Collisionless relaxation of the energy gap in superconductors

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Equations are derived for equal-time Green's functions and describe the dynamics of superconductors during a time that is short in comparison with the electron energy relaxation times τ_{ph} and τ_{ee}. The time evolution of small initial perturbations of the order parameter Δ is investigated. It is established that following initial perturbations of a definite type the energy gap relaxes only as a result of inelastic collisions of the electrons during a time on the order of τ_{ph}, τ_{ee}. In the general case, the order parameter at $t \ll \tau_{ph}$, τ_{ee} oscillates with a frequency $\sim 2\Delta$ and with an amplitude that attenuates asymptotically with time in accord with a power law.

1. INTRODUCTION

Much progress has been made recently in the problem of the nonstationary phenomena in superconductors. Equations describing the kinetics of superconductors were obtained. These equations are of simplest form in the case of gapless superconductivity[1-3]. In this case they are closed equations for the order parameter Δ and generalize the Ginzburg-Landau equations. At the same time, there are many problems in which an important role is played precisely by the presence of the finite gap in the energy spectrum of the superconductor. One such problem arises in the problem of perturbation relaxation in superconductors. It is easy to verify that in gapless superconductors the perturbations of Δ decrease exponentially to zero. What remains unclear is the character of the relaxation of Δ in the absence of external field in superconductors with gaps. In particular, one can conceive of a situation in which the perturbations of Δ experience natural undamped oscillations[4] (within a time short in comparison with the inelastic-relaxation times). The kinetics of superconductors with finite gaps is described by sufficiently complicated equations, obtained by Éliashberg[5] for equal-time Green's functions.

In this paper we examine, on the basis of equations derived from the Éliashberg equations, the behavior of a small initial perturbation of Δ in the absence of fields, and in a time interval small in comparison with the electron energy-relaxation times.

$$\tau_{ph} \approx \hbar\Theta_D^2/T^3, \quad \tau_{ee} \approx \hbar\varepsilon_F/T^2,$$

which reach 10^{-8} sec in order of magnitude.

It will be shown below that for a definite type of perturbation, the characteristic time of variation of the initial perturbation is Δ^{-1}, which is small in comparison with τ_{ph} and τ_{ee}. Then the inelastic collisions of the electrons do not influence the evolution of the perturbation during the course of times that are short in comparison with τ_{ph} and τ_{ee}, and the corresponding collision integrals can be neglected. This enables us to simplify greatly the initial Eliashberg equations for the equal-time Green's functions and to change over to equations relative to Green's functions with coinciding times (accurate to $\sim\omega_D^{-1}$). The derived equations can be regarded as "collisionless" kinetic equations for the superconductor.

In Sec. 2 we derive first the Éliashberg equations by a different method, which in essence is a generalization of the Keldysh technique[6] to the case of superconductors. This approach is simplest and most lucid.

2. DERIVATION OF THE FUNDAMENTAL EQUATIONS

Keldysh's method consists of ordering the second-quantization operators in the Green's functions on a double time-dependent contour consisting of two axes, one extending from $-\infty$ to $+\infty$, and the other from $+\infty$ to $-\infty$. The time on the second axis is assumed to be longer (in the sense of ordering of the operators) than any time on the first axis, while the ordering of the operators on the second axis is in antichronological order, i.e., the operator with the smallest time, closest to $-\infty$, is located to the left. In accordance with the fact that the time of each of the two ψ operators in the Green's function can be either on the first or on the second axis of the contour, four functions making up the matrix are possible. In the case of generalization to include superconductors, it is necessary to introduce also the spin indices of the operators (the Gor'kov-Nambu technique[7]):

$$\psi_\uparrow(1_i) \equiv \psi_1(1_i), \quad \psi_\downarrow^+(1_i) \equiv \psi_2(1_i),$$

where 1 is the aggregate of the spatial coordinates and time, and i is the index of the temporal axis. Thus, the single-particle Green's function of the electrons

$$G_{\alpha\beta}^{ik}(11') = i^{-1}\langle T\psi_\alpha(1_i)\psi_\beta^+(1_k')\rangle, \tag{1}$$

constitutes a matrix both in the spin indices (α, β) and in the temporal indices (i, k), while the angle brackets denote averaging with the density matrix taken at the instant t_0 at which the Heisenberg operators coincide with the Schrödinger operators.

