$$\dot{\zeta} = w, \qquad Ce^{-\zeta} = x, \quad \dot{v} = {}^1/_3(2w + x),$$
$$\ddot{v} = -{}^1/_3 w(2x dw/dx + x),$$

reducing (24) to the form

$$-\frac{8}{3}wx\frac{dw}{dx} + \frac{4}{3}w^2 + \frac{x^2}{3} = -b_1\Big[-\frac{8}{3}wx\frac{dw}{dx} + \frac{4}{3}(1+\delta)w^2 + \frac{1}{3}(1-\delta)x^2\Big]^{1+\delta}.$$

Further lowering of the order is attained by means of the substitution

$$w^2 = x^2(1 + Av/x^2), \quad A = {}^3/_4(1+\delta)^{1+1/\delta}b_1^{1/\delta}.$$

If we again go over from x to ζ, we obtain in final form

$$\frac{dv}{d\zeta} + v = -\Big[\frac{1}{1+\delta}\frac{dv}{d\zeta} + v\Big]^{1+\delta}. \tag{27}$$

The integral of this equation is

$$\zeta = -\ln C_1\Big(v + \frac{k}{1+\delta}\Big), \qquad k = \frac{dv}{d\zeta}. \tag{28}$$

It is convenient to express the solution in parametric form, starting from the definition k = −sv. The limits of variation of s will be $1 + \delta \geq s \geq 1$. Then

$$v = (s-1)^{1/\delta}\Big(1 - \frac{s}{1+\delta}\Big)^{-1/\delta},$$
$$e^{\zeta} = C_1^{-1}\Big(v + \frac{k}{1+\delta}\Big)^{-1} = \Big(1 - \frac{s}{1+\delta}\Big)^{1/\delta - 1}(s-1)^{-1/\delta}. \tag{29}$$

The expression for the particle density is obtained in terms of the same parameter s, since

$$n \sim v + k/(1+\delta). \tag{30}$$

Thus, the product $ne^{\zeta} = n\sqrt{-g}$ remains constant, which is merely an expression for the conservation of number of particles. This requirement is contained in the very equation relating p and ϵ. To the contrary, the ultrarelativistic case does not incorporate this requirement, so that the product $\epsilon\sqrt{-g}$ does not remain constant and ϵ becomes infinite at a faster rate than the vanishing of $\sqrt{-g}$.

It is convenient to choose as the time origin the point $s = 1 + \delta$, where the particle density is infinite. Near this point we have

$$t = \Big(1 - \frac{s}{1+\delta}\Big)^{1/2\delta}. \tag{31}$$

For arbitrary values of δ we cannot, of course, go over from t to −t (although, for example, this can be done when $\delta = {}^2/_3$). But when the particle density tends to infinity, the equation of state goes over into $\epsilon = 3p$, which again permits the passage through zero time.

In conclusion we are sincerely grateful to Ya. B. Zel'dovich, E. M. Lifshitz, I. D. Novikov, and I. M. Khalatnikov for very valuable discussions.

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Translated by J. G. Adashko
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SOVIET PHYSICS JETP VOLUME 20, NUMBER 5 MAY, 1965

IONIZATION IN THE FIELD OF A STRONG ELECTROMAGNETIC WAVE

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Submitted to JETP editor May 23, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1945-1957 (November, 1964)

Expressions are obtained for the probability of ionization of atoms and solid bodies in the field of a strong electromagnetic wave whose frequency is lower than the ionization potential. In the limiting case of low frequencies these expressions change into the well known formulas for the probability of tunnel auto-ionization; at high frequencies they describe processes in which several photons are absorbed simultaneously. The ionization probability has a number of resonance maxima due to intermediate transition of the atom to an excited state. In the vicinity of such a maximum the ionization cross section increases by several orders of magnitude. The positions and widths of the resonances depend on the field strength in the wave. It is shown that for optical frequencies the mechanism under consideration, of direct ionization by the wave field, may be significant in the case of electric breakdown in gases, and especially in condensed media.

AN essential feature of the tunnel effect, a feature of importance in practical applications, is the practical absence of time lag. In other words, the probability of tunneling remains constant up to the highest frequencies of the radio band. The reason for this is that the tunneling time is determined essentially by the mean free time of the electron passing through a barrier of width

$$l = I/eF,$$

where I—ionization potential and F—electric field intensity. The average electron velocity is of the order of $(I/m)^{1/2}$ (m—electron mass). Therefore, up to frequencies on the order of

$$\omega_t = eF/\sqrt{2mI}$$

the tunnel effect is determined simply by the instantaneous value of the field intensity.

At higher frequencies there should appear a frequency dependence of the tunneling probability, since the electron does not have time to jump through the barrier within one cycle. Values typical of the tunnel effect in semiconductors are I ~ 1 eV, m ~ 10^{-28} g, F ~ 10^5 V/cm, yielding $\omega_t \sim 10^{13}$ sec^{-1}, i.e., the dispersion can be noticeable at infrared and optical frequencies. A similar estimate is obtained for atoms, where I ~ 10 eV, m = 10^{-27} g, and F ~ 10^7 V/cm.

With the appearance of lasers, the question of the tunnel effect at such frequencies became timely, since this is apparently the most effective mechanism for the absorption of high-power radiation in the transparency region. By transparency region we understand here the region of frequencies $\hbar\omega < I$, in which the substance is transparent to low-power radiation. Recent papers report gas breakdown in the focus of a laser beam[1,2]. This group of questions was considered theoretically by Bunkin and Prokhorov[3].

At first glance there exists at such high frequencies still another absorption mechanism, which competes with the tunnel effect. We have in mind the multi-photon absorption, in which the transition of the electron into a free state is accompanied by simultaneous absorption of several quanta. We shall show, however, that the nature of these two effects is essentially the same. We shall obtain a common formula which goes over into the usual formula for the tunnel effect[4-6] at low frequencies and very strong fields, when $\omega \ll \omega_t$, and describes multi-photon absorption when $\omega \gg \omega_t$.

