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Nuclear Physics A 598 (1996) 390–417

NUCLEAR PHYSICS A

Solution of the gap equation in neutron matter

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Received 17 November 1995

Abstract

The problem of solving the gap equation for S-wave pairing in pure neutron matter is considered for the case that the pairing matrix elements $V(p,p')$ are calculated directly from a realistic bare neutron–neutron potential containing a strong short-range repulsion. The original gap equation is replaced identically by a coupled set of equations: a non-singular quasilinear integral equation for the dimensionless gap function $\chi(p)$ defined by $\Delta(p) = \Delta_F \chi(p)$ and a non-linear algebraic equation for the gap magnitude $\Delta_F = \Delta(p_F)$ at the Fermi surface. This reformulation admits a robust and rapidly convergent iteration procedure for the determination of the gap function. The treatment may be extended to singlet or triplet pairing in non-zero angular momentum states. S-wave pairing is investigated numerically for the Reid-soft-core interaction. Although the pairing matrix elements of this potential are everywhere positive, non-trivial solutions of the gap equation are obtained on the range $0 < p_F < p_c = 1.7496\ldots$ fm^{-1} of Fermi momenta, with the gap parameter Δ_F reaching a maximum of some 3 MeV near $p_F = 0.85$ fm^{-1}. Numerical results are also provided for the highly realistic Argonne v_{14} and v_{18} interactions. Within the context of the new computational scheme, a condition for closure of the gap is derived in terms of the first zero p_0 of the gap function $\Delta(p)$. It is shown that Δ_F vanishes exponentially not only in the low-density limit $p_F \to 0$, but also as the Fermi momentum rises and approaches the upper critical value p_c specified by $p_F = p_0(p_F)$, beyond which there exists no non-trivial solution of the gap equation. The numerical results for the function $\Delta(p)$ in neutron matter display a remarkable universality of structure, visible especially in the stability of p_0 under variation of density. Upon renormalizing the gap equation in terms of the vacuum S-wave scattering amplitude, this behavior is seen to be a manifestation of the resonant nature of the neutron–neutron interaction at low energy, which leads to a scattering amplitude of nearly separable form.

1. Introduction

There is heightened interest in nuclear matter at large neutron excess. In the astrophysical context, this interest is driven by observations bearing on the rotational dynamics

SSDI 0375-9474(95)00477-7

and thermal evolution of neutron stars [1], and, in nuclear physics, by measurements on light nuclei with neutron halos [2] and by increasing experimental access to heavier nuclei near the neutron drip line [3]. A central theoretical issue is the nature of pairing in neutron-rich assemblies. In this article, we shall focus on pairing in dilute neutron matter, with a two-fold purpose. Our first aim is to introduce a new method for numerical calculation of the gap $\Delta(p)$ in the single-particle spectrum of a superfluid Fermi system, as determined by the standard BCS equation with a singular kernel. This method is designed to overcome certain technical problems that obstruct the solution of the gap equation in neutron matter at the relevant densities, especially when the pairing force is furnished by a realistic bare nucleon–nucleon (NN) interaction. The new procedure is also applicable to physically more realistic gap equations containing dressed versions of the pairing interaction and single-particle energies (for example, see Ref. [4]).

Our second aim is to expose several novel features of the pairing problem in dilute neutron matter, stemming mainly from the fact that the neutron–neutron (nn) interaction is almost attractive enough to produce a two-body bound state at zero energy. This property is reflected in the resonant structure of the vacuum S-wave nn-scattering amplitude $\Gamma^{(0)}$, which is dominated by a pole term that reflects the large negative nn scattering length in vacuum, $a_{nn} \approx -18.5$ fm. It will be one of the tasks of this article to make explicit the interplay between the contributions to the gap coming from this pole term and from the usual logarithmic term in the particle–particle propagator.

To provide a benchmark for studies of neutronic subsystems of nuclei and neutron stars, it is desirable to carry out a precise evaluation of the energetics of pairing in pure neutron matter. Such an investigation is stimulated, for instance, by the fact that some many-body treatments of the normal phase [5,6] show a significant flattening of the equation of state of the neutron system (i.e. of the energy per particle $\mathcal{E}$ versus baryon density ρ) relative to the ideal Fermi gas, in the density region corresponding to Fermi momenta $p_F \sim 0.2$ fm^{-1}. In addition, an effective particle–particle interaction derived microscopically within variational theory has been found to yield $^1S_0$ gap values $\Delta_F = \Delta(p_F)$ as large as half the Fermi energy ε_F (Ref. [4]). Consequently, the negative pairing contribution to $\mathcal{E}$ (i.e. *minus* the condensation energy per particle $\mathcal{E}_c$) might prove to be of qualitative importance, producing a noticeable reduction of the pressure $P(\rho)$ and the compressibility modulus $K(\rho)$ and significant further softening of the matter toward instability.

A minimal prerequisite for a quantitative investigation of pairing phenomena in neutron matter is the availability of an accurate method for solution of the BCS gap equation for "realistic" NN potential models that have been adjusted to fit free-space two-body data. The gap equation for singlet S-wave pairing can be written in the general form

$$\Delta(p) = -\int \frac{V(p,p')\Delta(p')}{2E(p')} d\tau' , \tag{1}$$

where $V(p,p')$ is the angle-average of the effective particle–particle interaction at zero total momentum and $E(p) = [\xi^2(p) + \Delta^2(p)]^{1/2}$ is the quasiparticle spectrum in the superfluid state, with $\xi(p) = \varepsilon_p - \mu$. In turn, ε_p is the single-particle spectrum in the

normal state and μ is the corresponding chemical potential. We adopt the abbreviation $d\tau' = (p')^2 dp'/2\pi^2$.

Commonly, a straightforward iteration scheme is applied to the solution of Eq. (1), starting with some initial guess $\Delta^{[0]}(p')$ for the gap function $\Delta(p')$ appearing on the r.h.s. of the equation. The simple ansatz $\Delta^{[0]}(p')$ = constant is often used, a popular estimate for this constant gap being provided by the BCS weak-coupling formula [7] $\Delta_{\rm BCS} = 2\omega_c \exp[-1/N(0)g]$, where g is a coupling constant and $N(0) = Mp_F/2\pi^2$ (with $\hbar = 1$) is the density of single-particle states at the Fermi surface. It is customary to take $\omega_c = \varepsilon_{p_F}$ and $g = -V(p_F, p_F)$.

Even if we exclude interaction models containing infinitely hard cores, difficulties may arise when one attempts such an iterative solution of the gap equation. The situation encountered for the Reid-soft-core (RSC) potential [8] – a staple of the nuclear many-body theory – is symptomatic of a more general problem. For the Reid case one must face the immediate obstacle that the bare pairing interaction $V(p, p')$ is positive on the Fermi surface, $p = p' = p_F$. (In fact, $V(p, p')$ is never negative!) This feature rules out $\Delta(p)$ = constant = Δ as a viable approximate solution of Eq. (1); in particular, the BCS weak-coupling ansatz becomes meaningless. Although "softer" NN interaction models – notably the Urbana/Argonne v_{14} and v_{18} [9,10] and Paris [11] potentials – do have negative values for $V(p_F, p_F)$, an intrinsic difficulty persists. As is now well known, the pairing matrix elements of realistic NN forces are such that the momentum integration in the gap equation must be extended to very high values of p' (some 10–50 fm^{-1}) to achieve satisfactory accuracy [12].

What is not generally appreciated is that the contribution from this high-momentum region, which is proportional to Δ under the presumption of a constant gap, can profoundly modify the conventional picture of pairing. Schematically, the gap equation (1) becomes

$$\Delta = -N(0)g\Delta \ln \Delta + \gamma\Delta,$$

with γ as the constant of proportionality, leading to the relation

$$1 - \gamma = -N(0)g \ln \Delta,$$

which would otherwise (i.e. for $\gamma = 0$) verify the weak-coupling formula. The high-momentum contribution can change the sign of the l.h.s. of this relation, presaging the possibility of a solution of (1) when the pairing interaction is positive on the Fermi surface.

This raises the following point, especially relevant for nuclear interactions. While negativity, in a given two-body channel, of the angle-average of the pairing interaction on the Fermi surface, $p = p' = p_F$, is *sufficient* for the existence of a pairing instability in that channel, *it is not necessary*. In particular, the gap equation for S-wave pairing, Eq. (1), may have solutions at densities for which $V(p = p_F, p' = p_F) \equiv V_F > 0$. Moreover, as we shall demonstrate for the Reid interaction, pairing may exist over a substantial density range even if $V_F > 0$ *for all* p_F. The door to pairing by a positive pairing force is opened by the argument we have just sketched, but the possibility of

this phenomenon may be understood in simpler terms by direct inspection of Eq. (1). Since the denominator of the integrand on the right-hand side is always positive, we can of course make the quantity $\Delta(p)$ on the left-hand side positive at p_F with a negative pairing interaction $V(p,p')$ and with $\Delta(p')$ consistently positive at all momenta. This corresponds to the familiar, textbook view of the gap equation. On the other hand, with $V(p,p')$ positive for $p = p_F$ within some range of p' including p_F, we can also achieve a positive gap at p_F if the gap function $\Delta(p')$ in the integrand exhibits a negative excursion over some range of momenta $p' > p_F$, so as to produce a net negative value for the integral in Eq. (1). Such a solution $\Delta(p)$ will be start out positive but must turn negative at some momentum $p_0 > p_F$. The occurrence of a finite-momentum zero in the gap function $\Delta(p)$ is thus an inevitable concomitant of the existence of a non-trivial solution of the gap equation for pairing forces typically encountered in infinitely extended nuclear systems. In detail, it is found that $\Delta(p)$ displays a shape like that of the curves shown in Fig. 4 or Fig. 10, with damped oscillatory structure and a substantial negative swing for p values in the approximate range 2–8 fm^{-1}. Such oscillatory behavior is not restricted to potentials having $V_F > 0$ at the density in question; indeed it is seen quite generally for realistic NN interactions, which necessarily have positive pairing matrix elements $V(p,p')$ over extended momentum domains. From these considerations, one naturally suspects that the Fermi momentum at which the gap Δ_F closes must be intimately related to the first zero p_0 of the gap function $\Delta(p)$. It will be one of our tasks to explicate this relation.