In addition to the functions (1), we define also the retarded and advanced functions, and also the function introduced by Keldysh[6]:

$$\begin{gathered} G_{\alpha\beta}^R(11') = i^{-1}\theta(t_1-t_1')\langle\{\psi_\alpha(1), \psi_\beta^+(1')\}_+\rangle, \\ G_{\alpha\beta}^A(11') = i\theta(t_1'-t_1)\langle\{\psi_\alpha(1), \psi_\beta^+(1')\}_+\rangle, \\ G_{\alpha\beta}^K(11') = G_{\alpha\beta}^{12}+G_{\alpha\beta}^{21} = i^{-1}\langle[\psi_\alpha(1), \psi_\beta^+(1')]_-\rangle. \end{gathered} \tag{2}$$

From the definitions (1) and (2) we get the following relations[6] (the spin indices have not been written out):

$$\begin{gathered} G^{11}+G^{22} = G^{12}+G^{21}, \\ G^A = G^{11}-G^{21} = G^{12}-G^{22}, \\ G^R = G^{11}-G^{12} = G^{21}-G^{22}. \end{gathered} \tag{3}$$

If the system contains no fields acting directly on the electron spins, and if the spin-orbit interaction can be neglected, then

$$G_{\alpha\beta}^{ij*}(11') = (-1)^{\alpha+\beta}G_{\bar\alpha\bar\beta}^{\bar i\bar j}(11'), \tag{4}$$

where the bar over the index denotes its replacement by the opposite: $\bar 1 = 2$ and $\bar 2 = 1$. From (4) and (3) it follows that

$$\begin{gathered} G_{\alpha\beta}^{A(R)*}(11') = -(-1)^{\alpha+\beta}G_{\bar\alpha\bar\beta}^{A(R)}(11'), \\ G_{\alpha\beta}^{K*}(11') = (-1)^{\alpha+\beta}G_{\bar\alpha\bar\beta}^{K}(11'). \end{gathered} \tag{5}$$

In addition, it is easy to verify that

$$\begin{gathered} G_{\alpha\alpha}^{K*}(11') = -G_{\alpha\alpha}^K(1'1), \\ G_{\alpha\beta}^K(11') = G_{\alpha\beta}^K(1'1), \quad \alpha \neq \beta. \end{gathered} \tag{6}$$

In Dyson's equation for the Green's function

$$\hat\sigma_z\hat G_0^{-1}(1)G(11') - \Sigma(12)G(21') = 1\cdot\delta(1-1') \tag{7}$$

the operator

$$(G_0^{-1}(1))_{\alpha\beta} = \delta_{\alpha\beta}i\partial/\partial t_1 - (H(1))_{\alpha\beta}$$

is a matrix in the space of the spin indices, with

$$\hat H = \begin{pmatrix} H & 0 \\ 0 & -H^* \end{pmatrix},$$

where H(1) is the single-electron operator of the energy, reckoned from the Fermi energy, with allowance for the external fields. In addition, $(\sigma_z)_{ik}$ in (7) is a Pauli matrix, while $(1)_{\alpha\beta}^{ik} = \delta_{\alpha\beta}\delta_{ik}$. In the phonon model, the mass operator is given by a well known sum of diagrams[8] in which, however, each vertex is set in correspondence with the matrix

$$(\mathcal{T}^{(0)}(12;3))_{\alpha\beta}^{ij;k} = (\sigma_z)_{ij}\delta_{jk}(\sigma_z)_{\alpha\beta}\delta(1-2)\delta(1-3).$$