In the simplest case of ionization of atoms, the general formula for the ionization probability is

$$w = A\omega\Big(\frac{I_0}{\hbar\omega}\Big)^{3/2}\Big(\frac{\gamma}{(1+\gamma^2)^{1/2}}\Big)^{5/2}S\Big(\gamma, \frac{I_0}{\hbar\omega}\Big) \times \exp\Big\{-\frac{2I_0}{\hbar\omega}\Big[\sinh^{-1}\gamma - \gamma\frac{(1+\gamma^2)^{1/2}}{1+2\gamma^2}\Big]\Big\},$$
$$\gamma = \omega/\omega_t = \omega(2mI_0)^{1/2}/eF, \tag{1}$$

where the effective ionization potential is defined by

$$I_0 = I_0 + e^2F^2/4m\omega^2 = I_0(1 + 1/2\gamma^2); \tag{2}$$

$S(\gamma, \widetilde{I}_0/\hbar\omega)$ is a relatively slowly varying function of the frequency and of the field, defined by formula (18) below, and A is a numerical coefficient of the order of unity.

Formula (1) describes the indirect transition of an electron from the ground state of the atom to the free state. However, in the case when one of the higher harmonics of the incident monochromatic wave is close to resonance with the electronic transition, in which the atom goes over into the s-th excited state, an appreciable role can be assumed by a two-step process consisting of the excitation of the atom followed by ionization of the excited state. The probability of such a resonant process is described by the formula

$$w^{(r)} = \frac{1}{4}\sum_s \left| J_{n_s+1}\left(\frac{eF\sigma_s}{\hbar\omega}\right) + J_{n_s-1}\left(\frac{eF\sigma_s}{\hbar\omega}\right)\right|^2 \times \frac{|V_{0s}|^2 w_s}{(I_0 - I_s + {}^1/_2 e^2F^2\alpha_s - n_s\hbar\omega)^2 + \hbar^2(\omega_s+\gamma_s)^2}, \quad (3)$$

where J_n—Bessel functions; w_S—probability of ionization of the s-th excited state, described by formula (1) in which, however, I_0 must be replaced by I_S—ionization potential of the s-th level; σ_S and α_S—coefficients that describe the Stark shift of the s-th level in a static electric field:

$$I_s(F) = I_s - eF\sigma_s - {}^1/_2 e^2F^2\alpha_s; \quad (4)$$

v_{0S}—matrix element of the transition from the ground state to the s-th state in a homogeneous electric field:

$$V_{0s} = \int \psi_s^*(\mathbf{r})\, e\mathbf{F}\mathbf{r}\psi_0(\mathbf{r})\, d^3r;$$

n_S—integer closest to $(I_0 - I_S)/\hbar\omega$; γ_S—radiation width of the level.

We present now a detailed derivation and analysis of formulas (1) and (3). On the basis of the results we shall then derive briefly analogous expressions for the ionization probability in a solid.

The electric field of the wave exerts the strongest action on the states of the continuous spectrum and on the degenerate levels of the excited bound states. Therefore, as the first step, we take into account these effects. The wave function of the free electron in an electric field

$$\mathbf{F}(t) = \mathbf{F}\cos\omega t \quad (5)$$

is of the form

$$\psi_{\mathbf{p}}(\mathbf{r}, t) = \exp\left\{\frac{i}{\hbar}\left[\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega t\right)\mathbf{r} - \int_0^t \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega\tau\right)^2 d\tau\right]\right\}, \quad (6)$$

and for the s-th bound state

$$\psi_s(\mathbf{r}, t) = \psi_s(\mathbf{r})\exp\left\{\frac{i}{\hbar}\left(I_s t - \frac{eF\sigma_s}{\omega}\sin\omega t\right)\right\}. \quad (7)$$

We have in mind here the hydrogen atom, in which the Stark effect is nonlinear in the field. In the wave function of the final state (6), we have neglected for the time being the influence of the Coulomb field of the ionized atom. The role of this interaction, as well as of the quadratic Stark effect, will be discussed below.

Let us calculate now the probability of the transition from the ground state to a free-electron state of the type (6). The difference between our procedure and the usual perturbation theory lies thus only in the fact that we calculate the probability of transition not to a stationary final state, but to a state (6) that already takes exact account of the main effect of the electric field—the acceleration of the free electron. The matrix elements of the transition between the bound states, on the other hand, are taken into account only in the lower orders of perturbation theory, since they are proportional to eFa_0, and as will be shown below, the transition matrix elements which we take into account in the continuous spectrum are proportional to $eFa_0\sqrt{I_0/\hbar\omega}$. The ratio $\hbar\omega/I_0$ will be assumed to be sufficiently small. For the lasers presently in existence it is of the order of 0.1.

The probability of direct transition from the ground state to the continuous spectrum is of the form

$$w_0 = \frac{1}{\hbar^2}\lim_{T\to\infty}\mathrm{Re}\int\frac{d^3p}{(2\pi\hbar)^3}\int_0^T dt\cos\omega T\cos\omega t \times V_0\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega T\right)V_0\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega t\right) \times \exp\left\{\frac{i}{\hbar}\int_T^t\left[I_0 + \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega\tau\right)^2\right]d\tau\right\}, \quad (8)$$

$$V_0(\mathbf{p}) = \int e^{-i\mathbf{p}\mathbf{r}/\hbar}\, e\mathbf{F}\mathbf{r}e^{-r/a_0}\frac{d^3r}{(\pi a_0^3)^{1/2}} = 8(\pi a_0^3)^{1/2}e\hbar\mathbf{F}\nabla_{\mathbf{p}}(1 + p^2a_0^2/\hbar^2)^{-2}. \quad (9)$$