In earlier work, the difficulties and complications associated with pairing in nucleonic matter have been eliminated or suppressed by replacing the given interaction with a softer phase-shift-equivalent potential [13,14], or with a suitable microscopic [4,16,14] or (semi-)phenomenological [17–19] effective potential including medium effects, for which straightforward algebraic or iterative solution of the gap equation proceeds without much difficulty. (Also, a solution for a "hard" potential like the RSC interaction may sometimes be reached by using the solution for a soft phase-shift-equivalent potential as a starting point [13,20].) Another strategy, followed in Ref. [12], is to solve the gap equation for the reduced interaction introduced by Anderson and Morel [21], the input pairing interaction being renormalized by high-momentum effects to achieve a separation of low- and high-momentum regimes. (Within this approach one can still encounter problems for the RSC potential.) We shall instead formulate and apply an efficient and accurate method of solution that does not call for any alteration or renormalization of the original pairing force for use in the gap equation. The need for such a numerical approach is especially acute in the case of pairing in non-zero angular momentum states (namely in the $^3P_2$–$^3F_2$ channel), which may occur in neutron stars at densities beyond that of ordinary nuclear matter [22–27]. At these higher densities the superfluid properties become sensitive to the off-shell character of the assumed interaction, and hence replacement of the best available NN interactions (fitted to a variety of few-body data) by more tractable phase-shift-equivalent potentials becomes suspect.

In pure neutron matter, the unusually large *nn* scattering length has the important consequence that the limiting domain of low density, characterized by $p_F|a_{nn}| \ll 1$, is

strictly reached only at exceedingly low densities in more familiar terms, well below 10^{-5} times the equilibrium density ρ_0 of symmetrical nuclear matter. The pairing effect in the $^1S_0$ channel, which is the chief concern of this paper, is known to peak at a density that is far higher, yet still only about one tenth of ρ_0. The matter in this regime of maximal singlet pairing will be termed "dilute", acknowledging the fact that the mean particle spacing is much larger than the length scale set by the repulsive core of the NN interaction, while making a distinction from genuine low-density matter.

In Section 2 we reformulate the pairing problem in the $^1S_0$ channel and propose a new method for solution of the integral equation for the gap function. As explained in Section 3, the new perspective on the gap problem allows us to establish the anticipated link between the (density-dependent) first zero of the gap function $\Delta(p)$ and an upper critical density at which the gap Δ_F vanishes. Section 4 presents results from numerical application of the method to S-wave pairing, primarily for the test case of the Reid-soft-core interaction. Section 5 is devoted to a formal analysis of the nature of solutions of the singlet-S pairing problem, with specific attention to the observed structure of the gap function $\Delta(p)$ and its relation to the resonant character of the nn interaction. In Section 6 we summarize our findings and indicate several outstanding problems that must be overcome to arrive at a quantitative understanding of pairing in infinite nucleonic matter. Appendix A adapts the proposed strategy for the solution of gap equations to the special situation of vanishing V_F, while Appendix B shows how this new approach may be extended to examples of triplet pairing.

2. Procedure for solution of the gap equation

In this paper we shall concentrate on the problem of S-wave pairing in pure, homogeneous neutron matter, as described by the gap equation (1). To avoid unjustified complications, the particle–particle interaction V and the single-particle spectrum ε_p are taken in their simplest forms: we set V equal to the bare or free-space nn potential and ε_p equal to the free single-particle energy. Thus $\varepsilon_p = p^2/2M$, the chemical potential μ being just the Fermi energy $\varepsilon_F = \varepsilon_{p_F}$.

It is assumed that the interaction $V(p,p')$ has no zeros on the Fermi surface, that is $V(p_F,p_F) \neq 0$. (The special case of vanishing $V(p_F,p_F)$ is considered in Appendix A.) The key step of our method consists of decomposing the potential $V(p,p')$ into a separable part and a remainder $W(p,p')$ that vanishes when either argument is on the Fermi surface:

$$V(p,p') = V_F\phi(p)\phi(p') + W(p,p')\,. \tag{2}$$

The choice $\phi(p) = V(p,p_F)/V_F$, with $V_F = V(p_F,p_F) \neq 0$, meets the required conditions $W(p_F,p') = W(p,p_F) = 0$ for all $p\,,p'$. The gap equation then takes the form

$$\Delta(p) + V_F\phi(p)\int \phi(p')\frac{\Delta(p')}{2E(p')}d\tau' + \int W(p,p')\frac{\Delta(p')}{2E(p')}d\tau' = 0\,. \tag{3}$$

Defining a dimensionless gap function (or shape function) $\chi(p)$ through $\Delta(p) = \Delta_F\chi(p)$, and dividing Eqs. (1) and (3) by $\Delta_F \equiv \Delta(p_F)$, we have

$$\chi(p) + \int W(p,p')\frac{\chi(p')}{2\left[\xi^2(p') + \Delta_F^2\chi^2(p')\right]^{1/2}}d\tau' = \phi(p)\,. \tag{4}$$

(We note in passing that the function $\chi(p)$ represents an interaction amplitude calculated for the residual interaction potential W when one of the two momentum arguments of this amplitude lies on the Fermi surface.)

On the other hand, setting $p = p_F$ in (3) and remembering that $W(p_F,p)$ is identically zero while $\phi(p_F) = 1$, we obtain the equation

$$\Delta_F = -V_F\int \frac{\phi(p')\Delta(p')}{2\left[\xi^2(p') + \Delta_F^2\chi^2(p')\right]^{1/2}}d\tau' \tag{5}$$

for the gap value Δ_F at the Fermi surface, a result conveniently expressed as

$$1 + V_F\int \frac{\phi(p')\chi(p')}{2\left[\xi^2(p') + \Delta_F^2\chi^2(p')\right]^{1/2}}d\tau' = 0\,. \tag{6}$$

Thus, the original gap equation (1) has been replaced, without approximation, by the two equivalent equations (4) and (6). The essential effect of the substitution (2) is that for $p \to p_F$ the integrand in the second term of Eq. (4) vanishes and the near-singular situation is circumvented. The corresponding integral then becomes insensitive to any reasonable variation of $\Delta(p') = \Delta_F\chi(p')$ within the $E(p')$ denominator. In other words, the shape function $\chi(p)$ is practically independent of $\Delta(p')$. For specified Δ_F, Eq. (4) is an integral equation with non-singular kernel, effectively a linear integral equation ("quasilinear") because the non-linearity induced by the presence of $\chi(p')$ inside the square root is inconsequential. Eq. (6) is a non-linear algebraic equation that can be solved by Newton's method and related algorithms when a solution exists. For both formal and computational purposes, it is advantageous to rewrite Eq. (6) as

$$\int \frac{\phi^2(p)}{2\left[\xi^2(p) + \Delta_F^2\chi^2(p)\right]^{1/2}}d\tau = -\frac{1}{V_F} + \int \phi(p)\frac{\left[\phi(p) - \chi(p)\right]}{2\left[\xi^2(p) + \Delta_F^2\chi^2(p)\right]^{1/2}}d\tau\,, \tag{7}$$

observing that the identity $\chi(p_F) = \phi(p_F) = 1$ renders the right-hand side almost independent of the behavior of $\Delta(p)$ and hence of Δ_F.

A simple iteration scheme for determination of the gap function $\Delta(p) = \Delta_F\chi(p)$ is suggested by the extremely weak dependence of the function $\chi(p)$ on the behavior of the gap.

(i) Replace the quantity $\Delta_F\chi(p)$ in the denominator of the integrand in Eq. (4) by a (very small) constant scaling factor $\Delta_F^{[0]}$ and solve the resulting *linear* integral equation by matrix inversion on a suitable grid, to obtain a first approximation $\chi^{[1]}(p)$ for the shape function $\chi(p)$.

(ii) Substitute $\chi^{[1]}(p)$ for $\chi(p)$ in Eq. (7) (or Eq. (6)) and solve the resulting non-linear algebraic equation, to obtain a first approximation $\Delta_F^{[1]}$ for the scaling factor Δ_F.

(iii) Repeat step (i) with $\Delta_F^{[1]}\chi^{[1]}(p)$ as input for $\Delta_F\chi(p)$ in the integral term, generating a second approximation $\chi^{[2]}(p)$ to $\chi(p)$; and repeat step (ii) with $\chi^{[2]}(p)$ as input for $\chi(p)$, generating a second approximation $\Delta_F^{[2]}$ to Δ_F; and so on until satisfactory convergence has been achieved in both $\chi(p)$ and Δ_F.

In actual calculations with realistic *nn* interactions, the convergence of this algorithm is so rapid that accurate results for $\chi(p)$ and Δ_F are already achieved in the first iteration cycle (i), (ii). Moreover, the r.h.s. of Eq. (7) is so insensitive to $\Delta_F\chi(p)$ that it is immaterial in step (ii) whether, in the square root, this quantity is treated as $\Delta_F\chi^{[1]}$ or simply replaced by the constant $\Delta_F^{[0]}$.

The above iteration procedure exploits the quasilinear nature of Eq. (4) for $\chi(p)$, replacing this equation, in each iteration cycle, by a linear integral equation susceptible to elementary computational methods. The quasilinearity of Eq. (4), residing in the insensitivity of the integral term with respect to what one inserts for $\Delta_F\chi(p')$ in the denominator of the integrand, is emphasized by rewriting the equation in the generic linear form

$$\chi(p) + \int W(p,p')\frac{\chi(p')}{2\left[\xi^2(p') + \delta^2\right]^{1/2}}d\tau' = \phi(p)\,. \tag{4'}$$

The similar property of Eq. (7) is similarly highlighted by rewriting it as

$$\int \frac{\phi^2(p)}{2\left[\xi^2(p) + \Delta_F^2\chi^2(p)\right]^{1/2}}d\tau = -\frac{1}{V_F} + \int \phi(p)\frac{\left[\phi(p) - \chi(p)\right]}{2\left[\xi^2(p) + \delta^2\right]^{1/2}}d\tau\,. \tag{7'}$$

In writing (4′) or (7′), it is implied that any reasonable input for the quantity δ will suffice (e.g. $\Delta_F^{[n]}\chi^{[n]}$ from some iteration cycle n, and in many instances simply $\Delta_F^{[0]}$). The δ notation will prove useful in forthcoming arguments, serving as an economical reminder of the essential mathematical and numerical behavior of the reformulation of the gap problem expressed in Eqs. (4) and (7).

An analogous decomposition of the gap problem may be carried through in higher partial waves. The extension of our approach to triplet pairing is sketched in Appendix B.