In the first-order approximation we have

$$\Sigma_{\alpha\beta}^{ij}(12) = i(-1)^{\alpha+\beta+i+j}G_{\alpha\beta}^{ij}(12)D^{ji}(21). \tag{8}$$

The contribution made to Σ by the direct interaction is also given by the usual sum of diagrams[8], in which each elementary four-corner vertex is set in correspondence with the matrix

$$(\Gamma^{(1)}(12;34))_{\alpha_1\alpha_2\alpha_3\alpha_4}^{k_1k_2k_3k_4} = -iv(12)(\sigma_z)_{k_1k_2}\delta_{k_1k_3}\delta_{k_3k_4}\{(\sigma_z)_{\alpha_1\alpha_3}\delta(1-3)(\sigma_z)_{\alpha_2\alpha_4}\delta(2-4) - (\sigma_z)_{\alpha_1\alpha_4}\delta(1-4)(\sigma_z)_{\alpha_2\alpha_3}\delta(2-3)\}. \tag{9}$$

The mass operators are connected by a relation[6] that follows from (3):

$$\Sigma^{11}+\Sigma^{22}+\Sigma^{12}+\Sigma^{21} = 0.$$

The equations for G^R, G^A, and G^K follow from (7):

$$\begin{gathered} G_0^{-1}(1)G^{R(A)}(11') - \Sigma^{R(A)}(12)G^{R(A)}(21') = \delta(1-1'), \\ G_0^{-1}(1)G^K(11') - \Sigma^R G^K + \Omega G^A = 0, \end{gathered} \tag{10}$$

where

$$\begin{gathered} \Sigma^{R(A)} = \Sigma^{11}+\Sigma^{12(21)} = -\Sigma^{22}-\Sigma^{21(12)}, \\ \Omega = \Sigma^{12}+\Sigma^{21}. \end{gathered} \tag{11}$$

These equations differ from the corresponding equations of Keldysh's paper[6] only in that all the quantities are matrices in the spin indices.

Relations (4) and (5) for the Green's functions correspond to the following relations for the mass operators:

$$\begin{gathered} \Sigma_{\alpha\beta}^{ij*} = (-1)^{\alpha+\beta}\Sigma_{\bar\alpha\bar\beta}^{\bar i\bar j}, \\ \Sigma_{\alpha\beta}^{R(A)*} = -(-1)^{\alpha+\beta}\Sigma_{\bar\alpha\bar\beta}^{R(A)}, \quad \Omega_{\alpha\beta}^* = -(-1)^{\alpha+\beta}\Omega_{\bar\alpha\bar\beta} \end{gathered} \tag{12}$$

If we write out the spin indices in (10) explicitly and use (5) and (12) then, as can be easily verified, we obtain the Éliashberg equations[5]1). It is necessary to bear in mind here the following correspondence between the notation of our paper and[5]:

$$G_{11}^{R(A)} = -G^{R(A)}, \quad G_{12}^{R(A)} = -F^{R(A)}, \quad G_{11}^K = -G, \quad G_{12}^K = -F,$$

$$\Sigma_{11}^{R(A)} = -\Sigma_1^{R(A)}, \quad \Sigma_{12}^{R(A)} = \Sigma_2^{R(A)}, \quad \Omega_{11} = -\Sigma_1, \quad \Omega_{12} = \Sigma_2.$$

