_1-1} commutes with all other elements. We know that if x_1 appears only once in S and we add $(x_1 - 1)$ once, the dimension becomes $2d$.*

- *If x_1 appears twice and we add $(x_1 - 1)$ once, the dimension becomes $3d$.*
- *If x_1 appears twice and we add $(x_1 - 1)$ twice, the dimension becomes $4d$.*

Additionally, let M be a pairing matrix for a certain choice of basis for S. Then there exist a 3×3 matrix M_1 and a 4×4 matrix M_2, as well as a natural choice of basis for the index set, such that the new pairing matrix M' is $M \otimes M_1$ in the first case and $M \otimes M_2$ in the second case.

If this conjecture is found to be true, it will provide the dimensions for all sets of indices corresponding to valid elements $e_{\mu\lambda}$ and $f_{\lambda\mu}$ in $U(\mathfrak{so}_{2n})$, as well as a natural way to inductively compute all dual elements. For example, the dimension of the space for indices $(1, 2, 2, 3, 3, 4, 5)$ in $n = 5$ would then be $5 \cdot 4 \cdot 3 = 60$, because $(3, 3, 4, 5)$ has dimension 5 (analogous to $(1, 1, 2, 3)$ in $n = 3$), adding the 2s gives a factor of 4, and adding the 1 gives a factor of 3. However, computing the dual elements explicitly would still require a way to make a 3–to–1 and 4–to–1 correspondence of elements when adding the index $(x_1 - 1)$, respectively, much like the 2-to-1 correspondence from Proposition 9.

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Appendix A: Python code

The code below implements many of the calculations described in the paper, particularly in Sections 3 and 4. Here is a summary of the main functions:

- The `pair` function takes in two lists of indices and outputs the result of the q-pairing, except without the $(q - q)^{-1}$ factors. (For example, an output of $[0, -2]$ corresponds to $\left(1 + \frac{1}{q^2}\right)$ times some power of $-(q - q^{-1})$.
- The `mat` and `dual` functions take a basis of e's (and their respetcive f's) as described in Section 3, and they output M and M^{-1}, respectively.
- The `result` function computes a basis of e's or f's by enumerating a list of all possible permutations, using `reduce` to remove duplicates due to commuting elements, and using `perm` to confirm that the pairing matrix stays nonsingular.

The code runs down the left column completely before continuing in the right column.

```
from sympy import *
import itertools
from collections import deque

n = 4
var('q')

def a(i, j): # Cartan matrix lookup
    if (i == j):
        return 2
    if (abs(i-j) == 1 and max(i, j) <= n-1):
        return -1
    if (i == n-2 and j == n or i == n and j == n-2):
        return -1
    return 0

def pair(list1, list2):
    ret = []
    if(len(list1) != len(list2)):
        return []
    if(len(list1) == 1):
        if(list1[0] == list2[0]):
            return [0]
        else:
            return []
    first = list2[0]
    for i in range(len(list1)):
        if(list1[i] == first):
            inductive_list1 = list1.copy()
            inductive_list1.pop(i)
            inductive_list2 = list2.copy()
            inductive_list2.pop(0)
            ih = pair(inductive_list1, inductive_list2)
            prefactors = 0
            for j in range(i):
                prefactors += a(list1[j], list1[i])
            for j in range(len(ih)):
                ih[j] += prefactors
```

```
            ret.extend(ih)
    return ret

def mat(listofLists):
    M = zeros(len(listofLists))
    for i in range(len(listofLists)):
        for j in range(len(listofLists)):
            lst = pair(listofLists[i], listofLists[j])
            for k in lst:
                M[i, j] += q**k
    return M.applyfunc(simplify)

def dual(listofLists): # for computing specific dual elements
    M = zeros(len(listofLists))
    for i in range(len(listofLists)):
        for j in range(len(listofLists)):
            lst = pair(listofLists[i], listofLists[j])
            for k in lst:
                M[i, j] += q**k
    N= M.inv()
    return N.applyfunc(simplify)

def perm(tentlist): # given list of lists, removes linear dependence
    fin = []
    for i in range(len(tentlist)):
        fin1 = list(fin)
        fin1.append(tentlist[i])
        M = mat(fin1)
        if(M.det() != 0):
            fin.append(tentlist[i])
    return fin

def result(setofindices): # gives a basis of elements
    tentlist = list(set(itertools.permutations(setofindices)))
    for i in range(len(tentlist)):
        tentlist[i] = list(tentlist[i])
    tentlist = reduce(tentlist)
    return perm(tentlist)

def reduce(tentlist): # remove duplicates
    visited=[0 for i in range(len(tentlist))]
    adj = [[0 for i in range(len(tentlist))] for j in range(len(tentlist))]
    finlist = []
    for n in range(len(tentlist)):
        l = tentlist[n]
        for ind in range(len(l) - 1):
            if(a(l[ind], l[ind+1]) == 0):
                ledit = l[:]
                ledit[ind], ledit[ind+1] = ledit[ind+1], ledit[ind]
                m = tentlist.index(ledit)
                adj[m][n] = 1
                adj[n][m] = 1
    adjacents = [[] for i in range(len(tentlist))]
    for i in range(len(tentlist)):
        temp = []
        for j in range(len(tentlist)):
            if(adj[i][j] == 1):
                temp.append(j)
        adjacents[i] = temp
    for n in range(len(tentlist)):
        if(visited[n] == 0):
            finlist.append(tentlist[n])
            qu = deque([n])
            while(qu):
                curr = qu.popleft()
                for neigh in adjacents[curr]:
                    if(visited[neigh] == 0):
                        visited[neigh] = 1
                        qu.append(neigh)
    return finlist
```

Appendix B: Dual element expressions

We present here the full expressions for the A_i terms in the expression for the central element of $U_q(\mathfrak{so}_8)$. These expressions are certain dual elements $e^*_{\mu\lambda}$ and $f^*_{\lambda\mu}$, with factors of $\pm(q-1/q)^n$ removed (and already included in the expression for Theorem 8).

