CONTRIBUTION TO THE THEORY OF SUPERFLUIDITY IN AN IMPERFECT FERMI GAS

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The Cooper effect in a low-density Fermi gas is studied. The magnitude of the transition temperature into the superfluid state is determined.

IT is well known that the basis of superconductivity is the so-called Cooper effect[1] which consists in that the existence of an effective attraction between the electrons in a metal leads to the formation of electron pairs in a singlet state. The exchange of phonons between the electrons is the mechanism that causes the presence of such an attraction. We consider in the present paper the phenomenon of superfluidity in a low-density Fermi gas. The main methodological difference in this case consists in that in the theory of superconductivity the transition temperature is expressed in terms of a quantity of the order of the Debye temperature $\tilde{\omega} \ll \mu$, which is used as the natural parameter for a cutoff, while in a low-density Fermi-gas the only small parameter is its density, or more accurately, the quantity $f_0 p_0^2/v$, where f_0 is the s-wave scattering amplitude for interparticle scattering, and p_0 and v are respectively the limiting momentum and the velocity on the Fermi surface.

For our investigation it is convenient to use quantum field-theoretical methods. We shall assume, starting from Cooper's idea, that the attraction between the particles leads to the formation of bound pairs. Such a pair is a Bose formation. The temperature-dependent Green's function of a free boson $\mathfrak{G}(\mathbf{p}\omega)$ is of the form[2,3]

$$(i\omega_n - \varepsilon(\mathbf{p}) + \mu)^{-1}, \qquad \omega_n = 2\pi n T$$

and for $\omega_n = 0$ becomes infinite as soon as the so-called Bose condensation takes place $[\mu(T_c) = 0]$.

The two-particle Green's function plays the same role for a bound pair as the Green's function of a boson. If one studies the dependence of the two-particle Green's function on the total energy and momentum of the two particles (i.e., on the energy and momentum of the bound pair as a whole) one can find the temperature of the transition of the system into the superfluid state from the condition that this quantity should become infinite when the transition takes place. At lower temperatures bound pairs begin to form and to "condense." The normal ground state of attractive Fermi-particles is thus unstable. The tendency to form bound pairs will at absolute zero cause a change in the ground state of non-interacting fermions under the influence of their mutual attraction. We shall determine the temperature at which the transition to the superfluid state takes place and we shall also study the characteristic singularities of the vertex part of a Fermi system of weakly mutually attracting particles at T = 0 which appear because of the above-mentioned instability of the usual ground state. For the sake of convenience we shall start with the study of the vertex part at absolute zero.

The vertex function $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ is defined through the Fourier component of the two-particle Green's function

$$G_{\alpha\beta\gamma\delta}(x_1, x_2, x_3, x_4) = \langle T(\psi_{1\alpha}\psi_{2\beta}\psi_{3\gamma}^{+}\psi_{4\delta}^{+})\rangle \qquad (1)$$

by the following relation

$$\begin{aligned} G_{\alpha\beta\gamma\delta}&(p_1, p_2, p_3, p_4) \\ &= (2\pi)^8 G(p_1) G(p_2) [\delta(p_1 - p_3)\delta(p_2 - p_4)\delta_{\alpha\gamma}\delta_{\beta\delta} \\ &- \delta(p_1 - p_4)\delta(p_2 - p_3)\delta_{\alpha\delta}\delta_{\beta\gamma}] \\ &+ i(2\pi)^4 G(p_1)G(p_2)G(p_3)G(p_4)\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)\delta(p_1 \\ &+ p_2 - p_3 - p_4). \end{aligned} \qquad (2)$$

The Hamiltonian of the interparticle interaction is of the form

$$H_{int} = \frac{1}{2}\int V(x - y)\psi_1^{+}(x)\psi_2^{+}(y)\psi_2(y)\psi_1(x)\, d^4x\, d^3y,$$
$$V(x - y) = V(\mathbf{x} - \mathbf{y})\delta(x_0 - y_0). \qquad (2a)$$

It was shown by Landau[4] that the singularities of Γ for small values of the momentum transfer ($q = p_3 - p_1$ or $p_4 - p_1$) are connected with the existence of so-called "zero sound." We shall study the singularities of the vertex function* as far as the vari-

*A study of these singularities in general form in the theory of a Fermi liquid was made in an unpublished paper by A. A. Abrikosov, L. P. Gor'kov, L. D. Landau, and I. M. Khalatnikov. In the model considered the same approach allowed us to obtain more detailed results.

