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Best Arm Identification: Pure Exploration

Fixed confidence setting.

- $n$ stochastic arms, each with an associated Gaussian distribution $D_i = \mathcal{N}(\mu_i, 1)$.
- Each time we can choose an arm and take a sample from that distribution.
- Want the arm with largest mean.
- **Goal**: Succeed w.p. $1 - \delta$ and minimize the samples we need.
- $\mu_{[i]}$: $i^{th}$ largest mean, (Gap) $\Delta_{[i]} := \mu_{[1]} - \mu_{[i]}$. 
<table>
<thead>
<tr>
<th>Source</th>
<th>Sample Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even-Dar et al. [EDMM02]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \left( \ln \delta^{-1} + \ln n + \ln \Delta_{[i]}^{-1} \right)$</td>
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<td>Gabillon et al. [GGL12]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \left( \ln \delta^{-1} + \ln \sum_{i=2}^{n} \Delta_{[i]}^{-2} \right)$</td>
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<td>Jamieson et al. [JMNB13]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \left( \ln \delta^{-1} + \ln \left( \sum_{j=2}^{n} \Delta_{[j]}^{-2} \right) \right)$</td>
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<td>Kalyanakrishnan et al. [KTAS12]</td>
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<td>Jamieson et al. [JMNB13]</td>
<td>$\ln \delta^{-1} \cdot \left( \ln \ln \delta^{-1} \cdot \sum_{i=2}^{n} \Delta_{[i]}^{-2} + \sum_{i=2}^{n} \Delta_{[i]}^{-2} \ln \Delta_{[i]}^{-1} \right)$</td>
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<td>Karnin et al. [KKS13]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \left( \ln \delta^{-1} + \ln \Delta_{[i]}^{-1} \right)$</td>
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<td>Chen et al. [CL15]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \left( \ln \delta^{-1} + \ln \min(n, \Delta_{[i]}^{-1}) \right) + \Delta_{[2]}^{-2} \ln \Delta_{[2]}^{-1}$</td>
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**Table:** Sample complexity upper bounds. We omit the big-O notations.

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<td>Mannor et al. [MT04]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \ln \delta^{-1}$</td>
<td>instance-wise</td>
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<td>Farrell [Far64]</td>
<td>$\Delta^{-2} \ln \ln \Delta^{-1}$</td>
<td>worst-case, two-arm</td>
</tr>
<tr>
<td>Chen et al. [CL15]</td>
<td>$\sum_{i=2}^{n} \Delta_{[i]}^{-2} \ln \ln n$</td>
<td>worst-case</td>
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**Table:** Sample complexity lower bounds. We omit the big-Ω notations.
Two Types of Optimality

- **Instance-wise optimal**: can’t be improved on every instances up to a constant.

- All the algorithms listed above are worst case optimal.
  - Worst case optimal: can’t be improved on some instances up to a constant.

- We want an instance-wise optimal algorithm.
Gap Entropy Conjecture

- Subtly: Due to the $\Delta^{-2} \ln \ln \Delta^{-1}$ worst-case lower bound for two-arm by Farrell [Far64], there is no instance-wise algorithm even for the two-arm case.
- We conjecture that the two-arm case is the only obstruction!
- Define

$$G_k = \{i \in [2, n] \mid 2^{-k} \leq \Delta_i < 2^{-k+1}\}$$

$$H_k = \sum_{i \in G_k} \Delta_i^{-2} \quad p_k = H_k / \sum_j H_j.$$  

- Our new quantity, Gap entropy

$$\text{Ent}(I) = \sum_{G_k \neq \emptyset} p_k \log p_k^{-1}.$$  

- **Conjecture:** There is:
  - An upper bound: $O(\sum_{i=2}^{n} \Delta_i^{-2}(\text{Ent}(I) + \ln \delta^{-1}) + \Delta_2^{-2} \ln \ln \Delta_2^{-1}).$
  - An instance-wise lower bound: $\Omega(\sum_{i=2}^{n} \Delta_i^{-2}(\text{Ent}(I) + \ln \delta^{-1})).$

- The best we can hope for!
In a recent work [CL15], we obtain an upper bound for BEST-1-ARM.

Note that \( \text{Ent}(I) = O(\ln \ln n) \), our algorithm solves the case with maximum gap entropy.

A worst case lower bound

\[
\Omega \left( \sum_{i=2}^{n} \Delta_{[i]}^{-2} \ln \ln n \right)
\]

by constructing some instances with \( \text{Ent}(I) = \Theta(\ln \ln n) \).
Intuition: Upper bound

- In the framework of Karin, Koren and Somekh [KKS13].
- They assign $r^{th}$ round a confidence level $\delta_r$.
- Need to make sure $\sum_r \delta_r \leq \delta$.
- The complexity is then $O(\sum_r H_r \ln \delta_r^{-1})$.
- They set $\delta_r = \Theta(\delta/r^2)$, so their complexity is

$$O \left( \sum_r H_r \cdot (\ln \delta_r^{-1} + \ln r) \right) = O \left( \sum_i \Delta_{[i]}^{-2} \ln \ln \Delta_{[i]}^{-1} \right).$$

- In [CL15], we use a better way to assign $\delta_r$'s.
- $\sum_r H_r \ln \delta_r^{-1}$ is minimized when we set $\delta_r = \delta \cdot \frac{H_r}{\sum_k H_k}$, and we will get the running time $H \cdot (\text{Ent}(I) + \ln \delta^{-1})$.
- **Problem**: we don’t know $H_r$’s.
Ongoing Work

- We have some ideas on how to get an algorithm matching the upper bound.

- Despite that we are far from proving the conjectured lower bound, we have very strong evidence that it should be true.

- Joint work with Mingda Qiao (Tsinghua University).
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Asymptotic behavior of expected sample size in certain one sided tests.

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