Let us expand the expression

$$L(\mathbf{p}, t) = V_0\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega t\right) \times \exp\left\{\frac{i}{\hbar}\int_0\left[I_0 + \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}\sin\omega\tau\right)^2\right]d\tau\right\} \quad (10)$$

in a Fourier series in t:

$$L(\mathbf{p}, t) = \sum_{n=-\infty}^{\infty}\exp\left\{\frac{i}{\hbar}\left(I_0 + \frac{p^2}{2m} + \frac{e^2F^2}{4m\omega^2} - n\hbar\omega\right)t\right\}L_n(\mathbf{p}), \quad (11)$$

$$L_n(\mathbf{p}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V_0\left(\mathbf{p} + \frac{e\mathbf{E}}{\omega}\sin x\right) \times \exp\left\{\frac{i}{\hbar\omega}\left(n\hbar\omega x - \frac{e\mathbf{F}\mathbf{p}}{2m\omega}\cos x - \frac{e^2F^2}{8m\omega^2}\sin 2x\right)\right\}dx. \quad (12)$$

Substituting this expansion in (8), we obtain after the usual transformations

$$w_0 = \frac{2\pi}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sum_{n=-\infty}^{\infty}\frac{1}{4}\left|L_{n+1}(\mathbf{p}) + L_{n-1}(\mathbf{p})\right|^2 \times \delta\left(I_0 + \frac{p^2}{2m} + \frac{e^2F^2}{4m\omega^2} - n\hbar\omega\right), \quad (13)$$

or, using the presence of δ-functions

$$w_0 = \frac{2\pi}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}|L(\mathbf{p})|^2 \times \sum_{n=-\infty}^{\infty}\delta\left(I_0 + \frac{p^2}{2m} + \frac{e^2F^2}{4m\omega^2} - n\hbar\omega\right), \quad (14)$$

$$L(\mathbf{p}) = \frac{1}{2\pi}\oint V_0\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}u\right)\exp\left\{\frac{i}{\hbar\omega}\int_0\left[I_0 + \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}v\right)^2\right]\frac{dv}{(1-v^2)^{1/2}}\right\}du. \quad (15)$$

The integral with respect to u in (15) is taken along a closed contour which encloses the segment $(-1, 1)$.

Formula (14) has the explicit form of the sum of multi-photon processes. The exponential in (15) is rapidly oscillating, so that the integral can be calculated by the saddle-point method. The saddle points are determined by the condition

$$I_0 + \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}u_s\right)^2 = 0.$$

It is easy to verify, however, that for the hydrogen atom the matrix element $V_0(p + eFu/\omega)$ has a pole at the same points, by virtue of the relation $2mI_0 = \hbar^2/a_0^2$. The presence of this pole is neither accidental nor typical of the hydrogen atom, but reflects the universally known fact that the scattering amplitude has poles at complex momentum values corresponding to bound states[5]. Taking these singularities into account, we can easily verify that the contribution of each saddle point to (15) is equal to

$$2\sqrt{\pi a_0^3}\frac{I_0}{eFa_0}\frac{\hbar\omega}{(1-u_s^2)^{1/2}} \times \exp\left\{\frac{i}{\hbar\omega}\int_0^{u_s}\left[I_0 + \frac{1}{2m}\left(\mathbf{p} + \frac{e\mathbf{F}}{\omega}v\right)^2\right]\frac{dv}{(1-v^2)^{1/2}}\right\}.$$

The positions of the saddle points depend on p. However, contributions to the total probability of ionization (14) are made only by small p, satisfying the condition $p^2 \ll 2mI_0$. Consequently we can put $\mathbf{p} = 0$ in the pre-exponential factor, and we can expand in the exponential in powers of p up to second order inclusive. Substituting then the resultant expression in (14) and integrating with respect to p, we transform (14) to

$$w_0 = \sqrt{\frac{2I_0}{\hbar}}\omega\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)^{3/2}S\left(\gamma, \frac{I_0}{\hbar\omega}\right) \times \exp\left\{-\frac{2I_0}{\hbar\omega}\left[\sinh^{-1}\gamma - \gamma\frac{\sqrt{1+\gamma^2}}{1+2\gamma^2}\right]\right\}; \quad (16)$$

Here

$$\gamma = \omega\sqrt{2mI_0}/eF, \quad (17)$$

and the function $S(\gamma, \widetilde{I}_0/\hbar\omega)$, which varies slowly compared with an exponential function, is of the form

$$S(\gamma, x) = \sum_{n=0}^{\infty}\exp\left\{-2[\langle x+1\rangle - x + n] \times \left(\sinh^{-1}\gamma - \frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right\} \times \Phi\left\{\left[\frac{2\gamma}{\sqrt{1+\gamma^2}}(\langle x+1\rangle - x + n)\right]^{1/2}\right\}. \quad (18)$$

The symbol $\langle x\rangle$ denotes the integer part of the number x, and the function $\Phi(z)$, defined by

$$\Phi(z) = \int_0 e^{y^2-z^2}dy, \quad (19)$$

is expressed in terms of the well known probability integral.

The function $S(\gamma, \widetilde{I}_0/\hbar\omega)$ describes the spectrum structure connected with the discreteness of the number of absorbed photons. It obviously has characteristic threshold singularities of the type $(\widetilde{I}_0 - n\hbar\omega)^{1/2}$. The effective threshold energy of absorption $\widetilde{I}_0$, defined by formula (2), exceeds the ionization potential I_0 by the value of the average oscillation energy of the electron in the field of the wave. This is precisely the quantity which enters into the δ-function that expresses the law of energy conservation in the general formula (14).