3. Condition for gap closure

The reformulation of the gap equation expressed by Eqs. (4) and (7) is fruitful not only as a platform for efficient numerical solution but also as vehicle for the exploration of certain fundamental aspects of the gap problem that transcend the application to neutron matter.

One important analytical result that can be extracted is a condition relating the critical density at which the gap closes to the first zero of the shape function $\chi(p)$. This condition determines the upper extent of the density range in which a positive pairing force – more specifically an interaction having $V_F > 0$ – can still yield non-trivial solutions of the gap equation.

Thinking for the time being in terms of the Reid interaction, for which $V(p,p')$ is everywhere positive, and recalling the argument given in Section 1, it is clear that in order to satisfy (1) or (6) and obtain a positive (or non-zero) value of Δ_F at a given p_F, the functions $\Delta(p)$ of (1) and $\chi(p)$ of (6) must pass through zero at some point p_0 and become negative. In deriving a condition for gap closure in terms of the position of this first zero, it is important to realize that, given any reasonable choice of $V(p,p')$, the integral Eq. (4) will admit a solution for $\chi(p)$ at any density, whether or not Eq. (6) has a non-trivial solution for the gap amplitude Δ_F.

Now consider the recast version (7) of Eq. (6). As we have already stressed, the right-hand side of this equation is virtually independent of the gap Δ_F, a fact reflected in the surrogate form (7′), which captures the quantitative behavior of (7) with any reasonable input δ. On the other hand, the left-hand side of Eq. (7) is a positive, monotonically decreasing function of Δ_F, dropping from infinity to zero as Δ_F ranges from zero to infinity. Therefore a positive solution of (7) exists if and only if the right-hand side of this equation is positive.

In outline, the situation is as follows: The sign of the right-hand member of Eq. (7) depends primarily on the sign of $\chi(p)$ in the p domain relatively near the Fermi surface that is essential for the integration on the r.h.s. of Eq. (7). The latter sign, in turn, depends on the location of p_F with respect to p_0. In the lower density region defined by $p_F < p_0$, the identity $\chi(p_F) = 1$ imposes a positive sign on $\chi(p)$ for $p < p_0$ and a negative sign for $p > p_0$ (within the relevant p domain). The contribution to the integral from the negative-going portion of $\chi(p)$ prevails and ensures a positive sign for the r.h.s. of (7). In the higher density region where $p_F > p_0$, the function $\chi(p)$ must instead be positive for $p > p_0$ (and in the relevant p range), since it is constrained to equal 1 at $p = p_F > p_0$. Consequently, as p_F increases past p_0, the sign of the r.h.s. of Eq. (7) changes and a non-trivial solution of the gap equation can no longer exist. The behavior of $\chi(p)$ that is responsible for the shift of sign is documented numerically in the next section.

The operative behavior of $\chi(p)$ as a function of momentum p and density $\rho = p_F^3/3\pi^2$, in the neighborhood of the zero $p = p_0$, can be characterized mathematically by

$$\chi(p;\rho) = x(p;\rho)\frac{p - p_0(\rho)}{p_F - p_0(\rho)}. \tag{8}$$

In this expression, $x(p;\rho)$ is a smooth function of p near $p = p_0$ and incorporates the property $\chi(p_F) = 1$ through the normalization $x(p = p_F;\rho) = 1$. If $x(p;\rho)$ is also a smooth function of ρ, then $\chi(p;\rho)$ will have a simple pole at a density corresponding to $p_F = p_0(\rho)$.

Now suppose that, as the density rises and approaches a critical value ρ_c, the gap Δ_F approaches zero (gap closure). Then the l.h.s. of Eq. (7) diverges logarithmically as $\rho \to \rho_c$. Hence a non-trivial solution of this equation can exist only if its r.h.s. also diverges. Noting that for any density the integrand on the r.h.s. has a finite value at $p = p_F$, we see that equality of left and right members of the equation can be guaranteed

only if $\chi(p;p_F)$, as a function of p_F, has a pole at $p_F = p_c \equiv (3\pi^2\rho_c)^{1/3}$. Assume for the sake of argument that $p_c \neq p_0$. This assumption immediately entails a contradiction, since it implies that $x(p;\rho)$ diverges when ρ approaches ρ_c and thereby violates the condition $x(p_F;\rho_c) = 1$. On the other hand, with $p_c = p_0$ the function $x(p;\rho)$ remains well behaved in the closure regime. We are forced to conclude that the critical Fermi momentum for gap closure coincides with the first zero of the shape function $\chi(p)$ (or the gap function $\Delta(p)$) at the corresponding density. Thus,

$$p_c = p_0(p_F = p_c)\,. \tag{9}$$

A further analytical result stemming from our reformulation of the gap equation establishes the functional form of the gap Δ_F as the Fermi momentum p_F approaches the critical point p_c from below. In solving Eq. (7) near the critical density, $x(p;\rho_c)$ may be used in place of $x(p;\rho)$ in expression (8) for $\chi(p)$. Adopting the δ notation introduced in Eq. (4′), we can determine $x(p;\rho_c)$ independently by means of the homogeneous equation

$$r(p) = \int W(p,p';p_F = p_c)\frac{1}{2\left[\xi^2(p') + \delta^2\right]^{1/2}} r(p')\,d\tau' \tag{10}$$

for the residue $r(p)$ of the function

$$\chi(p;\rho) = \frac{r(p)}{p_F - p_c} \tag{11}$$

at the simple pole $p_F = p_c$. The vanishing of $W(p,p')$ for p on the Fermi surface has the important consequence that $r(p = p_c) = 0$. On comparing (8) and (11) we infer that

$$x(p;\rho_c) = \frac{r(p)}{p - p_c}\,. \tag{12}$$

This relation determines the normalization of $r(p)$ via the condition $x(p = p_c;\rho_c) = 1$. On rewriting $r(p)$ in the form

$$r(p) = \left.\frac{dr(p)}{dp}\right|_{p_c}(p - p_c)\,, \tag{13}$$

applicable near $p = p_c$, it is seen that the required solution of Eq. (10) obeys

$$\left.\frac{dr(p)}{dp}\right|_{p_c} = 1\,. \tag{14}$$

Eq. (8), with $x(p;\rho)$ given by Eq. (12), may now be inserted into Eq. (7). Estimation of the gap Δ_F from the resulting expression is simplified by the fact that for $p_F \sim p_c$, where Δ_F is necessarily very small, only the momentum region close to the Fermi surface is pertinent. Thus we need only retain the portions of (7) that are divergent when Δ_F goes to zero. Introducing the integral

$$I = \frac{1}{2p_F^2}\int \frac{r(p)}{|p^2 - p_F^2|}p^2\,dp\,, \tag{15}$$

we arrive at the relation

$$\ln \frac{1}{\Delta_F} + \text{const.} \sim \frac{Ip_F}{p_c - p_F} \qquad (p_F - p_c \to 0^-)\,. \tag{16}$$

Therefore the gap vanishes exponentially as the critical density is approached, according to the asymptotic formula

$$\Delta_F \sim \eta_c \varepsilon_F \exp\left[-\frac{Ip_F}{p_c - p_F}\right] \qquad (p_F - p_c \to 0^-)\,, \tag{17}$$

where η_c is a prefactor of order unity. Patently, this formula only applies for $p_F < p_c$, since we have shown that $\Delta_F \equiv 0$ for $p_F \geqslant p_c$.

The foregoing analysis has been framed for the Reid interaction, whose V_F is always positive. However, it may readily be generalized to other realistic NN interactions, such as Argonne v_{14} and v_{18}, for which V_F is negative below some Fermi momentum (near 0.7 fm^{-1} in these examples). The association of the critical point p_c with the condition $p_F = p_0(p_F)$ still applies, since clearly V_F will be positive in the vicinity of the gap closure density. Likewise, the argument leading to an exponentially vanishing gap as $\rho - \rho_c \to 0^-$ may be carried over immediately, upon appealing once again to the fact that for $p_F \sim p_c$ only momenta in the immediate vicinity of the Fermi surface participate in the formation of the gap.

4. Numerical applications

The procedure delineated in Section 2 has been applied to several interactions with the intent of gaining a detailed numerical picture of the behavior of the gap function in dilute neutron matter. For the Reid-soft-core (RSC) interaction, calculations of $\Delta(p)$ and $\chi(p)$ have been carried out at some 200 Fermi momenta in the interval 0.01 fm^{-1} < p_F < 2 fm^{-1}. The integrations over p' and p in Eqs. (4) and (7) have been extended out to 50 fm^{-1}. The proposed algorithm is stable even in regions where Δ_F becomes exponentially small and thus facilitates a quantitative study of $\Delta(p)$ near the upper critical point $p_F = p_c$ where the gap closes. The basic results are displayed in Figs. 1–13.

The only other published results on $^1S_0$ neutron pairing for the bare Reid interaction are those of Takatsuka [13], which are available at points in the interval 0.14 fm^{-1} < p_F < 1.30 fm^{-1}. These results were obtained by an iteration scheme that starts with the solution of the gap equation for a Gaussian-core potential that is approximately phase-shift-equivalent to the RSC interaction. The results for Δ_F cannot be compared directly with those of Fig. 1, since the single-particle energy ϵ_p entering the quasiparticle energy $E(p)$ was parametrized by an effective mass M^* less than the bare mass. In particular, the effective mass in the ansatz $\varepsilon_p - \mu = (p^2 - p_F^2)/2M^*$ was determined from $p_F/M^* = (d\varepsilon_p/dp)|_{p=p_F}$ using a Brueckner–Hartree–Fock approximation to ε_p for the RSC potential. We repeated our gap evaluation for the same effective masses; the discrepancies between the predictions for Δ_F from the two approaches is nowhere larger than 1–2%.

