A phonon is customarily defined as an electron interacting with the non-dispersive lattice vibrations in an ionic crystal. The formulation of this problem is due to Peierls. The parameters of the problem are the electron mass \( m \) and charge \( e \), the frequency \( \omega_0 \) of the optical phonons, and the values of the crystal dielectric constant in the low- and high-frequency limits, \( \varepsilon_0 \) and \( \varepsilon_\infty \). The dimensionless combination of these quantities
\[
\alpha = \left( \frac{\omega_0}{2 \sqrt{\varepsilon_0 m e^2}} \right) \left( \frac{1}{\varepsilon_\infty} - \frac{1}{\varepsilon_0} \right)
\]
determines the strength of the coupling between the electron and the lattice.

We shall be solely interested in the case of strong coupling, \( \alpha \ll 1 \). In this limit Peierls found that the phonon's ground state energy was given by \( E_0 = -0.1094a \hbar \omega_0 \). The phonon mass, \( M = 0.023a \hbar^2 \), was calculated by Landau and Peierls. The numerical coefficients in the formulas for \( E_0 \) and \( M \) are determined from the dimensionless Schrödinger equation (see Eq. (4) below). It is necessary to assume that the lowering of the ground state energy and the increase of the mass of the electron plus the interaction with phonons takes the major part of the electron-phonon interaction into account. The remaining part of the interaction must determine the corrections to \( E_0 \) and \( M \), and it must also cause the scattering of the phonon by real phonons. It is necessary, in particular, to take this scattering into account in order to calculate the phonon mobility. A systematic strong-coupling theory of the phonons should give a method for calculating physical quantities in the form of a power series in the reciprocal coupling constant. Such an approach has been developed by Bogolyubov and Tsybul'kov. They reproduced the results of Peierls and Landau for the phonon's energy and mass as the first-approximation in their approach. The major difficulty in this approach has turned out to be the introduction of phonon coordinates together with the electron coordinates. The condition that the total number of independent dynamical variables must be conserved upon introducing phonon coordinates required the determination of three subsidiary conditions on the phonon coordinates. This fact led to very cumbersome calculations, and the investigation of the phonon's kinetics has not been completed. The articles by Peierls, Feynman, Heitler, Hjörnason, and Plattman, and Kadanoff, are devoted to the investigation of the phonon's kinetics. Without going into the details of these articles, we note the striking disagreement between the results obtained in these articles for the phonon mobility in the limit \( \alpha \ll 1 \). The indicated circumstances justify an attempt to construct a theory of the phonon by means of a systematic expansion in powers of the inverse coupling constant. The ultimate goal is to investigate the kinetics of the phonons.

In the present work a phonon-phonon Hamiltonian will be derived which is suitable for the description of weakly excited states of the crystal near the energy \( E_0 \) in the limit \( \alpha \ll 1 \), and the corrections, some of which do not have the form of a Hamiltonian, are estimated. The calculation of the correction to the ground state energy \( E_0 \) will be reduced to a variational principle. We shall show that the phonon mobility is determined by two-phonon processes, namely, the scattering of a phonon by the phonons. It will be shown that the momentum transfer associated with a collision is small, so that the phonon mobility is related to the amplitude for the scattering of a phonon by a phonon by a simple relationship. Analysis of the equation for the scattering amplitude enables us to obtain an upper bound for the phonon mobility.

**DERIVATION OF THE POLARON-PHONON HAMILTONIAN**

The Lagrangian of an electron interacting with the phonons in an ionic crystal has the form
\[
\mathcal{L} = \mathcal{L}_{\text{elec}} + \mathcal{L}_{\text{ph}} = \frac{m}{2} \ddot{z}^2 - U(z) - \frac{1}{2} \sum_{n} \dot{\varphi}_{n}^2 - \frac{1}{2} \sum_{n} \varphi_{n}^2 \mathcal{F}(\varphi_{n}).
\]
Here \( \mathcal{F} \) and \( \varphi \) are the effective mass and velocity of the electron, and \( \varphi_{n} \) and \( \dot{\varphi}_{n} \) represent the coordinate and velocity of a phonon with quasi-momentum \( k \) and \( \sigma \) on the electron-phonon constant. For simplicity we set \( \hbar = m = \omega = 1 \). The factor \( \mathcal{F}^{-1} \) in front of the last term in Eq. (3) has also been replaced by \( m \) so that
\[
\sum_{n} \varphi_{n}^2 \mathcal{F}(\varphi_{n}) = \sum_{n} \varphi_{n}^2 \mathcal{F}(\varphi_{n}).
\]

The quantum formalism in the Lagrangian formulation is described in (4) and requires an averaging of \( \langle \varphi_{n} \varphi_{n} \rangle \) over the classical trajectories, where the relative weight of each trajectory is determined by the classical action, calculated for our problem with the aid of the Lagrangian (3).