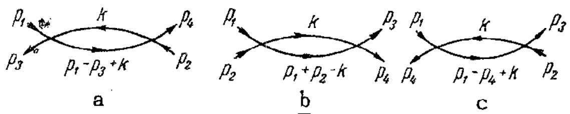

FIG. 1

able $q = p_1 + p_2$ is concerned. To do this we consider the diagrams of the first orders in perturbation theory (Fig. 1).

Singularities of the "zero-sound" type are connected with the diagrams of Fig. 1a, c. As far as the diagram of Fig. 1b is concerned, it contains an integral of two Green's functions of the form $f_0^2\int G(k)\,G(q-k)\,d^4k$. Substituting here for $G(k)$ the Green's function of a perfect Fermi gas

$$G(k) = [\omega + \mu - \varepsilon(\mathbf{k}) + i\delta \operatorname{sign}(\varepsilon(\mathbf{k}) - \mu)]^{-1},$$

we get after integrating over the frequency

$$-2\pi i f_0^2 \int \frac{d^3k}{\varepsilon + 2\mu - \varepsilon(\mathbf{k}) - \varepsilon(\mathbf{q}-\mathbf{k}) + i\delta},$$

$$\varepsilon(\mathbf{k}) > \mu, \quad \varepsilon(\mathbf{q}-\mathbf{k}) > \mu,$$

$$2\pi i f_0^2 \int \frac{d^3k}{\varepsilon + 2\mu - \varepsilon(\mathbf{k}) - \varepsilon(\mathbf{q}-\mathbf{k}) - i\delta},$$

$$\varepsilon(\mathbf{k}) < \mu, \quad \varepsilon(\mathbf{q}-\mathbf{k}) < \mu. \tag{3}$$

This integral diverges at large k; this is connected with the fact that in that range it is the same as the Born correction to the scattering amplitude of two particles in vacuo. One can thus remove the divergence for large k by renormalizing the scattering amplitude. However, at small $q \ll p_0$ and $\epsilon \ll \mu$ the result obtained diverges logarithmically also near the Fermi surface for $k \sim p_0$. The integration in (3) leads thus when q and ϵ are small to a logarithmic term of the order $(p_0^2 f_0^2/v) \times \ln\{qv/\mu, \epsilon/\mu\}$. The large magnitude of the logarithm can be compensated by the small parameter $p_0^2 f_0/v \ll 1$.

To evaluate the vertex part in that range of $\{qv, \epsilon\}$ values perturbation theory turns out to be insufficient and we must sum a whole "ladder" of diagrams shown in Fig. 1b. We write down the equation for the vertex part in such a way that we explicitly separate off special integrations of the type (3):

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = \widetilde{\Gamma}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) + \frac{i}{2(2\pi)^4}\int \widetilde{\Gamma}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)\,G(k)\,G(q-k)\,\Gamma_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k, \tag{4}$$

where $\widetilde{\Gamma}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ is the totality of those diagrams which are irreducible in the sense that they cannot be divided by a vertical line into two parts which are joint by two fermion lines directed to one side.

The above-mentioned divergence in the large momentum region can be removed as follows. We define a vertex function for particles, interacting in vacuo, using the equation (Belyaev[5] and Galitskii[6] introduced this quantity)

$$\Gamma^{(0)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = \Gamma^{(1)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) + \frac{i}{2(2\pi)^4}\int \Gamma^{(1)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)\,G^{(0)}(k)\,G^{(0)}(q-k)\,\Gamma^{(0)}_{\xi\eta\gamma\delta} \times (k, q-k, p_3, p_4)\,d^4k, \tag{5}$$

where $\Gamma^{(1)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ is the term of first order in the interaction, which is equal to $V(p_3 - p_1)\,\delta_{\alpha\gamma}\delta_{\beta\delta} - V(p_3 - p_2)\,\delta_{\alpha\delta}\delta_{\beta\gamma}$ while $G^{(0)}(k)$ is the vacuum particle Green function:

$$G^{(0)}(k) = (\omega - \varepsilon(\mathbf{k}) + i\delta)^{-1}.$$

If we write (5) in the form

$$\Gamma^{(1)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = L\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) \equiv \Gamma^{(0)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) - \frac{i}{2(2\pi)^4}\int \Gamma^{(1)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k) \times G^{(0)}(k)\,G^{(0)}(q-k)\,\Gamma^{(0)}_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k$$

and then subtract from both sides of Eq. (4) the integral

$$\frac{i}{2(2\pi)^4}\int \Gamma^{(1)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)\,G^{(0)}(k)\,G^{(0)}(q-k)\,\Gamma_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k,$$

we get

$$L\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = L\Gamma^{(0)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) + \widetilde{\Gamma}^{*}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) + \frac{i}{2(2\pi)^4}\int L\Gamma^{(0)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)\,[G(k)\,G(q-k) - G^{(0)}(k)\,G^{(0)}(q-k)]\,\Gamma_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k + \frac{i}{2(2\pi)^4}\int \widetilde{\Gamma}^{*}_{\alpha\beta\xi\eta}(p_1, p_2, p_3, p_4)\,G(k)\,G(q-k) \times \Gamma_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k, \tag{6}$$

where $\widetilde{\Gamma}^{*}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ stands for all irreducible diagrams from the second order in the interaction onwards.

We get the vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ up to terms of order $p_0^2 f_0^2/v$ inclusive. It is clear that as the diagrams for $\widetilde{\Gamma}$ do not contain logarithmic integrations we can restrict ourselves in this order in the expression for $\widetilde{\Gamma}^{*}$ to terms of second order in f_0 (Fig. 2)

Applying to both sides of Eq. (6) the operator L^{-1} we are led to the following equation

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = \Gamma^{(0)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) + \frac{i}{2(2\pi)^4}\int \Gamma^{(0)}_{\alpha\beta\xi\eta} \times (p_1, p_2, k, q-k)\,[G(k)\,G(q-k) - G^{(0)}(k)\,G^{(0)}(q-k)] \times \Gamma_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k + \frac{i}{2(2\pi)^4}\int \widetilde{\Gamma}^{*}_{\alpha\beta\zeta\eta}(p_1, p_2, k, q-k)\,G(k)\,G(q-k)\,\Gamma_{\xi\eta\gamma\delta}(k, q-k, p_3, p_4)\,d^4k. \tag{7}$$

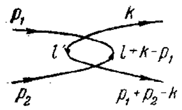

FIG. 2

Within the accuracy which we have adopted $L\widetilde{\Gamma}^* = \widetilde{\Gamma}^*$ and we have neglected in the terms which do not contain logarithmic integrations quantities of second order in f_0. As far as $\Gamma^{(0)}$ is concerned in the integral on the right hand side of (7), we must evaluate it up to terms in second order in f_0:

$$\Gamma^{(0)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k) = V(\mathbf{k}-\mathbf{p}_1)\delta_{\alpha\xi}\delta_{\beta\eta} - V(\mathbf{k}-\mathbf{p}_2)\delta_{\alpha\eta}\delta_{\beta\xi}$$
$$+\frac{i}{2(2\pi)^4}\int[V(\mathbf{l}-\mathbf{p}_1)V(\mathbf{k}-\mathbf{l})$$
$$+V(\mathbf{l}-\mathbf{p}_2)V(\mathbf{l}-\mathbf{q}+\mathbf{k})]G^{(0)}(l)G^{(0)}(q-l)\delta_{\alpha\xi}\delta_{\beta\eta}d^4l$$
$$-\frac{i}{2(2\pi)^4}\int[V(\mathbf{l}-\mathbf{p}_1)V(\mathbf{k}-\mathbf{q}+\mathbf{l})$$
$$+V(\mathbf{l}-\mathbf{p}_2)V(\mathbf{k}-\mathbf{l})]G^{(0)}(l)G^{(0)}(q-l)\delta_{\alpha\eta}\delta_{\beta\xi}d^4l. \qquad (8)$$