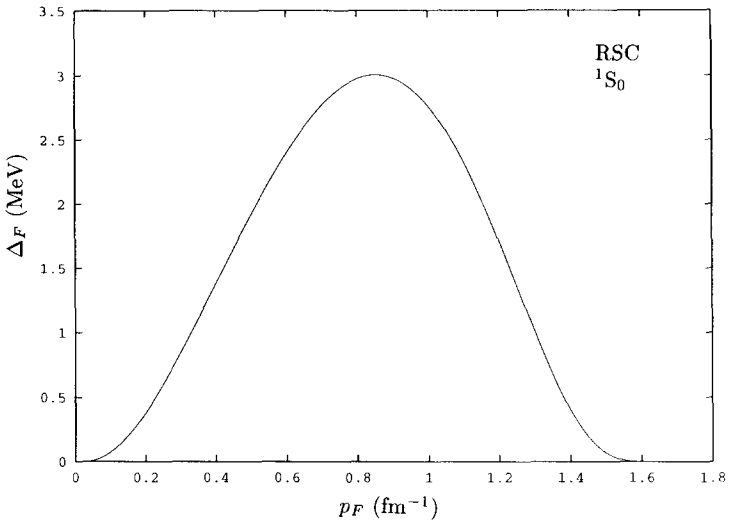

Fig. 1. Neutron-matter S-wave energy gap $\Delta_F = \Delta(p_F)$ at the Fermi surface, plotted against Fermi momentum p_F, for the Reid-soft-core (RSC) interaction as pairing force, a free single-particle spectrum $\varepsilon_p = p^2/2M$, and a chemical potential equal to the Fermi energy, $\mu = \varepsilon_{p_F} = \varepsilon_F$.

At this point we would like to inject a caveat regarding the use of an effective-mass approximation in the neutron-matter problem. While a reasonable variation of the density of states near the Fermi surface may be achieved in such a treatment, one must pay a price in terms of an incorrect behavior of the quasiparticle spectrum at large momenta and a significant departure from the vacuum scattering length. The complicated issue of the real behavior of the neutron mass operator at large momenta must be deferred until a future investigation, since the primary goal of the present article – the development of a new method for solution of the gap equation – is technical rather than physical. Accordingly, the general discussion to follow will focus on the case $M^* = M$.

Figs. 1 and 2 show plots of the gap amplitude $\Delta_F = \Delta(p = p_F)$ and the condensation energy per particle $\mathcal{E}_c$ as functions of the Fermi momentum p_F, calculated by our method for the RSC potential with $M^* = M$. The gap Δ_F reaches a maximum value of 3.00 MeV at $p_F = 0.85$ fm^{-1}, while the maximum value 0.29 MeV of $\mathcal{E}_c$ occurs at a somewhat lower density corresponding to $p_F = 0.63$ fm^{-1}. Pairing corrections to the pressure P and compressibility K differ from the ideal-gas values by less than 10%.

A non-trivial solution for the gap exists in the density interval $0 < p_F < p_c$, where p_c is determined by the location $p = p_0$ of the first zero of the gap function $\Delta(p;\rho)$ in momentum space. The location of this zero is almost independent of the density ρ (see Fig. 3). In the low-density regime $p_F \ll 1/a_{nn}$, the crossover occurs at $p = p_0 \simeq 1.90$ fm^{-1}. With increasing density, the first zero of $\Delta(p)$ is slowly driven toward smaller p, but accelerates noticeably as p_F approaches a critical value p_c for gap closure, specified by the condition $p_F = p_0(p_F) = p_c$. For the RSC interaction, we find $p_c \approx 1.7496$ fm^{-1}. At densities with $p_F \geqslant p_c$, there is no non-trivial solution for

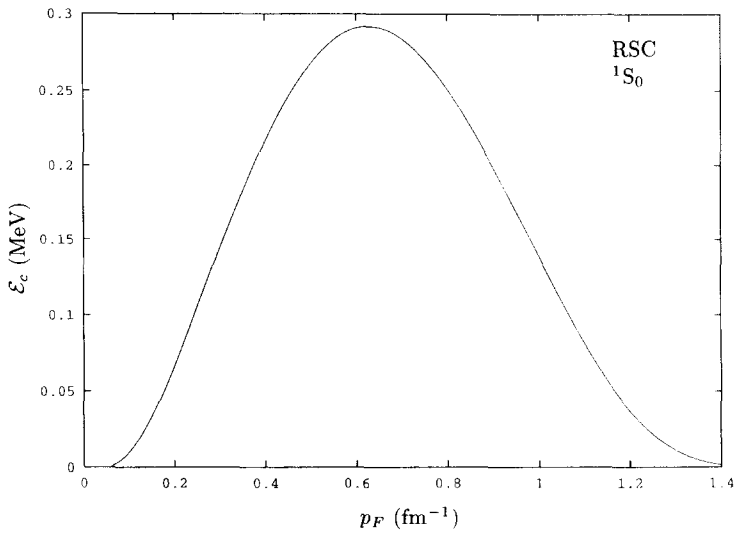

Fig. 2. Condensation energy per neutron, $\mathcal{E}_c$, plotted against Fermi momentum p_F, for the RSC interaction, $\varepsilon_p = p^2/2M$, and $\mu = \varepsilon_F$.

the gap Δ_F and $\Delta(p)$ vanishes identically. Because of the linear scale used for Δ_F in Fig. 1, it is not apparent that a finite gap survives beyond $p_F \approx 1.6$ fm^{-1}, decaying to exponentially small values in accord with Eq. (17) and reaching zero precisely at $p_F = p_c$.

Thus, our numerical findings for the RSC example bear out, in full detail, the analysis

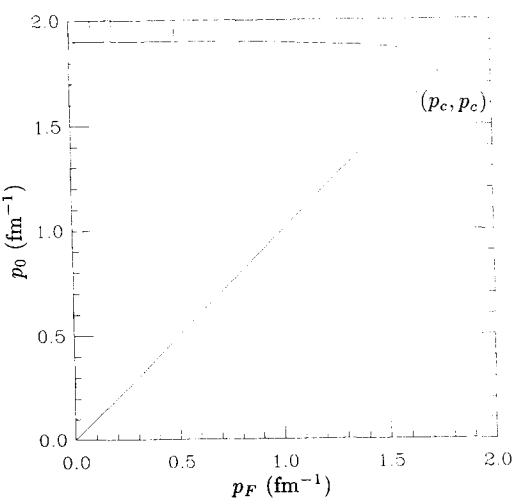

Fig. 3. Schematic behavior of the position p_0 of the first zero of the momentum-dependent S-wave gap function $\Delta(p)$ (or of the dimensionless gap function $\chi(p)$), as a function of Fermi momentum p_F. The critical point p_c is defined by the condition $p_0(p_F) = p_F = p_c$. The Reid-soft-core interaction is assumed, together with $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

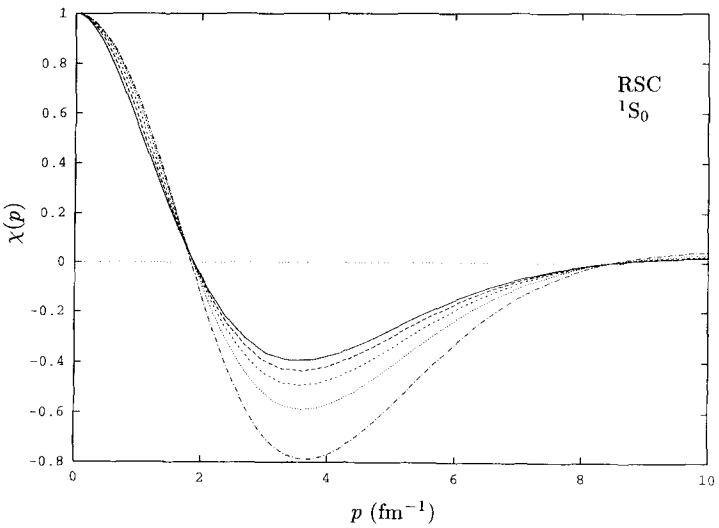

Fig. 4. Dimensionless gap function $\chi(p)$ versus momentum p, at different densities (with $p_F = 0.3$, 0.6, 0.9, 1.2 and 1.5 fm^{-1} for the solid, longer-dashed, short-dashed, dotted and dot-dashed curves, respectively). Here the shape function $\chi(p)$ is normalized to unity at $p = 0$. The Reid-soft-core (RSC) interaction is assumed, together with $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

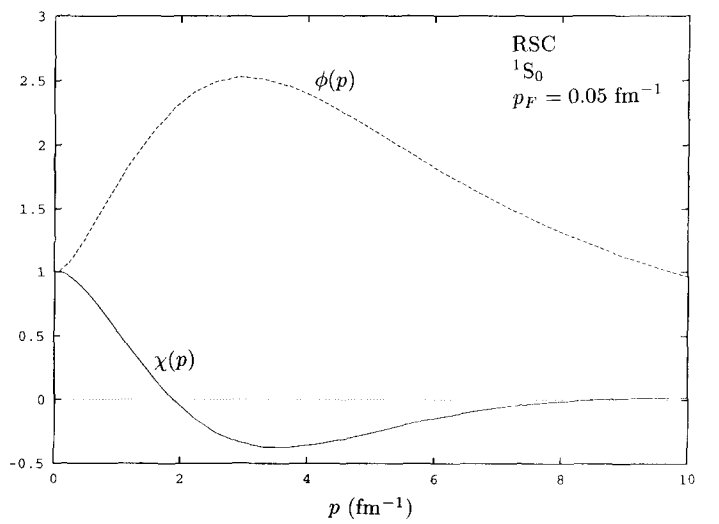

Fig. 5. Dimensionless gap function $\chi(p)$ (solid curve) and potential factor $\phi(p) = V(p, p_F)/V_F$ (dashed curve), as functions of momentum p, at a "low" density corresponding to Fermi momentum $p_F = 0.05$ fm^{-1}. Both $\chi(p)$ and $\phi(p)$ are normalized to unity at $p = p_F$. The RSC interaction is assumed, together with $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

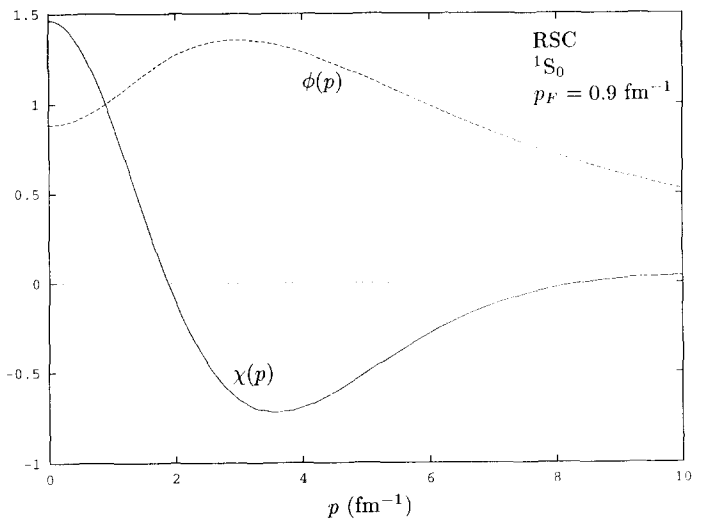

Fig. 6. Same as Fig. 5, but at a "moderate" density corresponding to $p_F = 0.9$ fm^{-1}, near the location of the peak of the gap Δ_F.