For the case of low frequencies and very strong fields, when $\gamma \ll 1$, the main contribution in expression (18) for S is made by large $n \sim \gamma^{-3}$. Consequently, going over from summation over n to integration, we obtain

$$S(\gamma, I_0/\hbar\omega) \approx \sqrt{3\pi}/4\gamma^2,$$

as a result of which the ionization probability is described by the formula

$$w_0 = \frac{\sqrt{6\pi}}{4}\frac{I_0}{\hbar}\left(\frac{eF\hbar}{m^{1/2}I_0^{3/2}}\right)^{1/2} \times \exp\left\{-\frac{4}{3}\frac{\sqrt{2m}I_0^{3/2}}{e\hbar F}\left(1-\frac{m\omega^2 I_0}{5e^2F^2}\right)\right\}. \quad (20)$$

As $\omega \to 0$, the exponential in (20) coincides with the known expression for the tunnel auto-ionization of atoms in an electric field[4,5], whereas the exponential factor is different from the correct one. This is connected with the fact that we have used for the wave functions of the final state the functions of the free electron (6), i.e., we have neglected the Coulomb interaction in the final state, which, as is well known, changes the power of F in the pre-exponential expression, without changing the exponential itself. For the same reason, the pre-exponential factor in (16) must also be corrected, but this is of less significance, since all the dependences are determined essentially by the exponential function. A crude quasiclassical analysis shows that, apart from a numerical coefficient, the inclusion of this interaction results in a correction factor $I_0\gamma/\hbar\omega\,(1+\gamma^2)^{1/2}$, which we have taken into account in (1).

In the opposite limiting case of high frequencies and not very strong fields $\gamma \gg 1$, the fundamental role in (18) is played by the zeroth term, and formula (1) describes in this case the probability of simultaneous absorption of several photons:

$$w_0 = A\omega\left(\frac{I_0}{\hbar\omega}\right)^{s_1/2} \exp\left\{2\left\langle\frac{\tilde{I}_0}{\hbar\omega}+1\right\rangle - \frac{\tilde{I}_0}{\hbar\omega}\left(1+\frac{e^2F^2}{2m\omega^2 I_0}\right)\right\} \times\left(\frac{e^2F^2}{8m\omega^2 I_0}\right)^{I_0/\hbar\omega+1}\Phi\left[\left(2\left\langle\frac{\tilde{I}_0}{\hbar\omega}+1\right\rangle - \frac{2I_0}{\hbar\omega}\right)^{1/2}\right]. \quad (21)$$

After introducing the correction for the Coulomb interaction in the final state, formulas (1) and (21) should be applicable, accurate to a numerical factor of the order of unity, to a description of the ionization of any atom, not only hydrogen. Indeed, as can be seen from the preceding deduction, this probability is determined essentially by the action of the field on the final state of the free electron, and not on the ground state of the atom. Therefore the use of another wave function for the ground state changes only the matrix element $V_0(\mathbf{p})$, i.e., the pre-exponential factor.

So far we have considered a transition from the ground state directly to the continuous-spectrum state. Let us consider now, in the next higher order of perturbation theory, a process in which the electron first goes over into an excited state (7), and then into the continuous spectrum. The probability of such a transition is equal to

$$w^{(r)} = \frac{1}{\hbar^2}\sum_s \lim_{T\to\infty}\mathrm{Re}\int\frac{d^3p}{(2\pi\hbar)^3}L_s^{(r)}(\mathbf{p},T)\cos\omega T \times\int_0^T L_s^{(r)*}(\mathbf{p},t)\cos\omega t\,dt, \quad (22)$$

$$L_s^{(r)}(\mathbf{p},t) = \frac{1}{i\hbar}V_s\left(\mathbf{p}+\frac{e\mathbf{F}}{\omega}\sin\omega t\right)\exp\left\{\frac{i}{\hbar}\int_0^t\left[I_s + \frac{1}{2m}\left(\mathbf{p}+\frac{e\mathbf{F}}{\omega}\sin\omega\tau\right)^2 - eF\sigma_s\cos\omega\tau\right]d\tau\right\} \times\int_0^t V_{0s}\cos\omega t'\exp\left\{\frac{i}{\hbar}\int_0^{t'}[I_0 - I_s + eF\sigma_s\cos\omega\tau]\,d\tau\right\}dt'. \quad (23)$$

After transformations analogous to (11)—(13), this expression reduces to the form

$$w^{(r)} = \frac{1}{4}\sum_{sn}\left|J_{n-1}\left(\frac{eF\sigma_s}{\hbar\omega}\right)+J_{n+1}\left(\frac{eF\sigma_s}{\hbar\omega}\right)\right|^2 \times\frac{|V_{0s}|^2}{(I_0-I_s-n\hbar\omega)^2}\frac{2\pi}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3} \times\sum_{n'}\frac{1}{4}\left|L_{n'-1}(\mathbf{p})+L_{n'+1}(\mathbf{p})\right|^2 \times\left[\delta\left(I_s+\frac{p^2}{2m}+\frac{e^2F^2}{4m\omega^2}-n'\hbar\omega\right) + \delta\left(I_0+\frac{p^2}{2m}+\frac{e^2F^2}{4m\omega^2}-(n'+n)\hbar\omega\right)\right], \quad (24)$$

where J_n—Bessel function of order n.

Carrying out summation over n′ and integration over $\mathbf{p}$, and retaining in the sum over n only the term n_s closest to resonance $|I_0 - I_s - n_s\hbar\omega| \ll \hbar\omega$, we obtain formula (3), but with a denominator in which we take account also of the shift of the atomic levels in the second order of the Stark effect, and of the attenuation of the level I_s due to its ionization w_s and its spontaneous emission γ_s. Formula (3) describes also the radiation of atoms at frequencies $\hbar\omega_s = I_0 - I_s$, if we replace w_s by γ_s in its numerator.