If we use the relation between the scattering amplitude for two particles in vacuo and the interaction potential

$$V(\mathbf{k}_1-\mathbf{k}_2) = f(\mathbf{k}_1, \mathbf{k}_2) + \frac{m}{(2\pi)^3}\int\frac{f(\mathbf{k}_1, \mathbf{l})f^*(\mathbf{k}_2, \mathbf{l})d^3\mathbf{l}}{l^2-k_2^2-i\delta} \qquad (9)$$

and the well-known relation between the imaginary part of the scattering amplitude and the effective scattering cross section

$$\operatorname{Im} f(\mathbf{k}_1, \mathbf{k}_2) = -\frac{mk_1}{16\pi^2}\int d\mathbf{n}\, f(\mathbf{k}_1, k_1\mathbf{n})f^*(\mathbf{k}_2, k_1\mathbf{n}), \qquad (10)$$

we get after substituting (9) and (10) into Eq. (8) an expression for $\Gamma^{(0)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)$:

$$\Gamma^{(0)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k) = f_0(1 - if_0p_0^2/4\pi v)(\delta_{\alpha\xi}\delta_{\beta\eta} - \delta_{\alpha\eta}\delta_{\beta\xi}). \qquad (11)$$

When one uses a Green's function in $G(p)$ one should, strictly speaking, evaluate its value up to terms of second order in f_0. Galitskii[6] has, however, shown that the correction of second order in f_0 to the perfect Fermi-gas Green's function leads merely to a renormalization of the Fermi energy. We shall thus in the following use the perfect Fermi-gas Green's functions assuming μ to be the renormalized Fermi energy.

The contribution to $\Gamma^*_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)$ in the intervals $\epsilon \ll \mu$ and $q \ll p_0$ which are of interest to us is equal to

$$\frac{if_0^2}{(2\pi)^4}\int G(l)G(p_1-k+l)d^4l\,(\delta_{\alpha\xi}\delta_{\beta\eta} - \delta_{\alpha\eta}\delta_{\beta\xi}).$$

Substituting $\Gamma^{(0)}_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)$ and $\widetilde{\Gamma}^*_{\alpha\beta\xi\eta}(p_1, p_2, k, q-k)$ into Eq. (7) and taking into account that of the terms of second order in f_0 we need only the logarithmic ones, we get the following expression for the vertex function:

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = \Gamma^{(0)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$$
$$\times\Big\{1 - \frac{if_0}{(2\pi)^4}\Big(1 - \frac{if_0p_0^2}{4\pi v}\Big)\int[G(k)G(q-k)$$
$$-G^{(0)}(k)G^{(0)}(q-k)]d^4k$$
$$+\frac{f_0^2}{(2\pi)^8}\int G(l)G(p_1-k+l)G(k)G(q-k)d^4l\,d^4k\Big\}^{-1}. \qquad (12)$$

The first of these integrals gives in the denominator of (12) a term

$$-\frac{p_0^2f_0}{(2\pi)^2v}\Big(1-\frac{if_0p_0^2}{4\pi v}\Big)\Big(2+\ln\frac{\varepsilon^2-q^2v^2+i\delta}{64\mu^2}+\frac{\varepsilon}{qv}\ln\frac{\varepsilon+vq-i\delta}{\varepsilon-vq+i\delta}\Big).$$

As we need only take into account the logarithmic term we must, when evaluating the second integral, put $\mathbf{p}_1 = \mathbf{k} = \mathbf{p}_0$, $\epsilon_1 = \omega = \mu$ after integrating over l and we must average over the direction of the vector $\mathbf{k}$. We get

$$\frac{2f_0^2p_0^4}{3(2\pi)^4v^2}(1+2\ln 2)\ln\frac{\varepsilon^2-q^2v^2}{64\mu^2}.$$

Substituting the results of these calculations into (12) we get

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = -(2\pi)^2vp_0^{-2}(\delta_{\alpha\gamma}\delta_{\beta\delta}$$
$$-\delta_{\alpha\delta}\delta_{\beta\gamma})\{(2\pi)^2v/p_0^2|f_0| - i\pi + \frac{8}{3} - \frac{14}{3}\ln 2$$
$$+\ln\frac{\varepsilon^2-q^2v^2+i\delta}{\mu^2}+\frac{\varepsilon}{qv}\ln\frac{\varepsilon+qv-i\delta}{\varepsilon-qv+i\delta}\Big\}^{-1}. \qquad (13)$$