We write out explicitly only the equations for G_{11}^K and G_{12}^K, which we shall need later on (together with the conjugate equations):

$$\begin{gathered} (i\partial/\partial t_1 - H(1))G_{11}^K(11') - \Sigma_{11}^R G_{11}^K + \Sigma_{12}^R G_{12}^{K*} + \Omega_{11}G_{11}^A + \Omega_{12}G_{12}^{A*} = 0, \\ (i\partial/\partial t_1' + H^*(1'))G_{11}^K(11') + G_{11}^K\Sigma_{11}^A + G_{12}^K\Sigma_{12}^{A*} - G_{11}^R\Omega_{11} + G_{12}^R\Omega_{12}^* = 0, \end{gathered} \tag{13}$$

$$\begin{gathered} (i\partial/\partial t_1 - H(1))G_{12}^K(11') - \Sigma_{11}^R G_{12}^K - \Sigma_{12}^R G_{11}^{K*} + \Omega_{11}G_{12}^A - \Omega_{12}G_{11}^{A*} = 0, \\ (i\partial/\partial t_1' - H(1'))G_{12}^K(11') - G_{12}^K\Sigma_{11}^{A*} + G_{11}^K\Sigma_{12}^A - G_{12}^R\Omega_{11}^* - G_{11}^R\Omega_{12} = 0. \end{gathered} \tag{14}$$

As is well known, $(\Sigma_{12}^R + \Sigma_{12}^A)/2$ plays the role of the operator of the energy gap Δ of the superconductor. We represent Σ_{12}^R and Σ_{12}^A in the form

$$\Sigma_{12}^{R(A)}(12) = \Delta(12) + {}^1/_2(\Sigma_{12}^{R(A)}(12) - \Sigma_{12}^{A(R)}(12)). \tag{15}$$

We add Eqs. (13) for G_{11}^K and, in similar fashion, Eqs. (14) for G_{12}^K. The result can be represented in the following form (we neglect renormalization effects described by Re $\Sigma_{11}^{R(A)}$ [5]):

$$\begin{gathered} (i\partial/\partial t - H(1) + H^*(1'))G_{11}^K(11') \\ +\Delta(12)G_{12}^{K*}(21') + G_{12}^K(12)\Delta^*(21') = I_{11}(11'), \\ (i\partial/\partial t - H(1) - H(1'))G_{12}^K(11') - \Delta(12)G_{11}^{K*}(21') \\ +G_{11}^K(12)\Delta(21') = I_{12}(11'). \end{gathered} \tag{16}$$

Here $t = (t_1 + t_1')/2$, and the form of the "collision integrals" I_{11} and I_{12} can be easily obtained from (13)–(15). From the expressions for Σ_{12}^R and Σ_{12}^A (see (11) and (8)) follows the well known results[5]

$$\Delta(12) = {}^1/_4 i^{-1}G_{12}^K(12)[D^A(21) + D^R(21)].$$

Here the sum of the phonon functions $D^A + D^R$, regarded as the function $\tau = t_1 - t_2$, is a narrow peak of width $\sim\omega_D^{-1}$. Therefore the energy gap is

$$\Delta(\mathbf{R}t) = \int d\mathbf{r}\,d\tau\,\Delta(12) = \frac{ig^2}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\int_{-\omega_D}^{\omega_D}\frac{d\varepsilon}{2\pi}G_{12}^K(\mathbf{R}t;\mathbf{p}\varepsilon), \tag{17}$$

where

$$\mathbf{R} = (\mathbf{r}_1+\mathbf{r}_2)/2, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$

$$G_{12}^K(\mathbf{R}t;\mathbf{p}\varepsilon) = \int d\mathbf{r}\,d\tau\,e^{-i\mathbf{p}\mathbf{r}+i\varepsilon\tau}G_{12}^K(12).$$

In a number of problems, the reciprocal of relaxation times in a superconductor are much smaller than Δ and then the characteristic frequencies of all the quantities as functions of t. In this case I_{11} and I_{12} can be discarded in (16). We can simplify the obtained collisionless kinetic equations further by using the already mentioned narrowness of Δ (12) as a function of τ, and by using the fact that to determine $\Delta(\mathrm{R}t)$ we need the function G_{12}^K at practically equal times ($\tau \lesssim \omega_D^{-1}$). This makes the system of collisionless equations (Eqs. (16) without the right-hand sides) for G_{11}^K and G_{12}^K with $\tau \lesssim \omega_D^{-1}$, together with expression (17) for $\Delta(\mathrm{R}t)$, a closed system.