The quantity $eF\sigma_s/\hbar\omega$, which enters in (3), is of the order of $1/\gamma$, and therefore formally when $\gamma \gg 1$ the probabilities $w^{(r)}$ and w_0 can be of the same order of magnitude. Numerically, however, $w^{(r)}$ is in the mean very small compared with w_0, owing to the factor $1/n_s!$ in the definition of J_n. Directly near resonance, however, $w^{(r)}$ becomes many orders larger than w_0:

$$\left(\frac{w^{(r)}}{w_0}\right)_{max} \sim \left(\frac{1}{n_s!}\right)^2\left(\frac{m\hbar\omega^3}{e^2F^2}\right)^{2I_s/\hbar\omega}$$

This resonance is of course very narrow and

therefore difficult to observe. It is interesting to note that if the frequency of the incident radiation lies in a relatively broad region, such that $|I_0 - I_s - n_s\hbar\omega| \lesssim e^2F^2\alpha_s/2$, then we can observe just as sharp a resonant change in the ionization probability when the field intensity in the wave is changed, for the resonant frequency itself is displaced in this case.

We note, finally, that in the case of the purely quadratic Stark effect ($\sigma_s = 0$) the situation does not change qualitatively, although quantitatively $w^{(r)}$ becomes somewhat smaller. The Bessel-function arguments in (3) then contain in lieu of $eF\sigma_s$ the quantity $e^2F^2\alpha_s/4$, and their index $n_s - 1$ is replaced by $(n_s - 1)/2$ for odd n_s and $n_s/2 - 1$ for even n_s. In the case of even n_s, V_{0s} is replaced by $V_{0s}^{(1)}$—the correction to V_{0s} in first-order perturbation theory.

The formulas presented above pertained to the ionization of individual atoms, i.e., to gases. We now turn to ionization by a strong electromagnetic wave in a crystal. In this case the ionization process reduces to the transfer of the electron from the valence band into the conduction band, in other words, to the creation of an electron-hole pair. Therefore the energy of the final state is not simply the energy of the free electron, but is equal to the sum of the energies of the electron and of the hole:

$$\varepsilon(\mathbf{p}) = \varepsilon_c(\mathbf{p}) - \varepsilon_v(\mathbf{p}). \quad (25)$$

Here the indices c and v denote the conduction and valence bands, and $\epsilon_{c,v}(\mathbf{p})$ is the dependence of the energy on the quasimomentum in these bands. The quasimomenta of the initial and final states, should naturally be the same, since the homogeneous electric field cannot change the momentum of a system that is neutral on the whole.

The Bloch wave functions of an electron, accelerated by the field inside each of the bands, have a form analogous to (6):

$$\psi_{\mathbf{p}}^{c,v}(\mathbf{r},t) = u_{\mathbf{p}(t)}^{c,v}(\mathbf{r})\exp\left\{\frac{i}{\hbar}\left[\mathbf{p}(t)\,\mathbf{r} - \int_0^t \varepsilon_{c,v}(\mathbf{p}(\tau))\,d\tau\right]\right\}, \quad \mathbf{p}(t) = \mathbf{p} + (e\mathbf{F}/\omega)\sin\omega t, \quad (26)$$

where $u_{\mathbf{p}}^{c,v}(\mathbf{r})$ are periodic functions that have the translational symmetry of the lattice. Calculations perfectly similar to (8)—(15) lead to a general formula for the ionization probability

$$w = \frac{2\pi}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\left|L_{cv}(\mathbf{p})\right|^2\sum_n\delta(\overline{\varepsilon(\mathbf{p})} - n\hbar\omega), \quad (27)$$

where, however,

$$\overline{\varepsilon(\mathbf{p})} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon\left(\mathbf{p}+\frac{e\mathbf{F}}{\omega}\sin x\right)dx, \quad (2$$

$$L_{cv}(\mathbf{p}) = \frac{1}{2\pi}\oint V_{cv}\left(\mathbf{p}+\frac{e\mathbf{F}}{\omega}u\right) \times\exp\left\{\frac{i}{\hbar\omega}\int_0^n\varepsilon\left(\left|\mathbf{p}+\frac{e\mathbf{F}}{\omega}v\right|\right)\frac{dv}{(1-v^2)^{1/2}}\right\}du. \quad (2$$

The matrix element of the optical transition from the valence to the conduction band is determined in the following fashion:

$$V_{cv}(\mathbf{p}) = i\hbar\int u_{\mathbf{p}}^{c*}(\mathbf{r})\,e\mathbf{F}\nabla_{\mathbf{p}}u_{\mathbf{p}}^{v}(\mathbf{r})\,d^3\mathbf{r}. \quad (30)$$

The saddle points of the integral (29) are now determined by the relation

$$\varepsilon(\mathbf{p} + e\mathbf{F}u_s/\omega) = 0. \quad (31)$$

The matrix element $V_{cv}(\mathbf{p} + e\mathbf{F}u/\omega)$ has at these points, as for isolated atoms, a pole with a universal residue value[6,7]

$$\mathrm{res}\,V_{cv}(\mathbf{p} + e\mathbf{F}u_s/\omega) = \pm i\hbar\omega/4. \quad (32$$

The essential difference from the case of atoms is, however, the fact that the function $\epsilon(\mathbf{p} + e\mathbf{F}u/\omega)$ has at the point u_s not a simple zero, but a branch point of the root type. Calculating $L_{cv}(\mathbf{p})$ with allowance for these singularities, we find that the contribution to (29) made by each of the saddle points is equal to

$$\frac{\hbar\omega}{3}\exp\left\{\frac{i}{\hbar\omega}\int_0^{u_s}\varepsilon\left(\mathbf{p}+\frac{e\mathbf{F}}{\omega}u\right)\frac{du}{(1-u^2)^{1/2}}\right\}.$$