We shall consider (13) to be the analytical continuation of $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ in the upper half-plane of the variable ϵ. Let $q = 0$ to begin with. We see that $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ has a pole at $\epsilon_0 = i\Delta$, where $\Delta = \mu(2/e)^{7/3}\exp\{-2\pi^2v/|f_0|p_0^2\}$. If we express the vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4)$ in terms of ϵ_0 it has the form

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2, p_3, p_4) = -(2\pi)^2vp_0^{-2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})$$
$$\times\Big\{-2+\ln\frac{\varepsilon^2-q^2v^2+i\delta}{\varepsilon_0^2}+\frac{\varepsilon}{qv}\ln\frac{\varepsilon+vq-i\delta}{\varepsilon-vq+i\delta}\Big\}^{-1}.$$

For small $qv \ll \Delta$ the position of the pole $\epsilon(\mathbf{q})$ approaches the real axis: $\epsilon(\mathbf{q}) = i\Delta(1 - q^2v^2/6\Delta^2)$. When q increases further $|\epsilon(\mathbf{q})|$ decreases and it tends to zero when $vq_{max} = e\Delta$ as

$$\varepsilon = (2e\Delta i/\pi)\ln(e\Delta/vq) \approx 2e\Delta i\,(q_{max} - q)/\pi q_{max}.$$

The Cooper phenomenon, i.e., the instability of a fermion system when there are attractions between the particles, leads thus to the occurrence in the vertex part of poles in the upper half-plane of the complex variable $\epsilon = \omega_1 + \omega_2$. Only those particles for which the total momentum is relatively small ($q \ll p_0$) will here display a tendency to form bound pairs. The quantity Δ has clearly the meaning of the reciprocal of the relaxation time of the system.

We turn now to the problem of determining the temperature at which the system makes the transition to the superfluid state. Because we cannot apply the usual technique at finite temperatures we must use the Matsubara quantities.[2,3] In the Matsubara method we have instead of (2) the following connection between the two-particle Green's function and the vertex part

$$\mathfrak{G}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_3\mathbf{p}_3,\omega_4\mathbf{p}_4) = (2\pi)^3T^{-1}\{(2\pi)^3T^{-1}[\mathfrak{G}(\omega_1\mathbf{p}_1)\mathfrak{G}(\omega_2\mathbf{p}_2) \times \delta_{\omega_1\omega_4}\delta(\mathbf{p}_1-\mathbf{p}_4)\delta_{\alpha\delta}\delta_{\beta\gamma} - \mathfrak{G}(\omega_1\mathbf{p}_1)\mathfrak{G}(\omega_2\mathbf{p}_2)\delta_{\omega_1\omega_3}\delta(\mathbf{p}_1 - \mathbf{p}_3)\delta_{\alpha\gamma}\delta_{\beta\delta}] - \tfrac{1}{2}\mathfrak{G}(\omega_1\mathbf{p}_1)\mathfrak{G}(\omega_2\mathbf{p}_2)\mathfrak{G}(\omega_3\mathbf{p}_3)\mathfrak{G}(\omega_4\mathbf{p}_4) \times \mathfrak{T}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_3\mathbf{p}_3,\omega_4\mathbf{p}_4)\} \times \delta_{\omega_1+\omega_2-\omega_3-\omega_4}\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4). \quad (14)$$

At the beginning of this paper we have already remarked that one can define the critical temperature as the temperature at which the Bose "condensation" of bound pairs sets in. At that point the two-particle Green's function. or what amounts to the same, according to (14), the vertex part, tends to infinity for the first time. We note that the dependence of the temperature dependent quantity $\mathfrak{T}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1, \omega_2\mathbf{p}_2, \omega_3\mathbf{p}_3, \omega_4\mathbf{p}_4)$ on the variable $\epsilon = \omega_1 + \omega_2$ is defined only in the discrete points $\epsilon_n = 2\pi nTi$, $n = 0, \pm 1, \pm 2, \ldots$ At the critical temperature the position of the pole is $\epsilon = 0$. Moreover, it is clear from similarity considerations that the Fourier component of the Matsubara quantity $\mathfrak{T}_{\alpha\beta\gamma\delta}(\omega_1 + \omega_2 = 0,\ \mathbf{p}_1 + \mathbf{p}_2 = 0)$ becomes infinite at the transition.