offered in Section 3 – although the remarkable insensitivity of p_0 to the density calls for further explanation (see Section 5). The evolution of the solution of the gap equation and the ultimate quenching of the gap at $p_F = p_c$ may be tracked by studying the evolving behavior of $\chi(p)$ and $\phi(p)$, which is illustrated in Figs. 5–8 for a low density, an intermediate density and densities just below and just above the critical point. One should especially take note of the dramatic change in the structure of $\chi(p)$ as the

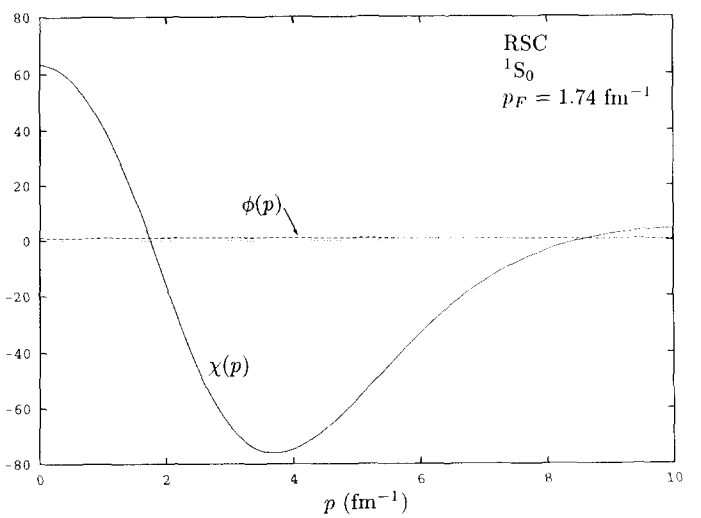

Fig. 7. Same as Fig. 5, but at a "high" density corresponding to $p_F = 1.74$ fm^{-1}, just below the critical Fermi momentum $p_F = 1.7496\ldots$ beyond which the gap is found to vanish.

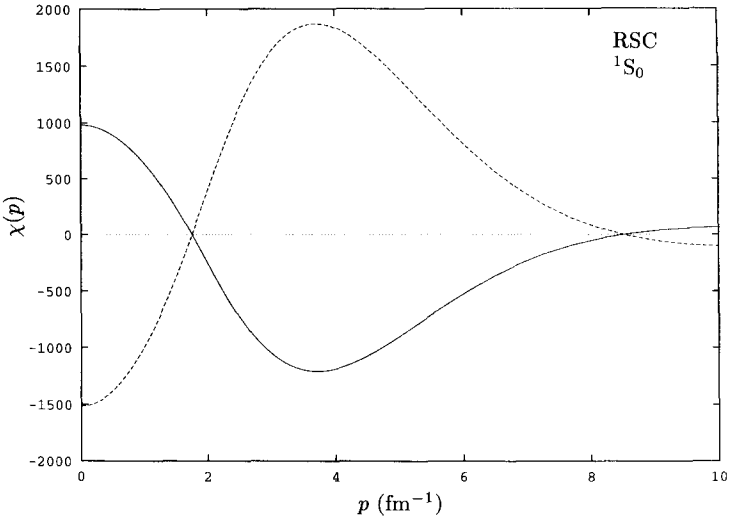

Fig. 8. Dimensionless gap function $\chi(p)$ versus momentum p, calculated at $p_F = 1.7490$ fm^{-1} (solid curve) and at $p_F = 1.75$ fm^{-1} (dashed curve), respectively just below and just above the critical Fermi momentum $p_F = p_c = 1.7496\ldots$ beyond which the gap is found to vanish. The function $\chi(p)$ is normalized to unity at $p = p_F$. The RSC interaction is assumed, together with $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

critical point is passed.

Numerical results for the RSC singlet-S pairing gap Δ_F closely similar to those of Fig. 1 have been recently been obtained by Pieper and Wiringa [20], based on the original form (1) of the gap equation (with the baseline choices $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$). They applied an iteration procedure in which the $\Delta(p')$ input for the nth iteration is given by a linear admixture of the outputs of the $(n-1)$th and $(n-2)$th iterations, the output from the $(n-1)$th step entering only in small proportion. For convergence of this approach, it is necessary to choose a reasonable initializing ansatz for $\Delta(p)$.

In Figs. 9–11 we present selected results from application of the method of Section 2 to the Argonne v_{14} and v_{18} interactions. Some calculated values of the gap at the Fermi surface are plotted in Fig. 9. The computational scheme should be modified near the Fermi momentum where V_F passes through zero (see Appendix A). The numerical results are consistent with the analytical findings of Section 3. We note that the function $\Delta(p)$, plotted in Fig. 10 for the v_{18} case at $p_F = 0.8$ fm^{-1}, again displays a first zero around $p_0 = 1.7 - 1.8$ fm^{-1}. The shift of sign of the function $\chi(p)$ as p_F passes through $p_0(p_F)$ is illustrated for the v_{18} potential in Fig. 11. Further studies within our approach have been carried out for the soft-core potential "V" constructed by Malfliet and Tjon [28] (MT). The functions $\phi(p)$ and $\chi(p)$ corresponding to this example are shown in Figs. 12 and 13, for a density near the peak of the gap.

The most important message to be drawn from these numerical exercises is that the structure of the solution for $\Delta(p)$ below the critical point is nearly independent of density, as demonstrated by the plots of $\chi(p)$ for the Reid case (Fig. 4). This structure

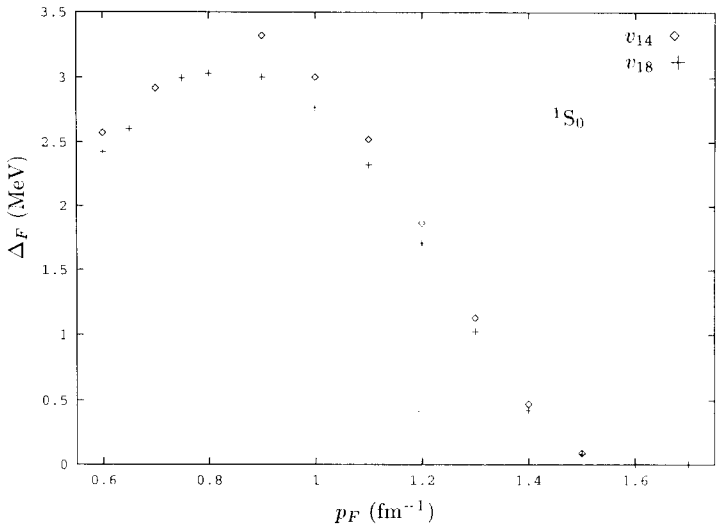

Fig. 9. Neutron-matter S-wave energy gap $\Delta_F = \Delta(p_F)$ at the Fermi surface, plotted against Fermi momentum p_F, for the Argonne v_{14} and v_{18} interactions as pairing force. Points indicated by a plus sign [by a diamond] refer to the v_{18} [respectively, v_{14}] interaction, with the standard choices $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$ of single-particle spectrum and chemical potential.

remains approximately the same if the RSC interaction is replaced by the Argonne v_{14} or v_{18} potential or the Malfliet–Tjon potential "V". The reason for the robustness of the structure of $\chi(p)$ and $\Delta(p)$ in dilute neutron matter will be revealed in the next section.

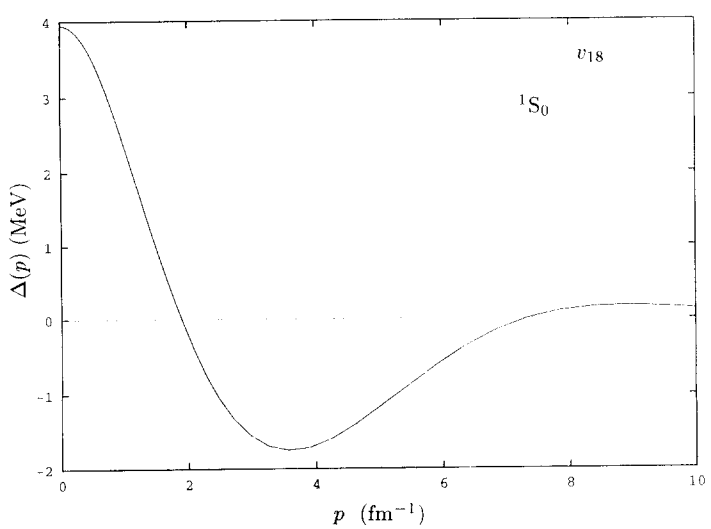

Fig. 10. S-wave gap function $\Delta(p)$ in neutron matter at $p_F = 0.8$ fm^{-1} as a function of momentum p, based on the Argonne v_{18} interaction as pairing force, $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

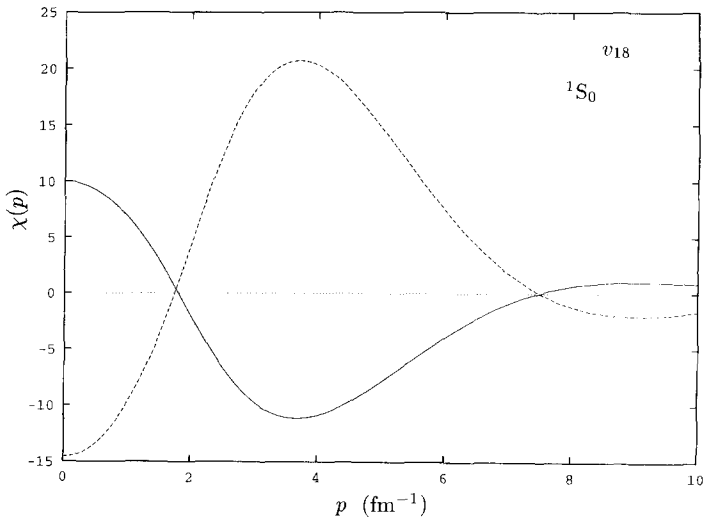

Fig. 11. Dimensionless gap function $\chi(p)$ versus momentum p, calculated at $p_F = 1.7$ fm^{-1} (solid curve) and at $p_F = 1.8$ fm^{-1} (dashed curve), respectively somewhat below and somewhat above the critical Fermi momentum $p_F = p_c$ beyond which the gap vanishes. The function $\chi(p)$ is normalized to unity at $p = p_F$. The Argonne v_{18} interaction is assumed, together with $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

5. Analysis of gap solutions

In this section we examine and explain some extraordinary features of the numerical results for $^1S_0$ neutron pairing. The analysis will lead to new insights into the pairing phenomenon that reflect fundamental properties of real interactions.