Further calculations are perfectly analogous to those that lead to formula (16), but call for the use of a concrete type of dispersion law $\epsilon(\mathbf{p})$. For typical semiconductors[6,7] we have

$$\varepsilon(\mathbf{p}) = \Delta(1 + p^2/m\Delta)^{1/2}, \quad (33)$$

where Δ—width of the forbidden band separating the valence band from the conduction band and m—reduced mass of the electron and the hole,

$$1/m = 1/m_e + 1/m_h.$$

In this case, having in mind small $\mathbf{p}$ (but not $\mathbf{p} + e\mathbf{F}u/\omega$) and introducing dimensionless variables x and y defined by

$$x^2 = p_\parallel^2/m\Delta; \quad y^2 = p_\perp^2/m\Delta, \quad (34)$$

where $p_\parallel$ and $p_\perp$ are the quasimomentum components parallel and perpendicular to the direction of the field $\mathbf{F}$, we obtain

$$\bar{\varepsilon}(\mathbf{p}) = \frac{2}{\pi}\Delta\left\{\frac{\sqrt{1+\gamma^2}}{\gamma}E\left(\frac{1}{\sqrt{1+\gamma^2}}\right) + \frac{\gamma}{\sqrt{1+\gamma^2}}E\left(\frac{1}{\sqrt{1+\gamma^2}}\right)\frac{x^2}{2} + \frac{\gamma}{\sqrt{1+\gamma^2}}K\left(\frac{1}{\sqrt{1+\gamma^2}}\right)\frac{y^2}{2}\right\} \quad (35)$$

and for the argument of the exponential in (2.9) we get

$$\frac{i}{\hbar\omega}\int_0^{u_s}\varepsilon\left(\mathbf{p}+\frac{e\mathbf{F}}{\omega}u\right)\frac{du}{\sqrt{1-u^2}} = -\frac{\Delta}{\hbar\omega}\left\{\frac{\sqrt{1+\gamma^2}}{\gamma}\left[K\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)-E\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right] + \frac{x^2}{2}\frac{\gamma}{\sqrt{1+\gamma^2}}\left[K\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)-E\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right] + \frac{y^2}{2}\frac{\gamma}{\sqrt{1+\gamma^2}}K\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right) \pm i\gamma x\left[1-2\sqrt{1+\gamma^2}E\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right) + \frac{2}{\sqrt{1+\gamma^2}}K\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right]\right\}. \quad (36)$$

The functions K and E in (35) and (36) are complete elliptic integrals of the first and second kind. The term in (36), which is linear in x, will henceforth be left out, for when account is taken of both saddle points it gives rise in L(p) to a rapidly oscillating factor of the type 2cos x, which reduces after squaring and integrating with respect to x to a factor 2, which we can take into account directly in the final answer. The quantity γ in (35)—(42) differs from (17) by a factor $1/\sqrt{2}$:

$$\gamma = \omega\sqrt{m\Delta}/eF.$$

The ionization probability is given by the expression

$$w = \frac{2\omega}{9\pi}\left(\frac{\sqrt{1+\gamma^2}}{\gamma}\frac{m\omega}{\hbar}\right)^{3/2}Q\left(\gamma, \frac{\tilde{\Delta}}{\hbar\omega}\right)\exp\left\{-\pi\left\langle\frac{\tilde{\Delta}}{\hbar\omega}+1\right\rangle \times\left[K\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)-E\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right]\Big/E\left(\frac{1}{\sqrt{1+\gamma^2}}\right)\right\}, \quad (37)$$

where $\tilde{\Delta}$—effective ionization potential:

$$\tilde{\Delta} = \frac{2}{\pi}\Delta\frac{\sqrt{1+\gamma^2}}{\gamma}E\left(\frac{1}{\sqrt{1+\gamma^2}}\right). \quad (38)$$

The symbol $\langle x\rangle$ again denotes the integer part of the number x, and $Q(\gamma, \tilde{\Delta}/\hbar\omega)$ is a function analogous to S in (16):

$$Q(\gamma, x) = \left[\pi/2K\left(\frac{1}{\sqrt{1+\gamma^2}}\right)\right]^{1/2} \times\sum_{n=0}^{\infty}\exp\left\{-\pi\left[K\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right) - E\left(\frac{\gamma}{\sqrt{1+\gamma^2}}\right)\right]n\Big/E\left(\frac{1}{\sqrt{1+\gamma^2}}\right)\right\} \times\Phi\left\{\left[\pi^2(2\langle x+1\rangle - 2x+n)\Big/2K\left(\frac{1}{\sqrt{1+\gamma^2}}\right) \times E\left(\frac{1}{\sqrt{1+\gamma^2}}\right)\right]^{1/2}\right\}. \quad (39)$$

In the derivation of (37) we have made use of an identity known from the theory of elliptic integrals:

$$K(x)E(\sqrt{1-x^2}) + K(\sqrt{1-x^2})E(x) - K(x)K(\sqrt{1-x^2}) = \pi/2.$$

Asymptotically, the behavior of the ionization probability in a solid (37) is similar to the variation of the ionization probability in gases (16), both for low and for high frequencies. In the case of low frequencies and strong fields $\gamma \ll 1$, the ionization probability reduces to the formula for the tunnel effect[6,8]:

$$w = \frac{2}{9\pi^2}\frac{\Delta}{\hbar}\left(\frac{m\Delta}{\hbar^2}\right)^{3/2}\left(\frac{e\hbar F}{m^{1/2}\Delta^{3/2}}\right)^{5/2} \times\exp\left\{-\frac{\pi}{2}\frac{m^{1/2}\Delta^{3/2}}{e\hbar F}\left(1-\frac{1}{8}\frac{m\omega^2\Delta}{e^2F^2}\right)\right\}. \quad (40)$$

At first glance (40) differs from the corresponding formula (19) in[6] by a factor of the order $(e\hbar F/m^{1/2}\Delta^{3/2})^{1/2}$ in front of the exponential. This is connected with the fact that (40), like the general formula (27), describes the average ionization probability over a time which is much larger than the period of the external field $2\pi/\omega$, while formula (19) of[6] yields the instantaneous tunneling probability in a slowly varying field. If we introduce in this last formula a field of the type F(t) = F cos ωt and average it over the time, then the result agrees with (40).