We introduce the notation

$$\begin{gathered} g(\mathbf{p};\mathbf{R}t) = \frac{1}{2\pi}\int_{-\omega_D}^{\omega_D}d\varepsilon\,G_{11}^K(\mathbf{R}t;\mathbf{p}\varepsilon), \\ f(\mathbf{p};\mathbf{R}t) = \frac{1}{2\pi}\int_{-\omega_D}^{\omega_D}d\varepsilon\,G_{12}^K(\mathbf{R}t;\mathbf{p}\varepsilon). \end{gathered} \tag{18}$$

Taking into account the approximations made above, we obtain from (16)

$$\begin{gathered} [i\partial/\partial t - H(\mathbf{r}_1t) + H^*(\mathbf{r}_1't)]g(\mathbf{r}_1\mathbf{r}_1't) + \Delta(\mathbf{r}_1t)f^*(\mathbf{r}_1\mathbf{r}_1't) + \Delta^*(\mathbf{r}_1't)f(\mathbf{r}_1\mathbf{r}_1't) = 0, \\ [i\partial/\partial t - H(\mathbf{r}_1t) - H(\mathbf{r}_1't)]f(\mathbf{r}_1\mathbf{r}_1't) - \Delta(\mathbf{r}_1t)g^*(\mathbf{r}_1\mathbf{r}_1't) + \Delta(\mathbf{r}_1't)g(\mathbf{r}_1\mathbf{r}_1't) = 0. \end{gathered} \tag{19}$$

We note that these equations are obtained in the BCS model without additional assumptions. In accordance with formula (9) for $\Gamma^{(1)}$ in the self-consistent field approximation we have

$$\Sigma_{\alpha\beta}^{ij}(12)=-i(\sigma_z)_{ij}\{(\sigma_z)_{\alpha\beta}v(13)G_{\gamma\gamma}^{ij}(33)-(-1)^{\alpha+\beta}G_{\alpha\beta}^{ij}(12)v(21)\}.$$

In this approximation $\sum_A^{ij}=0$ at $i\neq j$ and $\Omega=0$, and the mass operators $\Sigma_{\alpha\beta}^R$ and $\Sigma_{\alpha\beta}^A$ reduce to $\Sigma_{\alpha\beta}^{11}$ or $-\Sigma_{\alpha\beta}^{22}$. Next, as usual, we include Σ_{11}^R and Σ_{11}^A in the chemical potential, and express $\Delta(12)=\frac{1}{2}(\Sigma_{12}^{11}-\Sigma_{12}^{22})$ in the form

$$\Delta(12)=-\tfrac{1}{2}iG_{12}^K(12)v(21)=\tfrac{1}{2}ig\delta(12)G_{12}^K(11)=\Delta(1)\delta(12).$$

When relations (5) are taken into account, we arrive at (19).

It should be noted that the obtained equations (19) describe the behavior of the Green's functions integrated with respect to ϵ (see (18)), and not with respect to $\xi=(p^2-p_0^2)/2m$ as in Éliashberg's paper[5]. Therefore the first equation of (19) goes over at $\Delta=0$ into the usual collisionless kinetic equation.

3. COLLISIONLESS EVOLUTION OF PERTURBATIONS IN A SUPERCONDUCTOR

We use Eqs. (19) to investigate the evolution of the initial perturbations in a superconductor during times that are short in comparison with the energy-relaxation times τ_{ph} and τ_{ee}. We confine ourselves to the spatially homogeneous case and assume that all the quantities are independent of the direction of p, and that there are no fields. After applying a Fourier transformation with respect to the difference coordinate $\mathbf{r}-\mathbf{r}_1-\mathbf{r}_1'$, Eqs. (19) and (17) take the form

$$i\partial g(\xi t)/\partial t+\Delta(t)f^*(\xi t)+\Delta^*(t)f(\xi t)=0, \qquad (20a)$$