The first few expressions follow the pattern discussed in Proposition 9, and thus they should look familiar:

$$A_1 = q^3F_{1234} - q^2F_{2314} - q^2F_{3124} - q^2F_{1423} + qF_{4213} + qF_{3241} + qF_{4132} - F_{4321},$$

$$A_2 = q^3E_{1234} - q^2E_{2314} - q^2E_{3124} - q^2E_{1423} + qE_{4213} + qE_{3241} + qE_{4132} - E_{4321},$$

$$A_3 = q^3F_{4321} - q^2F_{4132} - q^2F_{4213} - q^2F_{3241} + qF_{1423} + qF_{2314} + qF_{3124} - F_{1234},$$

$$A_4 = q^3E_{4321} - q^2E_{4132} - q^2E_{4213} - q^2E_{3241} + qE_{1423} + qE_{2314} + qE_{3124} - E_{1234}$$

The next expressions are the A_is corresponding to the index set $(1,2,2,3,4)$:

$$\begin{aligned}A_5 &= q^4F_{12342} + F_{23421} + q^2F_{42132} + q^2F_{24123} - qF_{23214} + q^2F_{23124} - (q^3+q)F_{23142} \\ &\quad - \tfrac{q^4}{q^2+1}F_{12243} - q^3F_{13242} + q^2F_{32142} + \tfrac{q^4-q^2}{q^2+1}F_{42213} - q^3F_{42123} - \tfrac{q^2}{q^2+1}F_{34221}, \\ A_6 &= q^4E_{12342} + E_{23421} + q^2E_{42132} + q^2E_{24123} - qE_{23214} + q^2E_{23124} - (q^3+q)E_{23142} \\ &\quad - \tfrac{q^4}{q^2+1}E_{12243} - q^3E_{13242} + q^2E_{32142} + \tfrac{q^4-q^2}{q^2+1}E_{42213} - q^3E_{42123} - \tfrac{q^2}{q^2+1}E_{34221}, \\ A_7 &= F_{12342} + q^4F_{23421} + q^2F_{42132} + q^2F_{24123} - q^3F_{23214} + q^2F_{23124} - (q^3+q)F_{23142} \\ &\quad - \tfrac{q^2}{q^2+1}F_{12243} - qF_{13242} + q^2F_{32142} - \tfrac{q^4-q^2}{q^2+1}F_{42213} - qF_{42123} - \tfrac{q^4}{q^2+1}F_{34221}, \\ A_8 &= E_{12342} + q^4E_{23421} + q^2E_{42132} + q^2E_{24123} - q^3E_{23214} + q^2E_{23124} - (q^3+q)E_{23142} \\ &\quad - \tfrac{q^2}{q^2+1}E_{12243} - qE_{13242} + q^2E_{32142} - \tfrac{q^4-q^2}{q^2+1}E_{42213} - qE_{42123} - \tfrac{q^4}{q^2+1}E_{34221}\end{aligned}$$

Finally, here are the expressions corresponding to the index set $(1,1,2,2,3,4)$: they are identical except with F's versus E's.

$$\begin{aligned} A_9 = (-q^3-q)F_{121342} - \tfrac{q^4}{(q^2+1)^2}F_{223141} + q^2F_{143122} - q^2F_{122341} - \tfrac{q^2(q^4+q^2+1)}{(q^2+1)^2}F_{412231} + q^2F_{241312} \\ -\tfrac{q^3}{q^2+1}F_{131242} - \tfrac{q^3}{q^2+1}F_{421231} + \tfrac{q^3}{q^2+1}F_{232141} - (q^3+q)F_{413212} - q^2F_{312241} + q^2F_{132412} \\ +\tfrac{q^3}{q^2+1}F_{114232} + q^2F_{421321} + q^2F_{123124} + q^2F_{214231} - (q^3+q)F_{231421} + (q^4+q^2+1)F_{124321}, \\ A_{10} = (-q^3-q)E_{121342} - \tfrac{q^4}{(q^2+1)^2}E_{223141} + q^2E_{143122} - q^2E_{122341} - \tfrac{q^2(q^4+q^2+1)}{(q^2+1)^2}E_{412231} + q^2E_{241312} \\ -\tfrac{q^3}{q^2+1}E_{131242} - \tfrac{q^3}{q^2+1}E_{421231} + \tfrac{q^3}{q^2+1}E_{232141} - (q^3+q)E_{413212} - q^2E_{312241} + q^2E_{132412} \\ +\tfrac{q^3}{q^2+1}E_{114232} + q^2E_{421321} + q^2E_{123124} + q^2E_{214231} - (q^3+q)E_{231421} + (q^4+q^2+1)E_{124321}. \end{aligned}$$

The final six expressions listed have some denominators of (q^2+1), which are absent in all of the other elements. This factor comes from the Serre relation, and with a different choice of basis for the indices $(1,2,2,3,4)$ and $(1,1,2,2,3,4)$, it may be possible to make these expressions cleaner. Ideally, a resolution of Conjecture 12 would provide a way to compute these last six expressions without needing to manually invert the corresponding 15×15 and 20×20 M matrices, respectively.

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