In the opposite limiting case $\gamma \gg 1$, we again obtain a formula that describes the probability of multi-quantum absorption:

$$w = \frac{2}{9\pi}\omega\left(\frac{m\omega}{\hbar}\right)^{3/2}\Phi\left[\left(2\left\langle\frac{\tilde{\Delta}}{\hbar\omega}+1\right\rangle - \frac{2\tilde{\Delta}}{\hbar\omega}\right)^{1/2}\right] \times\exp\left\{2\left\langle\frac{\tilde{\Delta}}{\hbar\omega}+1\right\rangle\left(1-\frac{e^2F^2}{4m\omega^2\bar{\Delta}}\right)\right\}\left(\frac{e^2F^2}{16m\omega^2\bar{\Delta}}\right)^{\langle\tilde{\Delta}/\hbar\omega+1\rangle}, \quad (41)$$

$$\bar{\Delta} = \Delta + e^2F^2/4m\omega^2. \quad (42)$$

Ionization in solids can proceed also via intermediate states, similar to the process described by formula (3). The role of the excited states can then be played by the excitons, if their binding energy is commensurate with or larger than the energy of the quantum of incident radiation $\hbar\omega$, or else by the local level of the impurities and lattice defects, if their concentration is sufficiently high. The corresponding formula is perfectly analogous to (3) and will not be written out here.

Owing to the presence of the dielectric probability, the influence of the Coulomb interaction of the free electron and hole on the ionization probability is negligibly small[6].

We now present some quantitative estimates. Having in mind the hydrogen atom (I_0 = 13.6 eV) and emission in the red region of the spectrum at frequency $\omega \approx 3\times10^{15}$ sec^{-1}, we obtain $\langle I_0/\hbar\omega + 1\rangle = 8$. The field intensity is best expressed in terms of the power of the incident wave W and the transverse cross section of the beam[1)] R^2:

$$\frac{1}{\gamma^2} = \frac{e^2F^2}{2m\omega^2I_0} = \frac{2\pi e^2}{mc\omega^2I_0}\frac{W}{R^2} \approx 2.5\cdot10^{-22}\frac{W}{R^2}.$$

After substituting in (21) we obtain

$$\omega_0 \approx 10^{17}(2\cdot10^{-9}W/R^2)^8\,\text{sec}^{-1}.$$

Thus, for example, for W = 50 MW and $R^2 = 10^{-6}$ cm^2 we get $\omega_0 \approx 10^9$ sec^{-1}, i.e., within a time on the order of 0.001 μsec the gas is completely ionized. However, at lower values of the power or in the case of poor focusing, the effect decreases very rapidly. Thus, for W = 50 MW but $R^2 = 10^{-4}$ cm, the frequency is $\omega_0 \approx 10^{-7}$ sec^{-1}.

Analogously, for ionization of the first excited level of hydrogen, $I_1 = 0.25\,I_0$, we get

$$\omega_1 \approx 10^{16}(8\cdot10^{-9}W/R^2)^2\,\text{sec}^{-1}.$$

The Stark splitting for this level is $3eF\hbar^2/me^2$. Substituting these expressions in (3) we obtain for the resonant part of the ionization probability the following estimate:

$$w^{(r)} \approx \frac{w_1}{[(n_1-1)!]^2}\left(\frac{eF\sigma_1}{2\hbar\omega}\right)^{2n_1}\frac{(\hbar\omega)^2}{(I_0-I_1-n_1\hbar\omega)^2} \sim 10^{12}\left(6\cdot10^{-9}\frac{W}{R^2}\right)^8\left(\frac{\hbar\omega}{I_0-I_1-6\hbar\omega}\right)^2.$$

If, for example, the distance from resonance in the denominator of this formula is of the order of 2×10^{-2} eV, then $w^{(r)}$ exceeds w_0 by more than four orders of magnitude.

In the case of a molecular gas, the condition of resonance can be assumed to be satisfied almost always, since the energy which is lacking for resonance can be obtained from (or given up to) the vibrational degrees of freedom, which are strongly excited in some fields. The total ionization probability is in this case of the order of

$$w \sim \omega\left(\frac{eF\sigma_s}{\hbar\omega}\right)^{2n_s}\frac{w_s}{w_s+\gamma_s}$$

The main contribution to the ionization is made by the excited states with the highest ionization potentials.

In a real situation, the resonant ionization is decisive in a much wider frequency range, since the power of the incident radiation varies with time, and the resonant frequencies in (3) are shifted by an amount on the order of

$$\Delta(\hbar\omega_r) = \frac{1}{2}e^2F^2\alpha_s = 2\pi\frac{e^2}{c}\alpha_sW.$$

If $|I_0 - I_S - n_S\hbar\omega| < \Delta(\hbar\omega_r)$, then at some instant of time the resonance condition is satisfied exactly, and this instant makes the main contribution to the total ionization. The total ionization probability is determined by the time integral of formula (3):

$$\int w^{(r)}\,dt = \frac{2\pi}{e^2\alpha_s}\frac{|V_{0s}|^2}{\hbar}\left(\frac{dF^2}{dt}\right)^{-1}\frac{1}{4}\left|J_{n_s-1}\left(\frac{eF\sigma_s}{\hbar\omega}\right)\right|^2. \quad (43)$$

In the right side of this formula there should be substituted the resonant value of the field.