The remainder of our discussion will refer to the case $\epsilon = \omega_1 + \omega_2 = 0$, $\mathbf{q} = \mathbf{p}_1 + \mathbf{p}_2 = 0$.

The diagram of Fig. 1b contains in second order in f_0 an integral over the product of two Matsubara Green's functions $\mathfrak{G}(\omega\mathbf{k})$ for the perfect Fermi gas of the form

$$2\pi i f_0^2 T\sum_{\omega_n}\int \mathfrak{G}(\omega_n\mathbf{k})\,\mathfrak{G}(-\omega_n,-\mathbf{k})\,d^3\mathbf{k},$$

which after summation over ω_n becomes

$$\pi i f_0^2\int\frac{d^3\mathbf{k}}{\mu-\varepsilon(\mathbf{k})}\,\mathrm{th}\,\frac{\mu-\varepsilon(\mathbf{k})}{2T}. \quad (15)^*$$

The integral (15) diverges for large k. The divergence of this integral can be removed in the same way as at absolute zero by renormalizing the scattering amplitude. The integral diverges, however, also near the Fermi surface where its magnitude is $\sim (f_0^2p_0^2/v)\ln(T/\mu)$ which is compensated by the smallness of the parameter $f_0p_0^2/v \ll 1$. To evaluate $\mathfrak{T}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1, \omega_2\mathbf{p}_2, \omega_3\mathbf{p}_3, \omega_4\mathbf{p}_4)$ we must therefore again sum a "ladder" of diagrams such as Fig. 1b. We write down an equation for $\mathfrak{T}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1, \omega_2\mathbf{p}_2, \omega_3\mathbf{p}_3, \omega_4\mathbf{p}_4)$ which is similar to Eq. (5):

*th = tanh.

$$\mathfrak{T}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_3\mathbf{p}_3,\omega_4\mathbf{p}_4) = \widetilde{\mathfrak{T}}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_3\mathbf{p}_3,\omega_4\mathbf{p}_4) - \frac{T}{2(2\pi)^3}\sum_{\omega_n}\int\widetilde{\mathfrak{T}}_{\alpha\beta\xi\eta}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_n\mathbf{k},\varepsilon-\omega_n,\mathbf{q}-\mathbf{k}) \times \mathfrak{G}(\omega_n\mathbf{k})\,\mathfrak{G}(\varepsilon-\omega_n,\mathbf{q}-\mathbf{k}) \times \mathfrak{T}_{\xi\eta\gamma\delta}(\omega_n\mathbf{k},\varepsilon-\omega_n,\mathbf{q}-\mathbf{k},\omega_3\mathbf{p}_3,\omega_4\mathbf{p}_4)\,d^3\mathbf{k}. \quad (16)$$

We obtain an equation to determine the temperature at which the transition into the superfluid state takes place from (16) if we take into account that $\widetilde{\mathfrak{T}}$ has no singularities of the kind (15) and that we need only to know of the terms in $\widetilde{\mathfrak{T}}^{(2)}$ that one which gives a contribution $\sim (f_0^2p_0^4/v^2)\times\ln(T/\mu)$. Near the point where $\mathfrak{T}_{\alpha\beta\gamma\delta}(\epsilon, \mathbf{q})$ becomes infinite Eq. (16) becomes of the form

$$1 = -\frac{T_c}{(2\pi)^3}\sum_{\omega_n}\int V(\mathbf{p}_1-\mathbf{k})\,\mathfrak{G}(\omega_n\mathbf{k})\,\mathfrak{G}(-\omega_n,-\mathbf{k})\,d^3\mathbf{k} - \frac{T_c}{(2\pi)^3}\sum_{\omega_n}\int\widetilde{\mathfrak{T}}^{(2)}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_n\mathbf{k},-\omega_n,-\mathbf{k})\,\mathfrak{G}(\omega_n\mathbf{k}) \times \mathfrak{G}(-\omega_n,-\mathbf{k})\,d^3\mathbf{k}. \quad (17)$$

We have here taken into account that $\mathfrak{T}^{(1)}_{\alpha\beta\gamma\delta}(\omega_1\mathbf{p}_1, \omega_2\mathbf{p}_2, \omega_n\mathbf{k}, -\omega_n, -\mathbf{k})$ is independent of the fourth components of the momenta and is thus the same as the quantity $\Gamma^{(1)}_{\alpha\beta\gamma\delta}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}, -\mathbf{k})$.