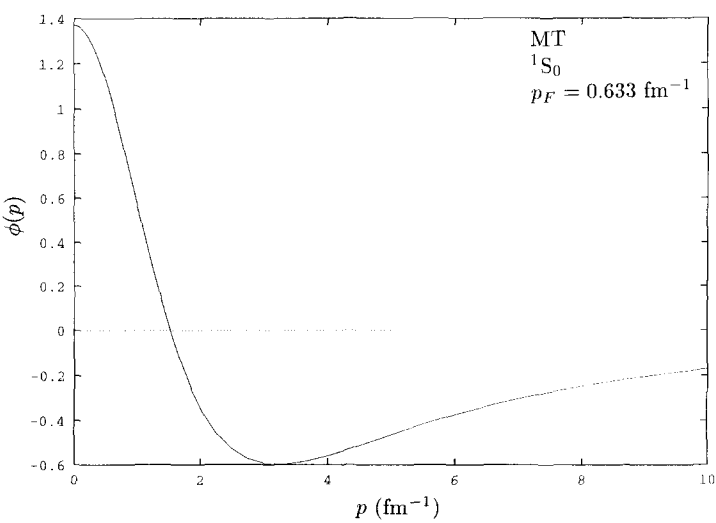

Fig. 12. Potential factor $\phi = V(p, p_F)/V_F$ for the Malfliet-Tjon (MT) potential "V" (Ref. [28]), plotted against momentum p at a Fermi momentum $p_F = 0.633$ fm^{-1} somewhat below the peak of the gap Δ_F.

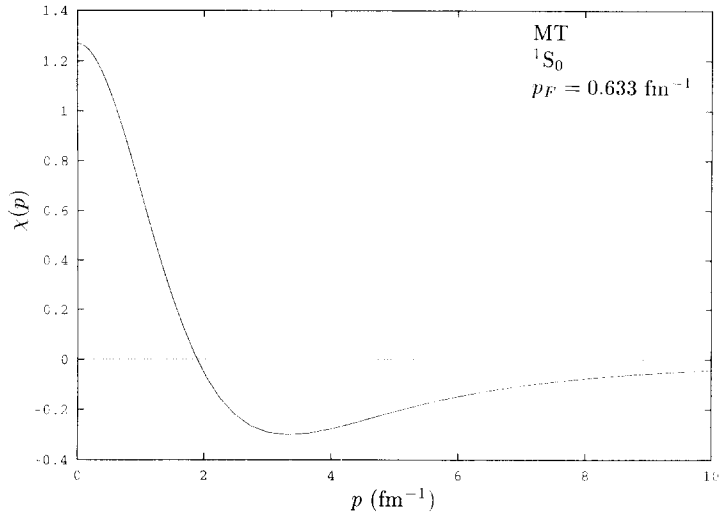

Fig. 13. Dimensionless gap function $\chi(p)$ for the Malfliet–Tjon (MT) potential "V", plotted against momentum p at a Fermi momentum $p_F = 0.633$ fm^{-1} somewhat below the peak of the gap Δ_F. The single-particle spectrum and chemical potential are taken as $\varepsilon_p = p^2/2M$ and $\mu = \varepsilon_F$.

It will be helpful first to rewrite the original equation (1) in terms of the vacuum S-wave scattering amplitude $\Gamma^{(0)}$, which is determined by the usual integral equation

$$\Gamma^{(0)}(p,p') = V(p,p') - \int V(p,p'')\frac{1}{2\varepsilon(p'')}\Gamma^{(0)}(p'',p')d\tau'', \tag{18}$$

where $\varepsilon(p) = p^2/2M$ as before. The objective of this renormalization is to obtain a closed equation for $\Delta(p)$ in which the momentum integration is concentrated near the Fermi surface. Multiplying the integral equation for $\Gamma^{(0)}(p,p_1)$ by $\Delta(p_1)/2E(p_1)$ and integrating over the momentum variable p_1 while performing a simple transformation using Eq. (1), we arrive at [29]

$$\Delta(p) = -M\int \Gamma^{(0)}(p,p_1)\left\{\frac{p_1^2/p_F^2}{[(p_1^2/p_F^2-1)^2 + d^2(p_1)]^{1/2}} - 1\right\}\Delta(p_1)\frac{dp_1}{4\pi^2}, \tag{19}$$

with $d(p) = \Delta(p)/\varepsilon_F$ and $\varepsilon_F = p_F^2/2M$. At large momenta, $p_1 \gg p_F$, the factor in curly brackets falls as $1/p_1^4$, ensuring rapid convergence of the integral in (19).

Consider now an expansion of the vacuum S-wave nn scattering amplitude $\Gamma^{(0)}(p,p')$ in terms of unit-norm eigenfunctions $\psi_n(p)$ of the Schrödinger kernel, satisfying

$$\psi_n(p) = -\nu_n\int \frac{1}{2\varepsilon(p_1)}V(p,p_1)\psi_n(p_1)d\tau_1 = -\nu_n M\int V(p,p_1)\psi_n(p_1)\frac{dp_1}{2\pi^2}. \tag{20}$$

The eigenfunction $\psi_n(p)$ is the Fourier transform of the $(n+1)$th Schrödinger S-state wave function $\psi_n(r)$ with zero energy. The eigenvalue ν_n appearing in Eq. (20) is the enhancement factor that must be applied to the potential V to make it support

a marginally bound, discrete S-state orthogonal to states of the same character with smaller n and fewer zeros. We know that the lowest eigenvalue, ν_0, lies very close to 1, since only a small change of the $^1S_0$ nn potential is enough to move the dineutron pole down to zero energy and produce a bound state. Such behavior is indicative of a nearly separable form for the scattering amplitude $\Gamma^{(0)}(p_1, p_2)$. To demonstrate this feature, we invoke the Hilbert–Schmidt expansion for the kernel of (20),

$$\frac{1}{2\pi^2} MV(p_1, p_2) = -\sum_n \frac{1}{\nu_n} \psi_n(p_1)\psi_n(p_2)\,, \tag{21}$$

which may be verified in a straightforward manner. After substituting this representation into Eq. (18), simple algebra yields

$$\begin{aligned} \frac{1}{2\pi^2} M\Gamma^{(0)}(p_1, p_2) &= \sum_n \frac{1}{\nu_n - 1} \psi_n(p_1)\psi_n(p_2) \\ &= \frac{1}{\nu_0 - 1} \psi_0(p_1)\psi_0(p_2) + \text{background}\,. \end{aligned} \tag{22}$$

Insertion of the resonant term of (22) into Eq. (19) leads to the key result $\chi(p) \propto \psi_0(p)$. This result holds to an excellent approximation for Fermi momenta p_F sufficiently less than the momentum p_0 where the eigenfunction $\psi_0(p)$ first changes sign, i.e. sufficiently below the critical momentum p_c. The nearly separable form of the nn scattering amplitude $\Gamma^{(0)}$ therefore provides a natural explanation of the striking degree of universality of the structure of the gap function in momentum space, observed in the all of the published calculations on singlet-S nucleon–nucleon pairing based on realistic forces. In particular, for the case of the RSC interaction the first zero of $\Delta(p)$ is closely tethered to a momentum value in the range $p_0 \simeq 1.75 - 1.90$ fm^{-1}, being pulled down very slowly toward $p_c = 1.7496\ldots$ fm^{-1} as the density increases, with a noticeable speedup as p_F nears p_c. The quantitative details of the function $p_0(p_F)$ and thus the critical Fermi momentum p_c are presumably determined by the structure of the repulsive core of the two-body interaction. In this connection, we observe that $\chi(p)$ shows no finite-p zero for a rectangular potential well, a Malfliet–Tjon potential without inner repulsion [28] or a one-term Yamaguchi separable potential [30].

As a dividend, the formula (22) provides a link between the near-resonance scattering length $a_{nn} = M\Gamma^{(0)}(0,0)/4\pi$ and the enhancement factor $(\nu_0 - 1)^{-1}$, namely

$$a_{nn} = -\frac{\pi}{2} \frac{\psi_0^2(0)}{\nu_0 - 1}\,. \tag{23}$$

We may now proceed to investigate the nature of solutions for the gap Δ_F in three density regimes. These are the limiting domain of low density, the intermediate or moderate-density region wherein the gap reaches a maximum, and the "high-density" regime where the gap closes.

Low density

The low-density region is defined by the condition that the mean particle spacing is large compared to the scattering length, or equivalently by $p_F|a_{nn}| \ll 1$. In this regime the gap $\Delta(p)$ varies slowly with p in the vicinity of the Fermi surface. Appealing to the rapid decay of the integrand of (19) as p_1 moves away from p_F, we may neglect the momentum dependence of both $\Delta(p_1)$ and $\Gamma^{(0)}(p, p_1)$ and work with the simplified equation

$$1 = -a_{nn}p_F \int \left\{ \frac{z^2}{[(z^2-1)^2 + d_F^2]^{1/2}} - 1 \right\} \frac{dz}{\pi}, \tag{24}$$

where $d_F = d(p_F)$. Since the gap becomes exponentially small in the low-density regime, a quantitative result can be achieved by evaluating only the leading logarithmic term in (24), which can be done analytically. Routine manipulations yield the asymptotic result

$$\Delta_F \sim 8\varepsilon_F \exp\left[-1/\lambda - 2\right] \qquad (p_F \to 0), \tag{25}$$

where we have introduced the effective coupling constant $\lambda = -2p_F a_{nn}/\pi$. This simple analytical formula, which gives reasonable results even if the coupling λ tends to infinity (and a_{nn} to $-\infty$), becomes exact in the low-density limit, $\lambda \sim 0$. In the context of the $^1S_0$ neutron superfluid, this formula is obviously preferable to the BCS weak-coupling estimate of the gap based on $\omega_c = \varepsilon_F$ and $g = -V(p_F, p_F)$. For the RSC interaction, which has $a_{nn} = -17.1$ fm, a comparison with results from numerical solution of (4) and (6) shows that the approximation (25) retains good accuracy up to $p_F \sim 0.05$ fm^{-1}, deviations between the two evaluations amounting to at most a few percent. The accuracy of the low-density analytical treatment can be improved by including the corrections linear in d, at the expense of a more cumbersome expression. Also, simple numerical calculations based on Eq. (24) can improve upon results from the analytical formula (25). For example, in the limiting case $a_{nn} = -\infty$ (or $\lambda = \infty$), a numerical calculation gives $\Delta_F \simeq 1.16\,\varepsilon_F$ instead of $\Delta_F \simeq 1.08\,\varepsilon_F$ from (25).