Now let us describe a method of overcoming the principal difficulty in the phonon problem, which is the transition from electron coordinates to phonon coordinates. In addition to an integration over the trajectories of the electron, we introduce some integration—over the trajectories \( R(t) \). Since \( R(t) \) is not contained in expression (3), the integration will give a numerical factor, which cancels out upon evaluating the thermodynamic averages. One further transformation, which does not change the value of the path-integrals, consists in replacing \( \varphi_{n}(t) \) by \( \varphi_{n}(t) + \varphi_{n}(t) \), where \( \varphi_{n}(t) \) is expected to be \( \varphi_{n}(t) \) with a different time. Now the Lagrangian explicitly depends on \( R(t) \) and \( \varphi(t) \), but this dependence is fictitious—it corresponds to the transformation to a moving reference frame and can be eliminated by the inverse of shift of \( \varphi(t) \). Let us choose

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In such a way that the individual terms in the Lagrangian (or the corresponding Hamiltonian) can be classified according to the powers of the coupling constant α. Hence, the second term in \( \mathcal{L} \) is of the order \( \alpha^2 \), and we construct a perturbation theory based on the functional \( \mathcal{L}^0 + \mathcal{L}_1 \), where \( \mathcal{L}_1 \) is a second-order correction to \( \mathcal{L} \). Let us return the phonon operator in \( \mathcal{L} \) to its standard form by a canonical transformation with the aid of the operator \( \exp[-i\mathcal{L}^0] \). This transformation depends on \( \alpha \) and \( \beta \) and leads to the following terms in the Hamiltonian:

\[
\hat{H} = \hat{H}^0 + \alpha \hat{\mathcal{L}}^0 + \beta \hat{\mathcal{L}}^1 + \alpha \beta \hat{\mathcal{L}}^0 \hat{\mathcal{L}}^1
\]

The left-hand side of the equation (11) can be recognized as the effective mass of the polariton, which is the mass of the polariton. The right-hand side of the equation is the expression for the effective mass of the polariton. The effective mass of the polariton is given by

\[
m_{\text{eff}} = m_0 + \frac{\alpha \beta}{2} \frac{1}{\sqrt{1 - \alpha^2}}
\]

The correct expression for the effective mass of the polariton is given by

\[
m_{\text{eff}} = m_0 + \frac{\alpha \beta}{2} \frac{1}{\sqrt{1 - \alpha^2}}
\]

The result for the effective mass of the polariton is obtained by

\[
m_{\text{eff}} = m_0 + \frac{\alpha \beta}{2} \frac{1}{\sqrt{1 - \alpha^2}}
\]

The next step in the derivation is to evaluate the effective mass of the polariton in the weak-coupling limit. The effective mass of the polariton is given by

\[
m_{\text{eff}} = m_0 + \frac{\alpha \beta}{2} \frac{1}{\sqrt{1 - \alpha^2}}
\]

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\]
Polaron Mobility

The mobility of the polaron is determined by its interaction with real phonons. According to the Hamiltonian (14), processes are possible involving the simultaneous emission or absorption of a pair of phonons and the scattering of a phonon by the polaron with a change of the momentum. The correlation (11) leads to the absorption or emission of a phonon and the correlation (16) to the Hamiltonian gives rise to three-phonon processes. The interaction of a polaron with two quasiclassical forms was treated in a different notation in work by the author and Rashba. It was shown that an infinite number of local phonon modes are present when the polaron is below the dispersion. The variational principle was formulated to calculate the wave functions and frequencies of these levels. In this version, it was possible to transform the variational functional in such a way that it was expressed only in terms of the polaron's ground state wave function \( \psi \):

\[
\frac{1}{2\hbar^2} \int \nabla \psi^* \nabla \psi = \int \frac{1}{2} \left( \frac{\hbar}{2m} \right)^2 \left( \nabla \psi_0 \right)^2 - \int \frac{1}{2} \omega e \left( \nabla \psi_0 \right)^2 \psi_0^* \psi_0.
\]

Here \( \psi_0 \) is the variational function, and \( \omega \) is the frequency of the local vibration. The variational functional does not contain any singular parameters; therefore, \( \psi \) differs from \( \psi_0 \) numerically. The ground state energy of the polaron now takes the form

\[
E = \int \frac{1}{2} \left( \frac{\hbar}{2m} \right)^2 \left( \nabla \psi_0 \right)^2 - \int \frac{1}{2} \omega e \left( \nabla \psi_0 \right)^2 \psi_0^* \psi_0.
\]

Formulas (20) and (21) enable us to calculate the energy of Eq. (9), which is the energy of the polaron, and moreover, the frequencies \( \omega_0 \) reduce to a common form.

Now let us prove that the corrections to \( E_0 \) (21) are small. The corrections arising from the term containing \( (k \cdot k)^2 \) in Eq. (10) is of the order of

\[
\sum \frac{1}{k^4} \left( \frac{\hbar^2}{m} \omega_0^2 \right)^2 - \frac{1}{2} \sum \frac{1}{k^4} \left( \frac{\hbar^2}{m} \omega_0^2 \right)^2.
\]

It is assumed that the mean square fluctuation of the velocity \( \langle v^2 \rangle \sim \langle k^2 \rangle \), where \( k \) is the characteristic phonon momentum \( k \rightarrow \omega \). Taking into consideration that the characteristic frequencies are equal to unity, we have \( \langle v^2 \rangle \sim \langle k^2 \rangle \sim \omega_0^2 \) for the mean square fluctuations, so that the correlation to the energy coming from the term containing \( (k \cdot k)^2 \) in Eq. (10) is of the order of \( m \hbar^2 \omega_0^2 \). If the polaron is below the dispersion, the phase factor \( e^{i2kx} \) from the integral (19) serves (at \( P = 0 \)) as one more source of a correction to the energy; it is necessary to combine this term with the last term. The first nonvanishing correction is of order \( \langle v^2 \rangle \); finally, the cubic correction (in the phonon variables) to the Hamiltonian (16) gives a contribution \( \sim \langle v^2 \rangle \) in second-order perturbation theory.

The correction to the polaron mass is found, for example, if the term \( \frac{1}{2} \sum \frac{1}{k} \frac{1}{k'} \hbar^2 \omega_0^2 \langle k^2 \rangle \langle k'^2 \rangle \) in Eq. (19) is taken into account in perturbation theory. It has a relative order \( \langle v^2 \rangle \). The estimates presented here prove that one can re-