As at $T = 0$, the divergence in the first integral in (17) far from the Fermi surface can be removed by renormalizing the scattering amplitude; after that Eq. (17) is reduced to the form

$$1 = -\frac{2f_0}{(2\pi)^3}\left(1-\frac{if_0p_0^2}{4\pi v}\right)\int\left[\frac{\mathrm{th}\,(\mu-\varepsilon(\mathbf{k}))/2T_c)}{\mu-\varepsilon(\mathbf{k})} + \frac{1}{\mu-\varepsilon(\mathbf{k})+i\delta}\right]d^3\mathbf{k} - \frac{T_c}{(2\pi)^3}\sum_{\omega_n}\int\widetilde{\mathfrak{T}}^{(2)}(\omega_1\mathbf{p}_1,\omega_2\mathbf{p}_2,\omega_n\mathbf{k}, -\omega_n,-\mathbf{k})\,\mathfrak{G}(\omega_n\mathbf{k})\,\mathfrak{G}(-\omega_n,-\mathbf{k})\,d^3\mathbf{k}. \quad (18)$$

The contribution to $\widetilde{\mathfrak{T}}^{(2)}$ which contains all irreducible diagrams of second order in the interaction is given by the expression

$$-\frac{f_0^2T_c}{(2\pi)^3}\sum_{\varepsilon_n}\int\mathfrak{G}(\varepsilon_n,\mathbf{l})\,\mathfrak{G}(\varepsilon_n+\omega_1-\omega_n,\mathbf{l}+\mathbf{p}_1-\mathbf{k})\,d^3\mathbf{l}.$$

The first integral in (18) gives the contribution

$$-\frac{p_0^2f_0}{2\pi^2v}\left(1-\frac{if_0p_0^2}{4\pi v}\right)\left(\ln\frac{8\gamma\mu}{T_c\pi}-2-\frac{i\pi}{2}\right), \quad (19)$$

where $\ln\gamma$ is Euler's constant.

The evaluation of the second integral in (18) proceeds taking the same considerations into account as at the absolute zero. The result of these calculations is

$$-\frac{p_0^4 f_0^2}{12\pi^4 v^2}(1+2\ln 2)\ln\frac{8\gamma\mu}{T_c\pi}. \qquad (20)$$

Substituting (19) and (20) into (18) we get

$$-\frac{2\pi^2 v}{|f_0|\,p_0^2}=\ln\left[\frac{T_c\pi}{\gamma\mu}\left(\frac{e}{2}\right)^{7/3}\right].$$

The temperature of the transition into the superfluid state is thus determined by the relation

$$T_c=(\gamma\mu/\pi)(2/e)^{7/3}\exp\{-2\pi^2 v/p_0^2|f_0|\}$$

or, $T_c = \gamma\Delta/\pi$, where Δ is the previously determined quantity which has the meaning of the reciprocal of the relaxation time of the system.

It is of interest to note that as in the model considered the integration practically is performed near the Fermi surface the relation between the "gap" width and the transition temperature must be the same as in the theory of superconductivity, and it follows thus that the gap width at $T = 0$ is equal to the reciprocal of the relaxation time of the system.

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[1] L. Cooper, Phys. Rev. **104**, 1189 (1956).

[2] Abrikosov, Gor'kov, and Dzyaloshinskii, JETP **36**, 900 (1959), Soviet Phys. JETP **9**, 636 (1959).

[3] E. S. Fradkin, JETP **36**, 1286 (1959), Soviet Phys. JETP **9**, 912 (1959).

[4] L. D. Landau, JETP **35**, 97 (1958), Soviet Phys. JETP **8**, 70 (1959).

[5] S. T. Belyaev, JETP **34**, 417 (1958), Soviet Phys. JETP **7**, 328 (1958).

[6] V. M. Galitskii, JETP **34**, 151 (1958), Soviet Phys. JETP **7**, 104 (1958).

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