It is noteworthy that formula (25) is applicable even if the scattering length a_{nn} changes its sign from negative to positive, corresponding to the formation of a real two-particle bound state. The gap value smoothly increases with declining positive a_{nn}. The deeper the vacuum bound state lies, the more closely the pair wave function resembles the wave function of this state. A detailed analysis of the strong-coupling limit may be found in Ref. [31], along with a discussion of the transition between weak- and strong-coupling regimes of pairing (see also Ref. [32]).

Moderate density

Consider next the case of moderate densities, where the product $p_F|a_{nn}|$ exceeds unity while p_F remains safely below p_c. In this regime the scattering amplitude $\Gamma^{(0)}$ is still adequately described by its resonant portion, but we can no longer ignore the structure of $\psi_0(p)$. Eq. (19) leads to the approximate result

$$\Delta_F \simeq \eta_m \varepsilon_F \exp\left[-\frac{\pi\psi_0^2(0)}{2a_{nn}p_F\psi_0^2(p_F)}\right], \tag{26}$$

where η_m is a suitable prefactor of order unity. For the RCS potential, the ratio $\psi_0(0)/\psi_0(p) \simeq \chi(0)/\chi(p)$ may be extracted from Fig. 4. It is seen that the gap $\Delta_F(\rho)$ first grows with density due to the factor $1/p_F$ in the (negative) exponent and attains a maximum in the region $p_F = 0.8$–0.9 fm^{-1} where the product $p_F\psi_0^2(p_F)$ has its maximum. Thereafter the value of Δ_F drops off, as p_F continues to increase toward the point $p_0(p_F)$ at which $\chi(p)$ crosses the p axis. In the vicinity of the critical point, the separable, resonant form of $\Gamma^{(0)}$ ceases to apply and a different approach is required to establish the behavior of Δ_F (see Section 3).

"High" density

In the interest of completeness, we may reiterate here our earlier result (Section 3) for the behavior of Δ_F as a function of p_F in the higher-density region where the gap closes. Based on the reformulation of the gap problem embodied in Eqs. (4) and (7), we have shown that Δ_F vanishes exponentially as the Fermi momentum p_F approaches the critical point $p_c = p_0(p_F = p_c)$ from below,

$$\Delta_F \sim \eta_c \varepsilon_F \exp\left[-\frac{Ip_F}{p_c - p_F}\right] \qquad (p_F - p_c \to 0^-)\,. \tag{27}$$

The constant I has been determined through Eq. (15), but we have left a prefactor η_c of order unity undetermined. Our arguments show that the S-wave energy gap $\Delta_F(p_F)$ becomes exponentially small at both edges of the p_F range $(0, p_c)$ in which it is non-zero, a non-trivial solution of the gap equation normally being precluded for larger p_F.

6. Concluding remarks

In summary, considering the case of $^1S_0$ neutron pairing in pure neutron matter, we have divided the problem of solving the gap equation for $\Delta(p) = \Delta_F\chi(p)$ into two more elementary tasks: (i) solution of the non-singular quasilinear integral equation (4) for the dimensionless gap function $\chi(p)$ and (ii) solution of the non-linear algebraic equation (6) for the scaling factor $\Delta_F = \Delta(p_F)$. This reformulation has been achieved by splitting the pairing interaction $V(p,p')$, identically, into a separable part $V_F\phi(p)\phi(p')$ and a remainder $W(p,p')$ that vanishes when either p or p' lies on the Fermi surface. Joint solution of Eqs. (4) and (6) may be accomplished by a rapidly convergent iteration procedure that produces highly accurate results even for realistic neutron–neutron pairing interactions containing very strong repulsive cores. The new formulation and its sound computational setting facilitate a study of fundamental aspects of the pairing phenomenon in neutron matter. In particular, we have been able to trace the remarkably universal structure of the S-wave gap function $\Delta(p)$ – most notably the insensitivity of its first zero p_0 to changes of density and of (realistic) potential input –

to the resonant nature of the *nn* interaction at low energies. We have also been able to gain new analytical insights into the behavior of the gap magnitude Δ_F near the critical point $p_F = p_c = p_0(p_F)$ at which the gap closes, establishing an exponential approach to zero as $p_F - p_c \to 0^-$.

In this exploratory investigation, a robust and accurate procedure has been developed for numerical solution of the standard gap equation. The associated reformulation of the gap equation has served to illuminate the mathematical nature of the pairing problem and has led to new analytical results of broad applicability. We have given special consideration to the implications of the large negative scattering length of the *nn* interaction, but we have bypassed a number of other physical issues that are pivotal to a satisfactory understanding of nucleonic pairing in extended matter. The most important of these are the effects of (i) strong short-range geometric correlations in the nucleonic medium arising mainly from the inner repulsion of realistic nucleon–nucleon interactions and (ii) longer-range correlations arising from higher-order medium processes, most especially the virtual exchange of density and spin-density fluctuations. In general, the pairing interaction $V(p,p')$ and the single-particle energies $\varepsilon(p)$ will experience substantial in-medium modifications in the density range in question. Although available calculations indicate some degree of cancellation between the modifications due to short-range and long-range correlations [4,16,17], a convincing quantitative evaluation of vertex and mass-operator corrections is sorely needed. Another physical issue that needs to be addressed, and has begun to receive consideration [33], is the proper treatment of the neutronic superfluid in the highly inhomogeneous environment of the inner crust of a neutron star. The approach introduced herein should prove useful in dealing with gap equations that incorporate these diverse physical elaborations, as well as with the complex problem of triplet pairing in neutron-star interiors. Adaptation of the approach to the latter problem has been initiated in Appendix B.

Clearly, the procedure we have formulated (along with much of the analysis) is also applicable or adaptable to gap equations for nucleonic pairing in infinite nuclear matter with arbitrary neutron–proton asymmetry $\alpha = (N-Z)/A$. The case of symmetry ($\alpha = 0$) or of near-symmetry has been thought to be relevant to pairing in finite nuclei. However, it is well known (see e.g. Refs. [4,12]) that the primary effect of pairing in infinite nuclear matter occurs below the spinodal point of the uniform system. The relation of solutions of the gap equation for uniform nuclear matter to pairing in finite nuclei must therefore be regarded as problematic. In this connection we may call attention to a recent study [34] of pairing in semi-infinite nuclear matter, which has cast serious doubt on the validity of descriptions in which a local-density approximation is used to transfer uniform-matter gap results to finite nuclei. Simple local-density approaches must naturally be viewed with suspicion, since the superfluid coherence length $l = 2\varepsilon_F/\pi p_F \Delta_F$ for nucleonic pairing can easily exceed nuclear dimensions.

In closing we should point out that the purview of the methodological advances reported here extends beyond the domain of nuclear physics. In particular, similar techniques may be exploited in the study of strongly interacting electronic systems that exhibit high-temperature superconductivity. It is straightforward to extend our methods

to finite temperature, and opportunities are open for consideration of the highly structured single-particle spectra appropriate to complex materials. Of special interest is the prospect of deriving improved estimates of the key parameter $\Delta(0)/T_c$ for different types of pairing.

Acknowledgements

This research was supported in part by the US National Science Foundation under Grant No. PHY-9307484, by the Russian Foundation for Fundamental Investigations under Grant No. 94-02-04-665a and by the International Science Foundation under Grant No. Mh 8300. The authors are grateful to T. Alm, M. Baldo, R.U. Hafizov, S.C. Pieper, P. Ring, P. Schuck, M. Takano, E. Vigezzi and R.B. Wiringa for valuable information and helpful discussions. J.W.C. thanks the Physics Division of Argonne National Laboratory for hospitality during an important phase of the work.

Appendix A. The case of vanishing V_F

Here we consider the special case $V_F \equiv V(p_F, p_F) = 0$ that was excluded in the treatment of the gap problem developed in Section 2. In this case, there is no danger of a logarithmic divergence of the integral member of the gap equation (1) when it is used for the determination of $\Delta_F = \Delta(p = p_F)$. A straightforward iterative solution of the gap equation would therefore appear more promising than in the ordinary situation. However, an incipient logarithmic divergence of the p' integral persists for any $p \neq p_F$. Accordingly, it is of interest to develop a procedure for dealing with the case $V_F = 0$ that shares the computational advantages of the method of Section 2. To formulate such a procedure, we assert a different decomposition of the pairing interaction.

The potential V is split into a portion that is additively rather than multiplicatively separable in the variables p and p', plus a remainder $W(p,p')$ that again vanishes identically when either momentum argument lies on the Fermi surface:

$$V(p,p') = V(p,p_F) + V(p_F,p') + W(p,p')\,. \tag{A.1}$$

The requirement $W(p,p_F) = W(p_F,p') = 0$ is obviously ensured by $V(p_F,p_F) = 0$. Inserting the ansatz (A.1) into Eq. (1) and again writing $\Delta(p) = \Delta_F\chi(p)$, the shape function $\chi(p)$ and the gap magnitude Δ_F are found to obey the equations

$$\chi(p) = 1 - V(p,p_F)\int \frac{\chi(p')}{2E(p')}d\tau' - \int W(p,p')\frac{\chi(p')}{2E(p')}d\tau' \tag{A.2}$$

and

$$1 = -\int V(p_F,p)\frac{\chi(p)}{2E(p)}d\tau\,. \tag{A.3}$$

The function $\chi(p)$ falls off as $p \to \infty$ in the same way as the gap $\Delta(p)$, so the integrals in Eq. (A.2) converge at large p'. By virtue of $V(p_F, p_F) = 0$, we can simplify Eq. (A.3) by replacing $\Delta(p)$ in the denominator $E(p)$ with the generic input δ (cf. Section 2),

$$1 = -\int V(p_F, p) \frac{\chi(p)}{2\left[\xi^2(p) + \delta^2\right]^{1/2}} d\tau . \tag{A.4}$$

However, in contrast to the case $V_F \neq 0$, the first integral in Eq. (A.2) for $\chi(p)$ is logarithmically divergent if we set $\Delta(p') = 0$ in the $E(p')$ denominator, which prevents us from making the same substitution in this equation.