$$(i\partial/\partial t-2\xi)f(\xi t)+2\Delta(t)g(\xi t)=0, \qquad (20b)$$

$$\Delta(t)=i\frac{\lambda}{2}\int d\xi\, f(\xi,t). \qquad (20c)$$

We took into account here the fact that in accordance with (6) g is pure imaginary. Assume that arbitrary initial perturbations $\delta g_0(\xi)$ and $\delta f_0(\xi)$ have occurred in the superconductor and are described by the deviations of the functions f and g from their stationary values g_0 and f_0. The perturbations can be produced either by electromagnetic radiation or by injecting quasiparticles or pairs into the superconductor. Let us find the evolutions of the perturbations with the aid of the linearized system (20). We see that the problem is quite analogous in its formulation with the Landau problem of evolution of perturbations in a collisionless plasma[9]. The electric self-consistent field in the plasma corresponds in the superconductor to the "field" $\Delta(t)$.

The stationary functions f_0 and g_0 can be represented, according to (20), in the form

$$f_0(\xi)=-i\Delta\chi(\xi),\; g_0(\xi)=-i\xi\chi(\xi), \qquad (21)$$

where the form of $\chi=\chi^*$ is determined by the particular conditions for the production of the stationary state. Under thermodynamic equilibrium we have

$$\chi=[\mathrm{th}(\varepsilon/2T)]/\varepsilon,$$

where $\epsilon=(\xi^2+\Delta^2)^{1/2}$. We linearize (20) with respect to the stationary values in (21):

$$(i\partial/\partial t-2\xi)\delta f(\xi t)-2i\xi\chi\delta\Delta(t)+2\Delta\delta g(\xi t)=0,$$

$$i\partial\delta g/\partial t-2\Delta\chi\delta\Delta''+2\Delta\delta f'(\xi t)=0.$$

The prime and double prime denote here the real and imaginary parts of the function, respectively. We note that, at small perturbations, $\delta\Delta'$ is the change of the gap $|\Delta|$, while $\delta\Delta''/\Delta$ is the change of the phase of the order parameter.

We separate the imaginary and real parts of the equation:

$$\partial\delta f''/\partial t+2\xi\delta f'-2\xi\chi\delta\Delta''=0,$$

$$\partial\delta f'/\partial t-2\xi\delta f''-2\xi\chi\delta\Delta'+2\Delta\delta g''=0, \qquad (22)$$

$$\partial\delta g''/\partial t+2\Delta\chi\delta\Delta''-2\Delta\delta f'=0.$$

For the Laplace transforms of the sought functions $\delta f(s)$ we obtain from (22)

$$(s^2+4\varepsilon^2)\delta f'(s)-4\varepsilon^2\chi\delta\Delta''(s)+2\xi\chi s\delta\Delta'(s)=\delta\dot f_0'+s\delta f_0'-2\xi\chi\delta\Delta',$$

$$(s^2+4\varepsilon^2)\delta f''(s)-2\xi\chi s\delta\Delta''(s)+4\xi^2\chi\delta\Delta'(s) \qquad (23)$$

$$=-\frac{2\xi}{s}(\delta\dot f_0'+s\delta f_0')+\frac{4\xi^2\chi}{s}\delta\Delta_0'+\frac{\delta f_0''}{s}(s^2+4\varepsilon^2).$$

Here $\delta\Delta_0'=[\delta\Delta'(t)]_{t=0}$, etc. In the derivation of (23) we have expressed δg_0 with the aid of the section equation of (22) in terms of $[d\delta f'/dt]_{t=0}\equiv\delta\dot f_0'$ and $\delta f_0''$.