If we introduce a dimensionless coefficient of the order of unity

$$a_s = |V_{0s}|^2/e^2F^2\alpha_sI_0,$$

then (43) can be rewritten in a more illustrative form

$$\int w^{(r)}\,dt = 2\pi a_s\frac{I_0\tau}{4\hbar}\left|J_{n_s-1}\left(\frac{eF\sigma_s}{\hbar\omega}\right)\right|^2, \quad (44)$$

where τ—characteristic time of the order of the duration of the radiation pulse, determined by the relation

$$\frac{1}{\tau} = \frac{1}{F^2}\frac{dF^2}{dt}. \quad (45)$$

Under the same assumptions used to obtain the previous estimate, we obtain (n_S = 6)

$$\int w^{(r)}\,dt \sim 10^{12}\tau(6\cdot10^{-9}W/R^2)^5.$$

Thus, for a pulse duration $\tau \sim 100$ μsec, radiation of 50 MW power, focused in a region with dimensions $R \sim 3\times10^{-3}$ cm produces practically full ionization.

In breakdown of gases under real conditions, the role of the processes considered in the present

1)In all the formulas that follow W is in MW and R in cm.

paper is determined by their competition with the cascade multiplication of the free electrons. The field intensity at which the cascade breakdown occurs increases rapidly with decreasing dimensions of the region where the field is effective, or with decrease in the pulse duration. Therefore for a sufficiently focused beam $R \lesssim 10^{-3}$ cm, or for sufficiently short pulses, the breakdown should apparently be determined by the direct ionization of the atoms in the field of the wave.

In the case of condensed media, the situation is much more favorable for the mechanism in question, for owing to the rapid energy dissipation by the free electrons, the breakdown fields are much higher there, and the ionization potentials lower.

In conclusion I take the opportunity to thank G. A. Askar'yan, V. L. Ginzburg, and M. A. Krivoglaz for valuable remarks made during the discussion of the present paper.

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Translated by J. G. Adashko
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SOVIET PHYSICS JETP VOLUME 20, NUMBER 5 MAY, 1965

MAGNETIC SUSCEPTIBILITY IN STRONG MAGNETIC FIELDS

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Submitted to JETP editor May 26, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1958-1965 (November, 1964)

The magnetic moment of conduction electrons in strong magnetic fields is determined. It is found that the magnetic field strength dependence of the ground state energy and level density can be deduced from an experimental investigation of the magnetic susceptibility for an arbitrary dispersion law. It is shown that for a quadratic dispersion law in strong magnetic fields the total magnetic moment (diamagnetic or paramagnetic) approaches saturation.

1. INTRODUCTION

AT the present time, the oscillating part of the magnetic susceptibility (the de Haas-van Alphen effect) has been well studied theoretically and experimentally (see, e.g., [1]). The oscillations, as is well known, occur at low temperatures T ($T < \mu H$) in weak magnetic fields ($\mu H < \zeta_0$, where μ is the Bohr magneton for the conduction electron, and ζ_0 is the limiting Fermi energy; the temperature will always be measured in energy units). The situation is much worse with respect to the investigation of the monotonic part χ of the magnetic susceptibility (which is all that remains in strong magnetic fields and high temperatures). Experimentally, this is because it is difficult to separate the monotonic susceptibility of the conduction electrons from the susceptibility of the lattice. The theoretical determination of the susceptibility for a non-quadratic dispersion law for the electrons cannot be made because, as it turns out, knowledge of the exact rules of quantization is required even in weak magnetic fields (see [2]). In this connection Rumer's calculation [3] for a free electron gas[1)] is essential for an understanding of the dependence. Of particular interest, naturally, is the region of strong magnetic fields

$$\mu H \gg T, \zeta_0, \tag{1}$$

where the magnetic susceptibility of the electron gas depends on magnetic field (in zeroth approximation χ does not depend on H for $\mu H \ll \zeta_0$ and $\mu H \ll T$).

However, Rumer's treatment fails to give an understanding of the situation for the general case in precisely the region of Eq. (1). To show this, let us turn to the simplest case of a dispersion law $\epsilon = p^2/2m^*$ (ϵ is energy, p momentum), where the effective mass m^* is not the same as the mass m_0 of a free electron. Then the energy levels have the form (z is the magnetic field direction)

$$\varepsilon_n^{\pm} = p_z^2/2m^* + (n + 1/2) \times \mu H \pm \mu_0 H/2 = \varepsilon \pm \mu_0 H/2 \tag{2}$$

(the last term is associated with the spin paramagnetism), so that the energy of the ground state equals

$$\varepsilon_{min} = \frac{eH\hbar}{2c}\left(\frac{1}{m^*} - \frac{1}{m_0}\right) = \frac{1}{2}(\mu - \mu_0)H.$$

In magnetic fields of Eq. (1), when the separation between the levels is the greatest in energy, all the electrons accumulate near ϵ_{min}. From the expression for ϵ_{min} it is clear that the case of free electrons, when $m^* = m_0$, is a special one, for only in this case does the energy ϵ_{min} equal zero at any field H, whereas when $m^* \neq m_0$, the magnitude $|\epsilon_{min}| \to \infty$ as $H \to \infty$. As a result, it is natural to expect in the general case a substantially different dependence $\chi(T, H)$ than the one offered in [3].

The purpose of this paper is to elucidate the form of this dependence. First, in Sec. 2, we shall carry out the calculation of $\chi(H, T)$ for the case of Eq. (2) both with $\mu \sim \mu_0$ and with $\mu \gg \mu_0$ (the case $\mu \ll \mu_0$, i.e., $m^* \gg m_0$, is not met in real conductors) and estimate the magnetic fields that satisfy the condition (1). The basic problem of Sec. 2 is to obtain $\chi(H, T)$ in explicit form and to demonstrate the idea of the calculation. In Sec. 3 we discuss the case when the μH that satisfies the inequality (1) is significantly less than all

[1)] Also of great interest is the calculation of magnetic susceptibility for semiconductors with a loop of extrema in a perpendicular loop of magnetic field, [4] which has been carried out on the basis of exact quantization rules.