Thus a different strategy is needed for the solution of the equation for the shape function. Defining the constant

$$K = -\int \frac{\chi(p)}{2E(p)} d\tau , \tag{A.5}$$

we can rewrite Eq. (A.2) as

$$\chi(p) = 1 + KV(p, p_F) - \int W(p, p') \frac{\chi(p')}{2\left[\xi^2(p') + \delta^2\right]^{1/2}} d\tau' , \tag{A.6}$$

after appealing to the property $W(p, p_F) = 0$ and replacing the exact $\Delta(p')$ in the second integral of (A.2) by the generic input δ. Eq. (A.6) is effectively a standard linear integral equation whose solution can be expressed as

$$\chi(p) = \chi_1(p) + K\chi_2(p) , \tag{A.7}$$

where the functions $\chi_1(p)$ and $\chi_2(p)$ are determined respectively by the equations

$$\chi_1(p) = 1 - \int W(p, p') \frac{\chi_1(p')}{2\left[\xi^2(p') + \delta^2\right]^{1/2}} d\tau' \tag{A.8}$$

and

$$\chi_2(p) = V(p, p_F) - \int W(p, p') \frac{\chi_2(p')}{2\left[\xi^2(p') + \delta^2\right]^{1/2}} d\tau' , \tag{A.9}$$

and evidently obey the conditions $\chi_1(p_F) = 1$ and $\chi_2(p_F) = 0$. In turn, Eqs. (A.8) and (A.9) can be solved without difficulty. Inserting the decomposition (A.7) into Eq. (A.3), we have $1 = N_1 + KN_2$, so the quantity K of Eq. (A.5) may be evaluated from the solutions of Eqs. (A.8) and (A.9) through $K = (1 - N_1)/N_2$, where

$$N_i = -\int V(p_F, p) \frac{\chi_i(p)}{2\left[\xi^2(p) + \delta^2\right]^{1/2}} d\tau \qquad (i = 1, 2) . \tag{A.10}$$

Likewise, upon inserting (A.7) into Eq. (A.5) we have $K = L_1 + KL_2$, where

$$L_i = -\int \frac{\chi_i(p)}{2\left[\xi^2(p) + \Delta^2(p)\right]^{1/2}} d\tau \qquad (i = 1, 2) . \tag{A.11}$$

Since $\chi_2(p_F) = 0$, the dependence of the integral L_2 on the gap $\Delta(p)$ is very mild. Numerical computation of the gap magnitude Δ_F may therefore proceed efficiently based on the relation $(1 - N_1)/N_2 = L_1 + [(1 - N_1)/N_2]L_2$, recast in the form

$$-\int \frac{\chi_1(p)}{2\left[\xi^2(p) + \Delta^2(p)\right]^{1/2}} d\tau = \frac{(1 - N_1)(1 - L_2)}{N_2}. \tag{A.12}$$

Appendix B. Triplet pairing

In this appendix we take up the extension of the method of Section 2 to pairing in higher angular momentum states. Specifically, we shall focus on $^3P_2$ pairing, which is the most important example for neutron-star matter. In the interests of a simple presentation, it is assumed that the interaction V is central. For this problem, the gap matrix has 5 complex components $\Delta_{1,m}(p)$ corresponding to the allowed values ± 2, ± 1, 0 of the magnetic quantum number m. Time-reversal invariance implies that these components are related by $\Delta_{1,m}(p) = (-1)^m \Delta^*_{1,-m}$ and hence are determined by 5 independent real quantities. The simplest solutions among the 5 possible types discussed in Refs. [23,25] have azimuthal symmetry and involve only a single scalar gap equation. These solutions correspond to so-called maximum-$|m|$ coupling, for which only the gap components with $m = \pm 2$ differ from zero, and of course to the case where only an $m = 0$ component enters. The relevant gap equations are [23,25]

$$\Delta_2(p) = -\frac{2}{\pi} \int V_{11}(p, p') \frac{\Delta_2(p')(3 \sin^2 \theta'/8\pi)}{2\left[\xi^2(p') + \Delta_2^2(p')(3 \sin^2 \theta')/8\pi\right]^{1/2}} \times (p')^2 dp' \sin \theta' d\theta' d\varphi' \tag{B.1}$$

for the maximum-$|m|$ case and

$$\Delta_0(p) = -\frac{2}{\pi} \int V_{11}(p, p') \frac{\Delta_0(p')(1 + 3 \cos^2 \theta')/8\pi}{2\left[\xi^2(p') + \Delta_0^2(p')(1 + 3 \cos^2 \theta')/16\pi\right]^{1/2}} \times (p')^2 dp' \sin \theta' d\theta' d\varphi' \tag{B.2}$$

for pure $m = 0$, where θ' and φ' are the polar and azimuthal angles and $V_{11}(p, p') = \langle p' | V(^3P_2) | p \rangle$ is the $L = 1$ spherical harmonic of the interaction potential V, as defined by Takatsuka and Tamagaki [23,25].

We call attention to the fact that it is possible to perform the angular integrations in Eqs. (B.1) and (B.2) analytically. Thereupon Eq. (B.1) becomes

$$\Delta_2(p) = -\frac{2}{\pi} \int V_{11}(p, p') \Phi_2\left(\beta_2(p')\right) \frac{\Delta_2(p')}{2\left[\xi^2(p') + 3\Delta_2^2(p')/8\pi\right]^{1/2}} (p')^2 dp', \tag{B.3}$$

where

$$\Phi_2(\beta) = \frac{3}{2\beta}\sin^{-1}\beta + \frac{3}{4\beta^2}\left[\sqrt{1-\beta^2} - \frac{\sin^{-1}\beta}{\beta}\right] \tag{B.4}$$

with

$$\beta_2^2(p) = \frac{3\Delta_2^2(p)/8\pi}{\xi^2(p) + 3\Delta_2^2(p)/8\pi}. \tag{B.5}$$

Similarly, the angular integration in Eq. (B.2) yields

$$\Delta_0(p) = -\frac{2}{\pi}\int V_{11}(p,p')\Phi_0\big(\beta_0(p')\big)\frac{\Delta_0(p')}{2\left[\xi^2(p') + \Delta_0^2(p')/16\pi\right]^{1/2}}(p')^2dp' \tag{B.6}$$

with

$$\Phi_0(\beta) = \frac{1}{2\beta}\log\left(\beta + \sqrt{1+\beta^2}\right) + \frac{3}{4\beta^2}\left[\sqrt{1+\beta^2} - \frac{\log\left(\beta + \sqrt{1+\beta^2}\right)}{\beta}\right] \tag{B.7}$$

and

$$\beta_0^2(p) = 3\frac{\Delta_0^2(p)/16\pi}{\xi^2(p) + \Delta_0^2(p)/16\pi}. \tag{B.8}$$

The procedure outlined in Section 2 may be carried over to these two cases without significant modification. The pairing interaction $V_{11}(p,p')$ is again represented in the form (2), i.e. as $V_{11}(p,p') = V_F\phi(p)\phi(p') + W_{11}(p,p')$ with $\phi(p) = V_{11}(p,p_F)/V_F$. Inserting this decomposition into the triplet gap equation (B.1) or (B.2), we are again led to a quasilinear integral equation (cf. Eq. (4′))

$$\chi_s(p) + \frac{2}{\pi}\int W_{11}(p,p')\frac{\Phi_s\big(\beta_s(p')\big)\chi_s(p')}{2\left[\xi^2(p') + \delta^2\right]^{1/2}}(p')^2dp' = \phi(p) \tag{B.9}$$

for a dimensionless shape function $\chi_s(p)$, labeled now by $s = 2$ or 0 for the respective $|m| = 2$ and $m = 0$ solutions, and defined via $\Delta_s(p) = \Delta_{Fs}\chi_s(p)$ with $\Delta_{Fs} = \Delta_s(p_F)$. As before, the vanishing of the W function on the Fermi surface implies that the integral over p' in the equation for the shape function receives significant contributions only from momenta removed from p_F. Again the dependence of the square-root denominator on the gap may be replaced by a generic input δ. Here it is important to note further that for momenta departing from p_F the quantities β_s become quite small, and that in turn $\Phi_s(\beta_s \to 0) \to 1$. Thus, in practice the function $\Phi_s(\beta_s)$ has no influence on $\chi_s(p)$, and we may work with the simplified (and now linear) integral equation

$$\chi_s(p) + \frac{2}{\pi}\int W_{11}(p,p')\frac{\chi_s(p')}{\left[\xi^2(p') + \delta^2\right]^{1/2}}(p')^2dp' = \phi(p). \tag{B.10}$$

The solution of this equation is independent of the index s; hence, in effect, the two gap functions $\Delta_0(p)$ and $\Delta_2(p)$ differ only in scale. The corresponding scale factors

Δ_{F2} and Δ_{F0} are determined (in the sense of Section 2) by the non-linear algebraic equations (cf. Eq. (7′))

$$\frac{2}{\pi}\int \frac{\phi^2(p)\Phi_s\big(\beta(p)\big)}{2\big[\xi^2(p)+c_s\Delta_{Fs}^2\chi_s^2(p)\big]^{1/2}}p^2dp$$
$$=-\frac{1}{V_F}+\frac{2}{\pi}\int\frac{\phi(p)\big(\chi_s(p)-\phi(p)\big)}{2\big[\xi^2(p)+\delta^2\big]^{1/2}}p^2dp\,, \qquad \text{(B.11)}$$

where $s=0$ or 2 and the numerical factor c_s is respectively $1/16\pi$ or $3/8\pi$.

A more detailed treatment of the $^3P_2$–$^3F_2$ pairing problem for realistic non-central forces and the three other solution types (involving respectively $m=\pm1$ components, $m=\pm2,0$ components and *all* magnetic components) is reserved for a future paper devoted to a numerical evaluation of the relevant gap functions.

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