We multiply (23) by $\lambda/2$ and integrate with respect to ξ. We confine ourselves to the case of functions χ that are even in ξ. Then, taking (23) into account, we obtain

$$s^2F(s)\delta\Delta''(s)=\left\langle\frac{\delta\dot f_0'+s\delta f_0'}{s^2+4\varepsilon^2}\right\rangle, \qquad (24a)$$

$$(s^2+4\Delta^2)F(s)\left(\delta\Delta'(s)-\frac{\delta\Delta_0'}{s}\right)=\frac{1}{s}\left\langle\frac{2\xi(\delta\dot f_0'+s\delta f_0')}{s^2+4\varepsilon^2}\right\rangle. \qquad (24b)$$

We have introduced here the notation

$$F(s)=\left\langle\frac{\chi}{s^2+4\varepsilon^2}\right\rangle,\quad \langle\ldots\rangle=\frac{\lambda}{2}\int_{-\omega_D}^{\omega_D}d\xi\ldots$$

and used the equality $\chi=1$, which follows from (20c) and (21).

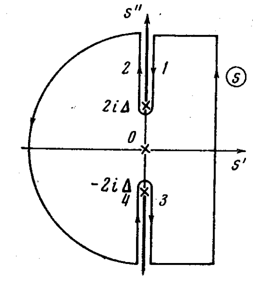
If $\delta\dot f_0'=0$ and $\delta f_0'=0$ (but $\delta f_0''\neq 0$) then as seen from (24b), $\delta\Delta'(s)=\delta\Delta_0'/s$, meaning that $\delta\Delta'(t)=\delta\Delta_0'$ = const and that the energy gap relaxes only as a result of inelastic collisions during a relaxation time much larger than Δ^{-1}. If $\Delta\dot f_0'+s\delta f_0'\neq 0$, then that part of this function which is even in ξ determines the time variation of $\delta\Delta''(t)$, i.e., the change of the phase of the order parameter, while the odd part determines the variation of $\delta\Delta'(t)$, i.e., of the modulus of the gap.

Let us investigate the analytic properties of F(s) in the complex s plane. Near the imaginary axis $s=i\Omega+\delta$, where $\delta\to 0$, we have

$$F(s)=\frac{\lambda}{2}\left\{\fint_{-\omega_D}^{\omega_D}\frac{d\xi\,\chi(\varepsilon)}{4\varepsilon^2-\Omega^2}-i\frac{\pi}{2}\mathrm{sgn}(\Omega\delta)\frac{\chi(\Omega/2)}{(\Omega^2-4\Delta^2)^{1/2}}\theta(\Omega^2-4\Delta^2)\right\}. \qquad (25)$$

At $|\Omega|>2\Delta$, the imaginary part of F(s) experiences a discontinuity when s crosses the imaginary axis. Therefore F(s) has a branch point at $s=\pm 2i\Delta$, and the single-valued branch F(s) should be chosen by drawing the cuts as shown in the figure.

The functions in the right-hand side of (24) have the same branch points as F(s). Therefore $\delta\Delta''(s)$ has in the general case branch points at $s=\pm 2i\Delta$, and in addition poles of first and second order at zero, while $\delta\Delta'(s)$ has, in addition to these branch points, also a first-order pole at zero. It is important to emphasize that $\delta\Delta'(s)$ has no poles at the point $s=\pm i2\Delta$. Indeed, the right-hand side of (24b) differs from zero if $\delta\dot f_0'+s\delta f_0'$ has a part that is odd in ξ. At the same time, from the continuity of this part at the point $\xi=0$ it



follows that it can be represented in the form $\xi\eta(\xi)$, where $\eta(\xi)$ is an even function having no singularities at zero. Then the right-hand part of (24b) can be transformed into

$$\langle\eta\rangle-(s^2+4\Delta^2)\left\langle\frac{\eta}{s^2+4\varepsilon^2}\right\rangle.$$

The first term does not depend in general on s, and the second factor at $s^2+4\Delta^2$ has the same singularities as F(s).

Since the function $\delta\Delta'(s)$ has no poles at the points $s=\pm i2\Delta$, it follows that the energy gap, even if we neglect the inelastic collisions, has no natural undamped oscillations with frequency 2Δ (cf.[4]).

Let us illustrate the behavior of $\delta\Delta(t)$ under the following concrete initial conditions

$$\delta f_0'=2\alpha_1\xi\chi+\alpha_2\chi,\; \delta\dot f_0'=2\beta_1\xi\chi+\beta_2\chi,$$

where α_i and β_i are small constants. It follows then from (24), with allowance for (23), that

$$\delta\Delta''(s)=\frac{\beta_2}{s^2}+\frac{\alpha_2}{s}=\frac{\dot\Delta_0''}{s^2}+\frac{\Delta_0''}{s},$$

$$\delta\Delta'(s)=\frac{\Delta_0'}{s}+\frac{\beta_1+\alpha_1 s}{s(s^2+4\Delta^2)F(s)}-\frac{\beta_1+\alpha_1 s}{s}$$

Taking the inverse Laplace transform, we find that the imaginary part of the gap, which is proportional to the phases, increases linearly with time:

$$\delta\Delta''(t)=\delta\dot\Delta_0''t+\delta\Delta_0'',$$

while the real part is given by

$$\delta\Delta'(t)=\delta\Delta_0'-\beta_1+\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\frac{ds(\beta_1+\alpha_1 s)e^{st}}{s(s^2+4\Delta^2)F(s)}. \qquad (26)$$

The integral can be represented here in the form of a difference of the integral over the entire closed contour in the figure and the integrals over sections 1, 2, 3, and 4 of the second contour. As a result we obtain

$$\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\frac{ds(\beta_1+\alpha_1 s)e^{st}}{s(s^2+4\Delta^2)F(s)}=\frac{\beta_1}{4\Delta^2F(0)}$$

$$-\frac{2}{\pi}\int_{2\Delta}^{\infty}d\Omega\frac{\mathrm{Im}\,F(\Omega)}{|F(\Omega)|^2}\frac{1}{\Omega^2-4\Delta^2}\left[\beta_1\frac{\cos\Omega t}{\Omega}-\alpha_1\sin\Omega t\right]. \qquad (27)$$

In the calculation we took into account the fact that the imaginary part of F(s) on opposite sides of the cut along the imaginary axis (see the figure) differ in sign, and that $F(s)=F(-s)$.

It is easy to verify that at $t=0$ the integral in (26) is equal to β_1 and that $\delta\Delta'(0)=\delta\Delta_0'$, as should be the case. To understand the character of the time variation of the gap perturbation, let us find the asymptotic value of the integral in (27) at $2t\Delta\gg 1$. It is determined by the behavior of the integrand at $\Omega\approx 2\Delta$. It can be shown that $\mathrm{Re}\,F(s)$ has no singularities as $s\to 2i\Delta$ (see (25)) and that $\mathrm{Im}\,F/|F|^2=(\mathrm{Im}\,F)^{-1}$ near this point. Substituting Im F from (25) and taking the equilibrium value of χ, we obtain

$$\delta\Delta'(t)=\delta\Delta_0'-\beta_1+\frac{\beta_1}{4\Delta^2F(0)}+\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\lambda\,\mathrm{th}(\Delta/2T)}$$

$$\times(2t\Delta)^{-1/2}\left[\beta_1\cos\left(2\Delta t+\frac{\pi}{4}\right)-2\alpha_1\Delta\sin\left(2\Delta t+\frac{\pi}{4}\right)\right].$$

We see that the perturbation of the gap, as a function of the time, take the form of oscillations having a frequency $\sim 2\Delta$ and an amplitude that attenuates like $t^{-1/2}$. It is important that with increasing t (but at $t\ll\tau_{ee},\tau_{ph}$) in the considered collisionless approximation the gap perturbation $\delta\Delta'(t)$ tends generally speaking neither to zero nor to $\delta\Delta_0'$.

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1)We note that the term $(2\pi)^4(\omega)\delta(k)$ in Eq. (13) of Éliashberg's paper [5] should be discarded.

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